Homological algebra/Functional analysis

The cyclic homology of crossed-product algebras, II ♠

Homologie cycliques des algèbres produits-croisés, II

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ARTICLE INFO

Article history:
Received 5 March 2017
Accepted 21 April 2017
Available online 10 May 2017
Presented by Alain Connes

ABSTRACT

In this note, we produce explicit quasi-isomorphisms computing the cyclic homology of crossed-product algebras associated with group actions on manifolds. We obtain explicit relationships with equivariant cohomology. On the way, we extend the results of the first part to the setting of group actions on locally convex algebras.

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RÉSUMÉ


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0. Introduction

In [13], referred hereafter as Part I, we constructed explicit quasi-isomorphisms for cyclic and periodic complexes of algebraic crossed-products $\mathcal{A} \rtimes \Gamma$, where $\Gamma$ is any group acting on a unital algebra $\mathcal{A}$ over a commutative ring $k \supseteq \mathbb{Q}$. In this note, we extend these results to actions on locally convex algebras where we use the cyclic space of completed chains. We then apply this results in the setting of group actions on manifolds to get explicit quasi-isomorphisms. For the finite-order components, the results are expressed in terms of what we call “mixed equivariant homology”. By using a cap product construction, this enables us to construct cyclic cycles out of equivariant characteristic classes. This improves the description of cyclic homology given in [5]. For the infinite-order components, we simplify the approach of [9] and correct the misidentification of the cyclic homology there. There are analogues of these results for group actions on varieties (see Remark 3.6).

Throughout this note we shall assume the notation, definitions and main results of Part I.

© Research partially supported by grants 2013R1A1A2008802 and 2016R1D1A1B01015971 of National Research Foundation of Korea.
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http://dx.doi.org/10.1016/j.crma.2017.04.013
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1. Actions on locally convex algebras

In this section, we extend the results of Part I to group actions on locally convex algebras. Given a unital locally convex algebra \( A \), we let \( C(A) \) be its cyclic space of completed chains \( (C_\bullet(A), d, s, t) \), where \( C(A) = \hat{A}^{0(m+1)} \) and \( \hat{\otimes} \) is the projective tensor product. The algebraic space of chains \( C_\bullet(A) \) is dense in \( C(A) \). The cyclic (resp., periodic cyclic) homology of \( C(A) \) is denoted by \( H_{C\bullet}(A) \) (resp., \( H_{C\bullet}(A) \)).

Let \( G \) be a group acting on \( A \) by continuous automorphisms. We endow the crossed-product algebra \( A_G := A \rtimes G \) with the weakest locally convex topology with respect to which the linear embeddings \( A \ni a \mapsto au_\phi \in A_{\phi} \) are continuous. With respect to this topology \( A_G \) is a locally convex algebra. In Part I we made use of the direct-sum of cyclic spaces \( C(A) = \oplus C(A_\phi) \), where the summation is over all conjugacy classes \([\phi]\) and \( C(A_\phi) \) is generated by chains \( d^i u_\theta \otimes \cdots \otimes d^j u_\phi \), with \( \theta_0 , \ldots , \theta_n \in [\phi] \). Let \( C_\bullet(A_\phi) \) be the closure of \( C_\bullet(A) \) in \( C_\bullet(A_\phi) \). We obtain a cyclic subspace and, as in the algebraic case, \( C(A_G) = \oplus C(A_\phi) \). Let us denote by \( H_{C\bullet}(A_G) \) (resp., \( H_{C\bullet}(A_G) \)) the cyclic (resp., periodic cyclic) homology of \( C(A_G) \). Then \( H_{C\bullet}(A_G) = \oplus H_{C\bullet}(A_\phi) \) and \( \oplus H_{C\bullet}(A_\phi) \subseteq H_{C\bullet}(A_G) \), where the inclusion is onto when \( G \) has finitely many conjugacy classes.

