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The cyclic homology of crossed-product algebras, II ☆

Homologie cycliques des algèbres produits-croisés, II

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ABSTRACT

In this note, we produce explicit quasi-isomorphisms computing the cyclic homology of crossed-product algebras associated with group actions on manifolds. We obtain explicit relationships with equivariant cohomology. On the way, we extend the results of the first part to the setting of group actions on locally convex algebras.

RÉSUMÉ

Dans cette note, on produit des quasi-isomorphismes explicites calculant l'homologie cyclique des algèbres produits-croisés provenant d'actions de groupes sur les variétés. On obtient des liens avec la cohomologie équivariante. On étend aussi les résultats de la première partie au cadre des actions de groupes sur les algèbres localement convexes.

0. Introduction

In [13], referred hereafter as Part I, we constructed explicit quasi-isomorphisms for cyclic and periodic complexes of algebraic crossed-products $\mathcal{A} \rtimes \Gamma$, where Γ is any group acting on a unital algebra $\mathcal{A}$ over a commutative ring $k \supset \mathbb{Q}$. In this note, we extend these results to actions on locally convex algebras where we use the cyclic space of completed chains. We then apply this results in the setting of group actions on manifolds to get explicit quasi-isomorphisms. For the finite-order components, the results are expressed in terms of what we call "mixed equivariant homology". By using a cap product construction, this enables us to construct cyclic cycles out of equivariant characteristic classes. This improves the description of cyclic homology given in [5]. For the infinite-order components, we simplify the approach of [9] and correct the misidentification of the cyclic homology there. There are analogues of these results for group actions on varieties (see Remark 3.6).

Throughout this note we shall assume the notation, definitions and main results of Part I.

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1. Actions on locally convex algebras

In this section, we extend the results of Part I to group actions on locally convex algebras. Given a unital locally convex algebra $\mathcal{A}$, we let $\mathbf{C}(\mathcal{A})$ be its cyclic space of completed chains $(\mathbf{C}_\bullet(\mathcal{A}), d, s, t)$, where $\mathbf{C}(\mathcal{A}) = \mathcal{A}^{\hat{\otimes}(m+1)}$ and $\hat{\otimes}$ is the projective tensor product. The algebraic space of chains $C_\bullet(\mathcal{A})$ is dense in $\mathbf{C}_\bullet(\mathcal{A})$. The cyclic (resp., periodic cyclic) homology of $\mathbf{C}(\mathcal{A})$ is denoted by $\mathbf{HC}_\bullet(\mathcal{A})$ (resp., $\mathbf{HP}_\bullet(\mathcal{A})$).

Let Γ be a group acting on $\mathcal{A}$ by continuous automorphisms. We endow the crossed-product algebra $\mathcal{A}_\Gamma := \mathcal{A} \rtimes \Gamma$ with the weakest locally convex topology with respect to which the linear embeddings $\mathcal{A} \ni a \to au_\phi \in \mathcal{A}_\Gamma$, $\phi \in \Gamma$, are continuous. With respect to this topology $\mathcal{A}_\Gamma$ is a locally convex algebra. In Part I we made use of the direct-sum of cyclic spaces $C(\mathcal{A}) = \oplus C(\mathcal{A}_\Gamma)_{[\phi]}$, where the summation is over all conjugacy classes $[\phi]$ and $C(\mathcal{A})_{[\phi]}$ is generated by chains $a^0 u_{\phi_0} \otimes \cdots \otimes a^m u_{\phi_m}$ with $\phi_0 \cdots \phi_m \in [\phi]$. Let $\mathbf{C}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ be the closure of $C_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ in $\mathbf{C}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$. We obtain a cyclic subspace and, as in the algebraic case, $\mathbf{C}(\mathcal{A}_\Gamma) = \oplus \mathbf{C}(\mathcal{A}_\Gamma)_{[\phi]}$. Let us denote by $\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ (resp., $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$) the cyclic (resp., periodic cyclic) homology of $\mathbf{C}(\mathcal{A}_\Gamma)_{[\phi]}$. Then $\mathbf{HC}_\bullet(\mathcal{A}_\Gamma) = \oplus \mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ and $\oplus \mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \subset \mathbf{HP}_\bullet(\mathcal{A}_\Gamma)$, where the inclusion is onto when Γ has finitely many conjugacy classes.