Given \( \phi \in G \), the structural operators \( (d_\phi, s_\phi, t_\phi) \) of the paracyclic \( C_\Gamma \)-module \( C_\Gamma(A) \) uniquely extends to continuous operators on \( C_\bullet(A) \) so that we obtain a paracyclic \( C_\Gamma \)-module \( C_\Gamma(A) := (C_\bullet(A), d_\phi, s_\phi, t_\phi) \). We denote by \( C^\phi(A) \) the cylindrical space \( C_\Gamma(A) \) as defined in Part I. This is just the tensor product over \( \Gamma\phi \) of the paracyclic \( C_\Gamma \)-modules \( C^\phi(\Gamma\phi) \) and \( C(A) \). The space of \((p,q)\)-chains is \( C^\phi_{p,q}(\Gamma\phi, A) := C_p(\Gamma\phi) \otimes_{\Gamma\phi} C_q(A) \). We equip it with the weakest locally convex topology with respect to which the linear embeddings \( C_\bullet(A_G) \supseteq c \mapsto (\psi_0, \ldots, \psi_p) \otimes_{\Gamma\phi} x \in C^\phi_{p,q}(\Gamma\phi, A) \), \( \psi_j \in \Gamma\phi \), are continuous. In Part I, we exhibited a cyclic space embedding and quasi-isomorphism \( \mu_\phi : \text{Diag}_\phi(C^\phi(\Gamma\phi,A)) \rightarrow C_\bullet(\Gamma\phi) \). It uniquely extends to a continuous embedding and quasi-isomorphism \( \mu_\phi : \text{Diag}_\phi(C^\phi(\Gamma\phi,A)) \rightarrow C_\bullet(\Gamma\phi) \). Therefore, we obtain quasi-isomorphisms of cyclic complexes,

\[
\text{Tot}_\bullet(C^\phi(\Gamma\phi,A)^\Sigma) \xrightarrow{\psi} \text{Diag}_\phi(C^\phi(\Gamma\phi,A)^\Sigma) \xrightarrow{\mu_\phi} C_\bullet(\Gamma\phi)^\Sigma.
\]

There are similar quasi-isomorphisms between the respective periodic cyclic complexes. The mixed complex \( \text{Tot}_\bullet(C^\phi(\Gamma\phi,A)^\Sigma) \) can be studied in the same way as in Part I. Thereon all the results of Section 4 and Section 5 of Part I for \( C(A_G) \) hold \emph{mutatis mutandis} for \( C(\Gamma\phi,A_G) \).

Suppose that \( \phi \) has finite order \( r \). As in Part I, given a \( \phi \)-invariant mixed complex \( C^\phi \) we denote by \( C^\phi(\Gamma\phi, C^\phi) \) the mixed bicomplex obtained as the tensor product over \( \Gamma\phi \) of the mixed complex \( C(\Gamma\phi) = (C(\Gamma\phi), \partial, 0) \) with \( C^\phi \). We refer to Part I for the definition of \( \phi \)-parachain complexes and \( \phi \)-cyclic spaces. If \( C^\phi \) is a \( \phi \)-parachain complex, then we denote by \( C^{\phi/\phi} \) its \( \phi \)-invariant subcomplex. This is a mixed complex, and so we may form the mixed bicomplex \( C^\phi(\Gamma\phi, C^{\phi/\phi}) \). As shown in Part I, we have an \( S \)-homotopy equivalence \( (\epsilon_\phi v) : C^\phi(\Gamma\phi)^\Sigma \rightarrow C^\phi(\Gamma\phi)^\Sigma \), where \( v^\phi : C^\phi(\Gamma\phi) \rightarrow C_\bullet(\Gamma\phi) \) and \( \epsilon : C_\bullet(\Gamma\phi) \rightarrow C^\phi(\Gamma\phi) \) are the parachain complex maps \( v_\phi(\psi_0, \ldots, \psi_m) = \frac{1}{m+1} \sum_{0 \leq s \leq m} \phi^s(\psi_0, \ldots, \phi^m(\psi_m)) \) and \( \epsilon(\psi_0, \ldots, \psi_m) = \frac{1}{(m+1)!} \sum_{\sigma \in S_m} \epsilon(\psi_{\sigma^{-1}(0)}, \ldots, \psi_{\sigma^{-1}(m)}) \). (Here \( S_m \) is the group of permutations of \([0, \ldots, m]\).)

**Theorem 1.1.** Let \( \phi \in G \) have finite order, and suppose we are given a quasi-isomorphism of \( \phi \)-parachain complexes \( \alpha : C^\phi(A) \rightarrow C^\phi_\perp(A) \). Then the following are quasi-isomorphisms of cyclic complexes,

\[
\text{Tot}_\bullet(C^\phi(\Gamma\phi,C^\phi/\phi)^\Sigma) \xleftarrow{(\epsilon_\phi v^\phi) \otimes \alpha} \text{Tot}_\bullet(C^\phi(\Gamma\phi,A)^\Sigma) \xrightarrow{\psi} \text{Diag}_\phi(C^\phi(\Gamma\phi,A)^\Sigma) \xrightarrow{\mu_\phi} C_\bullet(\Gamma\phi)^\Sigma.
\]

There are similar quasi-isomorphisms between the respective periodic cyclic complexes. This provides us with isomorphisms \( H_{C\bullet}(A_G) \cong H_{C\bullet}(\text{Tot}(C^\phi(\Gamma\phi,C^\phi/\phi))) \) and \( H_{C\bullet}(A_G) \cong H_{C\bullet}(\text{Tot}(C^\phi(\Gamma\phi,C^\phi/\phi))) \).

**Remark 1.2.** When \( \Gamma\phi \) is finite, there is an explicit \( S \)-homotopy equivalence between the cyclic complexes of \( \text{Tot}(C^\phi(\Gamma\phi,C^\phi/\phi)) \) and the \( \Gamma\phi \)-invariant mixed complex \( C^\phi_\perp(\Gamma\phi) \). We thus obtain explicit quasi-isomorphisms that identify \( H_{C\bullet}(A_G) \) and \( H_{C\bullet}(A_G) \) with \( H_{C\bullet}(C^\phi_\perp(\Gamma\phi)) \) and \( H_{C\bullet}(C^\phi_\perp(\Gamma\phi)) \).

Suppose now that \( \phi \) has infinite order. Set \( \Gamma_\phi = \Gamma\phi/\phi \), where \( \phi \) is the subgroup generated by \( \phi \). In addition, let \( u_\phi \in C^2(\Gamma\phi, C) \) be a group 2-cocycle representing the Euler class \( e_\phi \in H^2(\Gamma\phi, C) \) of the central extension \( 1 \rightarrow \phi \rightarrow \Gamma\phi \rightarrow \Gamma\phi \rightarrow 1 \). The cap product \( u_\phi \cap - : C_\bullet(\Gamma\phi) \rightarrow C_{\bullet-2}(\Gamma\phi) \) is a chain map, and so \( C^\phi_\perp(\Gamma\phi) := (C^\phi(\Gamma\phi), \partial, u_\phi \cap -) \) is an \( S \)-module in the sense of Jones–Kassel [11,12]. We refer to Part I for the definition of a triangular \( S \)-module. As in Part I, given any \( \phi \)-invariant mixed complex \( C^\phi = (C^\phi_\perp, b, B) \), we denote by \( C^\phi(\Gamma\phi, C^\phi) \) the triangular \( S \)-module given by the tensor product over \( \Gamma\phi \) of \( C^\phi(\Gamma\phi) \) and \( C^\phi \). Its total \( S \)-module is \( \text{Tot}(C^\phi(\Gamma\phi)) \), \( b^\phi(d_\phi u_\phi \cap -) \), where \( \text{Tot}(C^\phi(\Gamma\phi)) = \oplus_{\phi \in \Gamma_\phi} C_\bullet(\Gamma\phi) \otimes_{\Gamma\phi} C^\phi \) and \( d_\phi = \partial + (-1)^{\phi} B^\phi(u_\phi \cap -) \in C_\bullet(\Gamma\phi) \otimes_{\Gamma\phi} C^\phi \). In Part I, we constructed an explicit quasi-isomorphism \( \theta : \text{Tot}_\bullet(C^\phi(\Gamma\phi)) \rightarrow \text{Tot}_\bullet(C^\phi_{\perp}(\Gamma\phi, C^\phi)) \). We then have the following result.
Theorem 1.3. Let $\phi \in \Gamma$ have infinite order. Suppose we are given a quasi-isomorphism of parachain complexes $\alpha : C^p_\Gamma(A) \to \mathcal{C}_\bullet$, where $\mathcal{C}_\bullet$ is a $\phi$-invariant mixed complex. Then the following are quasi-isomorphisms of chain complexes,