Given $\phi \in \Gamma$, the structural operators (d_ϕ, s, t_ϕ) of the paracyclic $\mathbb{C}\Gamma_\phi$-module $C^\phi(\mathcal{A})$ uniquely extends to continuous operators on $\mathbf{C}_\bullet(\mathcal{A})$ so that we obtain a paracyclic $\mathbb{C}\Gamma_\phi$-module $\mathbf{C}^\phi(\mathcal{A}) := (\mathbf{C}_\bullet(\mathcal{A}), d_\phi, s, t_\phi)$. We denote by $\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})$ the cylindrical space $C^\phi(\Gamma_\phi, \mathbf{C}^\phi(\mathcal{A}))$ as defined in Part I. This is just the tensor product over Γ_ϕ of the paracyclic $\mathbb{C}\Gamma_\phi$-modules $C^\phi(\Gamma_\phi)$ and $\mathbf{C}^\phi(\mathcal{A})$. The space of (p,q)-chains is $\mathbf{C}^\phi_{p,q}(\Gamma_\phi, \mathcal{A}) := C_p(\Gamma_\phi) \otimes_{\Gamma_\phi} \mathbf{C}_q(\mathcal{A})$. We equip it with the weakest locally convex topology with respect to which the linear embeddings $\mathbf{C}_q(\mathcal{A}) \ni \xi \to (\psi_0, \ldots, \psi_p) \otimes_{\Gamma_\phi} \xi \in \mathbf{C}^\phi_{p,q}(\Gamma_\phi, \mathcal{A})$, $\psi_j \in \Gamma_\phi$, are continuous. In Part I, we exhibited a cyclic space embedding and quasi-isomorphism $\mu_\phi : \mathrm{Diag}_\bullet(C^\phi(\Gamma_\phi, \mathcal{A})) \to C_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$. It uniquely extends to a continuous embedding and quasi-isomorphism $\mu_\phi : \mathrm{Diag}_\bullet(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})) \to \mathbf{C}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$. Therefore, we obtain quasi-isomorphisms of cyclic complexes,

$$\mathrm{Tot}_\bullet(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A}))^\natural \underset{\mathrm{AW}^\natural}{\overset{\sqcup\!\sqcup^\natural}{\rightleftarrows}} \mathrm{Diag}_\bullet(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A}))^\natural \xrightarrow{\mu_\phi} \mathbf{C}_\bullet(\mathcal{A}_\Gamma)^\natural_{[\phi]}. \tag{1}$$

There are similar quasi-isomorphisms between the respective periodic cyclic complexes. The mixed complex $\mathrm{Tot}_\bullet\big(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\big)^\natural$ can be studied in the same way as in Part I. Thereon all the results of Section 4 and Section 5 of Part I for $C(\mathcal{A}_\Gamma)_{[\phi]}$ hold *mutatis mutandis* $\mathbf{C}(\mathcal{A}_\Gamma)_{[\phi]}$ by replacing $C^\phi(\Gamma_\phi, \mathcal{A})$ and $C^\phi(\mathcal{A})$ by their closures $\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})$ and $\mathbf{C}^\phi(\mathcal{A})$.

Suppose that ϕ has finite order r. As in Part I, given a ϕ-invariant mixed complex $\mathscr{C}$ we denote by $C^\flat(\Gamma_\phi, \mathscr{C})$ the mixed bicomplex obtained as the tensor product over Γ_ϕ of the mixed complex $C^\flat(\Gamma_\phi) = (C(\Gamma_\phi), \partial, 0)$ with $\mathscr{C}$. We refer to Part I for the definitions of ϕ-parachain complexes and ϕ-cyclic spaces. If $\mathscr{C}$ is a ϕ-parachain complex, then we denote by $\mathscr{C}^\phi$ its ϕ-invariant subcomplex. This is a mixed complex, and so we may form the mixed bicomplex $C^\flat(\Gamma_\phi, \mathscr{C}^\phi)$. As shown in Part I, we have an S-homotopy equivalence $(\varepsilon\nu_\phi) : C^\phi(\Gamma_\phi)^\natural \to C^\flat(\Gamma_\phi)^\natural$, where $\nu_\phi : C^\phi_\bullet(\Gamma_\phi) \to C_\bullet(\Gamma_\phi)$ and $\varepsilon : C_\bullet(\Gamma_\phi) \to C^\flat_\bullet(\Gamma_\phi)$ are the parachain complex maps $\nu_\phi(\psi_0, \ldots, \psi_m) = \frac{1}{r^{m+1}} \sum_{0 \le \ell_j \le r-1} (\phi^{\ell_0}\psi_0, \ldots, \phi^{\ell_m}\psi_m)$ and $\varepsilon(\psi_0, \ldots, \psi_m) = \frac{1}{(m+1)!} \sum_{\sigma \in \mathfrak{S}_m} (\psi_{\sigma^{-1}(0)}, \ldots, \psi_{\sigma^{-1}(m)})$. (Here $\mathfrak{S}_m$ is the group of permutations of $\{0, \ldots, m\}$.)