$$
\text{Tot}_\ast(C^\sigma(\Gamma', \mathcal{C}_\bullet)) \xrightarrow{\delta(1_{\mathcal{C}_\bullet})} \text{Tot}_\ast(C^\sigma(\Gamma', A)) \xrightarrow{\delta} \text{Diag}_\ast(C^\phi(\Gamma, A)) \xrightarrow{\phi^\ast} C_\ast(\Lambda^1 \mathcal{A}_{\Gamma^1})[\phi].
$$

This gives an isomorphism $H_\ast(C^\sigma(\Gamma, \mathcal{C}_\bullet)) \simeq H_\ast(C^\sigma(\Gamma, \mathcal{C}_\bullet))$, under which the periodicity operator of $H_\ast(C^\sigma(\Gamma', \mathcal{C}_\bullet))$ is the cap product $e_\phi \cap -$ : $H_\ast(\text{Tot}(C^\sigma(\Gamma', \mathcal{C}_\bullet))) \to H_{\ast-2}(\text{Tot}(C^\sigma(\Gamma', \mathcal{C}_\bullet))).$

In the same way as in Part I, the bi-paracyclic Alexander–Whitney map enables us to construct a differential graded bilinear map $\triangleright : C^p(\Gamma, k) \times \text{Tot}_\ast(C^\sigma(\Gamma, A)) \longrightarrow \text{Tot}_\ast(C^\sigma(\Gamma, A))^3$. For general infinite order actions, we then have the following result.

Theorem 1.4. Let $\phi \in \Gamma$ have infinite order.

1. Suppose we are given a quasi-isomorphism of parachain complexes $\alpha : C^p_\Gamma(A) \to \mathcal{C}_\bullet$, where $\mathcal{C}_\bullet$ is a $\phi$-parachain complex. Then we have spectral sequence $E^2_{p, q} = H_p(\Gamma, H_q(\mathcal{C}_\bullet)) \Rightarrow H_{p+q}(\mathcal{C}_{\Gamma^1}[\phi]).$

2. The bilinear map $\triangleright$ and the quasi-isomorphisms (1) give rise to an associative action of the cohomology ring $H^\ast(\Gamma, \mathcal{C})$ on $H_\ast(C^\sigma(\Gamma, \mathcal{C}_\bullet))$. The periodicity operator is given by the action of the Euler class $e_\phi \in H^2(\Gamma, \mathcal{C})$. In particular, $H^p(\mathcal{C}_{\Gamma^1}[\phi]) = 0$ whenever $e_\phi$ is nilpotent in $H^q(\Gamma, \mathcal{C})$.

2. Equivariant cohomology and mixed equivariant homology

From now on we assume that $\Gamma$ acts by diffeomorphisms on a manifold $M$. Let $\Omega(M) = (\Omega^\ast(M), d)$ be the de Rham complex of differential forms on $M$. Recall that the equivariant cohomology $H^\ast_T(\Gamma, M)$ is the cohomology of the total complex of Bott’s cochain bicomplex $C^\ast_T(M) = (C^\ast_p(\Gamma, M), \delta, d)$, where $C^\ast_p(\Gamma, M) := C^p(\Gamma, \Omega^\ast(M))$ consists of $\Gamma$-equivariant maps $\omega : \Gamma \to \Omega^\ast(M)$. In other words, $H^\ast_T(\Gamma, M)$ is the cohomology of the cochain complex $(\text{Tot}_\ast(C^\ast_T(M)))$, where $\text{Tot}_\ast(C^\ast_T(M)) = \bigoplus_{p+q=n} C^p_q(\Gamma, M)$ and $d = \delta + (-1)^p d$. It is isomorphic to the cohomology of the homotopy quotient $ET \times_T M$.

The even/odd equivariant cohomology $H^\ast_{T_{ev/odd}}(\Gamma, M)$ is the cohomology of the complex $C^\ast_{T_{ev/odd}}(M) = (C^p_{T_{ev/odd}}(M), d)$, where $C^p_{T_{ev/odd}}(M) = \prod_{p+q\text{ even/odd}} C^p_q(\Gamma, M)$. This is a natural receptacle for the construction of equivariant characteristic classes (cf. [2,10]). In particular, given any $\Gamma$-equivariant vector bundle $E$ over $M$, we have a well-defined equivariant Chern character $\chi(\Gamma, E) \in H^0(\Gamma, M)$ (see [2,10]).

We can define the equivariant homology $H_\ast^T(\Gamma, M)$ of the $\Gamma$-manifold $M$ by using a dual version of Bott’s bicomplex. For our purpose, we actually need to construct a “mixed complex” version of equivariant homology. More precisely, we introduce the equivariant mixed bicomplex $\mathcal{C}^\ast_{T_{ev/odd}}(M) := (\mathcal{C}^\ast_{T_{ev/odd}}(\Gamma, M), \delta, 0, 0, d)$, where $\mathcal{C}^p_{T_{ev/odd}}(\Gamma, M) = C_p(\Gamma) \otimes \Omega^p(M)$. Its total mixed complex is $\text{Tot}(\mathcal{C}^\ast_{T_{ev/odd}}(\Gamma, M)) = (\text{Tot}_\ast(\mathcal{C}(\Gamma, M)), \delta, 0, (-1)^pd)$.

Definition 2.1. The cyclic homology of the mixed complex $\text{Tot}(\mathcal{C}(\Gamma, M))$ is called the mixed equivariant homology of the $\Gamma$-manifold $M$ and is denoted by $H_\ast^T(\Gamma, M)$. Its periodic cyclic homology is called the even/odd equivariant homology of $M$ and is denoted by $H^T_{ev/odd}(\Gamma, M)$.

The mixed equivariant homology is the natural receptacle of the cap product between equivariant cohomology and group homology. Namely, the usual cap product $\wedge : C_{T_{ev/odd}}^p(M) \times C_m(\Gamma, C) \to C_{m-p-q}(\Gamma, M)$ is compatible with the differentials $\delta$ and $d$, and so it gives rise to a cap product $\wedge : H^T_{ev/odd}(M) \times H^T_{ev/odd}(\Gamma, C) \to H^T_{ev/odd}(M)^2$. In particular, caping equivariant characteristic classes with group homology provides us with a geometric construction of mixed equivariant homology classes.

3. The cyclic homology of $C^\infty(M) \times \Gamma$

In this section, we assume that $\Gamma$ is a group acting by diffeomorphisms on a compact manifold $M$. We get an action on the Fréchet algebra $\mathcal{A} := C^\infty(M)$. We shall now explain how to use the results of the previous sections for constructing explicit quasi-isomorphisms for the cyclic and periodic homologies of the crossed-product algebra $\mathcal{A}_\Gamma = \mathcal{A} \rtimes \Gamma$. Given $\phi \in \Gamma$, we denote by $M^\phi$ its fixed-point set in $M$. We shall say that the action of $\phi$ on $M$ is clean when, for every $x_0 \in M^\phi$, the fixed-point set $M^\phi$ is a submanifold of $M$ near $x_0$, and we have $T_{x_0}M^\phi = \ker(\phi(x_0) - 1)$ and $T_{x_0}M = T_{x_0}M^\phi \oplus \text{ran}(\phi(x_0) - 1)$. These conditions are satisfied when $\phi$ preserves a metric or more generally an affine connection. In particular, they are always satisfied when $\phi$ has finite order.

Suppose that $\phi$ acts cleanly on $M$. For $a = 0, 1, \ldots, \dim M$, set $M^\phi_a := \{x \in M^\phi : \dim \ker(\phi(x) - 1) = a\}$. Each subset $M^\phi_a$ is a submanifold of $M$, and so we have a stratification $M^\phi = \bigsqcup M^\phi_a$. This enables us to define the de Rham complex...
Theorem 3.1. Let $\phi \in \Gamma$ have finite order. Then the following are quasi-isomorphisms,

$$
\text{Tot}_* \left( C(\Gamma_0, M^\phi) \right)^\natural \xleftarrow{(\epsilon M)\otimes \varphi^\phi} \text{Tot}_* \left( C^\phi(\Gamma_0, A) \right)^\natural \xrightarrow{\text{Diag}_* (C^\phi(\Gamma_0, A))^\natural} H^\phi C_* (A^\Gamma, \phi)^\natural.
$$

(2)

There are similar quasi-isomorphisms between the respective periodic cyclic complexes. We thus obtain isomorphisms $H_* \left( C^\phi(\Gamma_0, \phi) \right)^\natural \simeq H^\phi_* C_* (A^\Gamma, \phi)^\natural \simeq H_{\text{ev/odd}}^\phi (M^\phi)^\natural$.

Remark 3.2. Brylinski–Nistor [5] (see also Crainic [9]) expressed $H_* \left( C^\phi(\Gamma_0, \phi) \right)^\natural$ and $H_* \left( C^\phi(\Gamma_0, \phi) \right)^\natural$ in terms of the equivariant homology of $M^\phi$. We obtain explicit quasi-isomorphisms with the equivariant homology complex by combining the quasi-isomorphisms (2) with the Poincaré duality for the de Rham complex $\Omega(M^\phi)$. In particular, this enables us to recover the aforementioned results of [5]. When $\Gamma_0$ is finite, by Remark 1.2 we have quasi-isomorphisms that allow us to express $H_* \left( C^\phi(\Gamma_0, \phi) \right)^\natural$ and $H_* \left( C^\phi(\Gamma_0, \phi) \right)^\natural$ in terms of the $\Gamma_0$-invariant de Rham cohomology $H^* (M)^\Gamma$. In particular, when $\Gamma$ is finite we recover the description of $H_* \left( C^\phi(\Gamma_0, \phi) \right)^\natural$ given by Baum–Connes [1].

Let $\eta^\phi : H_{\text{ev/odd}}^\phi (M^\phi)^\natural \to H_* \left( C^\phi(\Gamma_0, \phi) \right)^\natural$ be the isomorphism defined by the quasi-isomorphisms (2). Composing it with the cap product from Section 2 provides us with the following corollary.

Corollary 3.3. Let $\phi \in \Gamma$ have finite order. Then we have a graded bilinear grading map,

$$
\eta^\phi (\cdot \smile \cdot) : H_{\text{ev/odd}}^\phi (M^\phi)^\natural \times H_{\text{ev/odd}}^\phi (\Gamma_0, \phi) \to H_* \left( C^\phi(\Gamma_0, \phi) \right)^\natural.
$$

In particular, equivariant characteristic classes naturally give rise to classes in $H_* \left( C^\phi(\Gamma_0, \phi) \right)^\natural$.

The definition of the isomorphism $\eta^\phi$ involves the bi-paracyclic versions of the shuffle and Alexander–Whitney maps. As it turns out, we actually obtain a very simple formula when we pair $\eta^\phi$ with cochains arising from equivariant currents. To see this, let $\Omega^\Gamma_0 (M) = (\Omega^\Gamma_0 (M), \partial)$ be the cochain complex of equivariant currents, where $\Omega^\Gamma_0 (M), m \geq 0, \text{ consists of maps } C : \Gamma \to \Omega^m (M)$ that are $\Gamma$-equivariant in the sense that $C(\gamma^{-1} \gamma_0 \psi_j) = (\gamma_0)_* C(\gamma_0 \psi_j)$ for all $\gamma_j \in \Gamma$. (Here $\Omega^m (M)$ is the space of $m$-dimensional currents.) Any equivariant current $C \in \Omega^\Gamma_0 (M)$ defines a cochain $C(\psi) \in C^0 (\Gamma_0, \phi)$ by $C(\psi \cdot h_0 \partial \psi_1 \cdot \cdots \cdot \partial \psi_m) = \frac{1}{m!} C(\psi \cdot f^0 \partial f^1 \cdot \cdots \cdot \partial f^m)$, where we have set $\psi = \psi_0 \cdot \cdots \cdot \psi_m$ and $\tilde{f} = f(\psi_0 \cdot \cdots \cdot \psi_{j-1}) \cdot \cdots \cdot f(\psi_0 \cdot \cdots \cdot \psi_m)$. This provides us with a map of mixed complexes from $(\Omega^\Gamma_0 (M), \partial, 0)$ to $(C^\phi(\Gamma_0, \phi), B, b)$. Therefore, we obtain cochain maps that satisfy the respective periodic and cochain periodic complexes. Note that the periodic cyclic complex of $\Omega^\Gamma_0 (M)$ is just $(\Omega^\Gamma_0 (M), d)$. The transverse fundamental class cocycle of Connes [7] and the CM cocycle of an equivariant Dirac spectral triple [14] are examples of cocycles arising from equivariant currents. In what follows, given any equivariant chain $\omega = (\omega p, q), \omega p, q \in C_{p, q} (\Gamma, M), \text{ we denote by } \omega_0 \text{ its component in } C_{0, 0} (\Gamma, M) \simeq \Omega^\Gamma_0 (M)$.

Proposition 3.1. Let $\phi \in \Gamma$ have finite order. Then, for any closed equivariant current $C \in \Omega^\Gamma_0 (\Gamma_0, \phi, M)$ and any equivariant cycle $\omega \in C^\epsilon(\Gamma_0, M, \phi)$, we have $(\varphi C, \eta^\phi (\omega)) = (C(\phi), \omega_0)$, where $\omega_0 \in C^\epsilon (\Gamma, M)$ is such that $\omega_0 |_{M^\phi} = \omega_0$.

Let $E$ be a $\Gamma_0$-equivariant vector bundle over a submanifold component $M^\phi$. Given any connection $\nabla^E$ on $E$, the equivariant Chern character of $E$ is represented by a cocycle $\text{Ch}_{\Gamma_0} (\nabla^E) \in C^\epsilon_{\text{ev}} (M^\phi)$ [see (2.10)]. The space $C_0 (\Gamma_0, \phi, \mathbb{C}) \simeq \mathbb{C}$ is spanned by the cycle $1 := 1 \otimes \gamma_0$. It can be checked that $(\text{Ch}_{\Gamma_0} (\nabla^E) \otimes 1)_0 = \text{Ch} (\nabla^E)$, where $\text{Ch} (\nabla^E)$ is the Chern form of $\nabla^E$. Thus, for any closed equivariant current $C \in C_{\text{ev}} (M)$ such that $C(\phi) \subset M^\phi$, we have $(\varphi C, \eta^\phi (\text{Ch}_{\Gamma_0} (\nabla^E) \otimes 1)) = (C(\phi), \text{Ch} (\nabla^E))$. More generally, let $\xi = \sum \lambda \psi_0 \otimes \psi_2 \otimes \cdots \otimes \psi_{2q}$ be a cycle in $C_{2q} (\Gamma_0, \phi, \mathbb{C})$. Then have
\begin{align}
  \langle \psi_\phi \rangle^\phi (\text{Ch}_\phi (\nabla^E \wedge \xi)) &= \sum \langle \text{C} (\phi), \text{CS} \left( (\psi_\phi^1)^E \wedge \ldots (\psi_\phi^q)^E \right) \rangle, \quad \text{where } \text{CS} ((\psi_\phi^0)^E, \ldots, (\psi_\phi^q)^E) \text{ is the Chern–Simons form of the connections } ((\psi_\phi^0)^E, \ldots, (\psi_\phi^q)^E) \text{ as defined in } [10].
\end{align}

Suppose now that \( \phi \) has infinite order and acts cleanly on \( M \). As \( \Omega (M^\phi) \) is a \( \phi \)-invariant mixed complex, we may form the triangular \( S \)-module \( C^\phi (\Gamma_\phi, \Omega (M^\phi)) \) as in Section 1 and Part I. Its total \( S \)-module is \( \text{Tot}_* (C^\phi (\Gamma_\phi, \Omega (M^\phi))), \) \( d^1, \alpha \phi \wedge - \), where \( \text{Tot}_m (C^\phi (\Gamma_\phi, \Omega (M^\phi))) = \oplus_{p+q=m} C_p (\Gamma_\phi) \otimes \Omega^q (M^\phi) \) and \( d^1 = \partial + (-1)^p \alpha \phi \wedge - \) on \( C_p (\Gamma_\phi) \otimes \Omega^q (M^\phi) \). Applying Theorem 3.4 then gives the following result.

**Theorem 3.4.** Let \( \phi \in \Gamma \) have infinite order and act cleanly on \( M \). The following are quasi-isomorphisms,
\begin{align}
  \text{Tot}_* (C^\phi (\Gamma_\phi, \Omega (M^\phi))) \xrightarrow{(1 \otimes \alpha \phi)} \text{Tot}_* (C^\phi (\Gamma_\phi, A)) \xrightarrow{\text{Diag}_* (C^\phi (\Gamma_\phi, A)).} \xrightarrow{H_\phi} C_\phi (A, [\phi]) \xrightarrow{\gamma}.
\end{align}

This gives an isomorphism \( \text{HC}_\phi (A)_{[\phi]} \cong H_* (\text{Tot}(C^\phi (\Gamma_\phi, \Omega (M^\phi)))) \), under which the periodicity operator of \( \text{HC}_\phi (A)_{[\phi]} \) is the cap product \( e_\phi \wedge - : H_* (\text{Tot}(C^\phi (\Gamma_\phi, \Omega (M^\phi)))) \rightarrow H_{*-2} (\text{Tot}(C^\phi (\Gamma_\phi, \Omega (M^\phi)))) \).

**Remark 3.5.** The quasi-isomorphisms \( (3) \) and the filtration by columns of \( \text{Tot}_* (C^\phi (\Gamma_\phi, \Omega (M^\phi))) \) give rise to a spectral sequence \( E^p_{p,q} = H_p (\Gamma_\phi, \Omega^q (M^\phi)) \Rightarrow \text{HC}^{p+q} (A, \Gamma_\phi) \), where the \( E^2 \)-differential is given by \( (\Gamma_\phi, \Omega^{q+1} (M^\phi)) \Rightarrow H_{p-2} (\Gamma_\phi, \Omega^{q+1} (M^\phi)) \). Crainic [9] obtained such a spectral sequence, and inferred from this that \( \text{HC}_m (A)_{[\phi]} \cong \oplus_{p+q=m} H_p (\Gamma_\phi, \Omega^q (M^\phi)) \) (see [9, Corollary 4.15]). What we really have is the isomorphism \( \text{HC}_\phi (A)_{[\phi]} \cong H_* (\text{Tot}(C^\phi (\Gamma_\phi, \Omega (M^\phi)))) \) given by Theorem 3.4.

**Remark 3.6.** All the results of this section have analogues for group actions on smooth varieties by combining the results of Part I with the twisted Hochschild–Kostant–Rosenberg Theorem of [3]. When \( \Gamma_\phi \) is finite, we get explicit quasi-isomorphisms that enable us to recover the description of cyclic and periodic homology in terms of (algebraic) orbifold cohomology in [3]. More generally, when \( \phi \) has finite order, the results are expressed in terms of a mixed equivariant homology for smooth varieties. Furthermore, the framework of Section 1 enables us to extend those results to the \( 1 \)-adic completions considered in [3].

**Acknowledgements**

I wish to thank Alain Connes, Sasha Gorokhovsky, Masoud Khalkhali, Henri Moscovici, Victor Nistor, Markus Pflaum, Hessel Posthuma, Bahram Rangipour, Xiang Tang, and Hang Wang for various discussions related to the subject matter of this note.

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