Theorem 1.1. *Let $\phi \in \Gamma$ have finite order, and suppose we are given a quasi-isomorphism of ϕ-parachain complexes $\alpha : \mathbf{C}^\phi_\bullet(\mathcal{A}) \to \mathscr{C}_\bullet$. Then the following are quasi-isomorphisms of cyclic complexes,*

$$\mathrm{Tot}_\bullet\left(C^\flat(\Gamma_\phi, \mathscr{C}^\phi)\right)^\natural \xleftarrow{(\varepsilon\nu_\phi)\otimes\alpha} \mathrm{Tot}_\bullet\left(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \underset{\mathrm{AW}^\natural}{\overset{\sqcup\!\sqcup^\natural}{\rightleftarrows}} \mathrm{Diag}_\bullet\left(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \xrightarrow{\mu_\phi} \mathbf{C}_\bullet(\mathcal{A}_\Gamma)^\natural_{[\phi]}.$$

There are similar quasi-isomorphisms between the respective periodic cyclic complexes. This provides us with isomorphisms $\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \simeq \mathrm{HC}_\bullet(\mathrm{Tot}(C^\flat(\Gamma_\phi, \mathscr{C}^\phi)))$ and $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \simeq \mathrm{HP}_\bullet(\mathrm{Tot}(C^\flat(\Gamma_\phi, \mathscr{C}^\phi)))$.

Remark 1.2. When Γ_ϕ is finite, there is an explicit S-homotopy equivalence between the cyclic complexes of $\mathrm{Tot}\big(C^\flat(\Gamma_\phi, \mathscr{C}^\phi)\big)$ and the Γ_ϕ-invariant mixed complex $\mathscr{C}^{\Gamma_\phi}$. We thus obtain explicit quasi-isomorphisms that identify $\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ and $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ with $\mathrm{HC}_\bullet(\mathscr{C}^{\Gamma_\phi})$ and $\mathrm{HP}_\bullet(\mathscr{C}^{\Gamma_\phi})$.

Suppose now that ϕ has infinite order. Set $\overline{\Gamma}_\phi = \Gamma_\phi/\langle\phi\rangle$, where $\langle\phi\rangle$ is the subgroup generated by ϕ. In addition, let $u_\phi \in C^2(\overline{\Gamma}_\phi, \mathbb{C})$ be a group 2-cocycle representating the Euler class $e_\phi \in H^2(\overline{\Gamma}_\phi, \mathbb{C})$ of the central extension $1 \to \langle\phi\rangle \to \Gamma_\phi \to \overline{\Gamma}_\phi \to 1$. The cap product $u_\phi \frown - : C_\bullet(\overline{\Gamma}_\phi) \to C_{\bullet-2}(\overline{\Gamma}_\phi)$ is a chain map, and so $C^\sigma(\overline{\Gamma}_\phi) := (C_\bullet(\overline{\Gamma}_\phi), \partial, u_\phi \frown -)$ is an S-module in the sense of Jones–Kassel [11,12]. We refer to Part I for the definition of a triangular S-module. As in Part I, given any ϕ-invariant mixed complex $\mathscr{C} = (\mathscr{C}_\bullet, b, B)$, we denote by $C^\sigma(\overline{\Gamma}_\phi, \mathscr{C})$ the triangular S-module given by the tensor product over $\overline{\Gamma}_\phi$ of $C^\sigma(\overline{\Gamma}_\phi)$ and $\mathscr{C}$. Its total S-module is $(\mathrm{Tot}_\bullet(C^\sigma(\overline{\Gamma}_\phi, \mathscr{C})), d^\dagger, u_\phi \frown -)$, where $\mathrm{Tot}_m(C^\sigma(\overline{\Gamma}_\phi, \mathscr{C})) = \oplus_{p+q=m} C_p(\overline{\Gamma}_\phi) \otimes_{\overline{\Gamma}_\phi} \mathscr{C}_q$ and $d^\dagger = \partial + (-1)^p b + (-1)^p B(u_\phi \frown -)$ on $C_p(\overline{\Gamma}_\phi) \otimes_{\overline{\Gamma}_\phi} \mathscr{C}_q$. In Part I, we constructed an explicit quasi-isomorphism $\theta : \mathrm{Tot}_\bullet(C^\phi(\Gamma, \mathscr{C}))^\natural \to \mathrm{Tot}_\bullet(C^\sigma(\overline{\Gamma}, \mathscr{C}))$. We then have the following result.

Theorem 1.3. *Let $\phi \in \Gamma$ have infinite order. Suppose we are given a quasi-isomorphism of parachain complexes $\alpha : C^{\phi}_{\bullet}(\mathcal{A}) \to \mathscr{C}_{\bullet}$, where $\mathscr{C}$ is a ϕ-invariant mixed complex. Then the following are quasi-isomorphisms of chain complexes,*

$$\mathrm{Tot}_{\bullet}(C^{\sigma}(\overline{\Gamma}_{\phi}, \mathscr{C})) \xleftarrow{\theta(1\otimes\alpha)} \mathrm{Tot}_{\bullet}\left(\mathbf{C}^{\phi}(\Gamma_{\phi}, \mathcal{A})\right)^{\natural} \underset{\mathrm{AW}^{\natural}}{\overset{\text{Ш}^{\natural}}{\rightleftarrows}} \mathrm{Diag}_{\bullet}\left(C^{\phi}(\Gamma_{\phi}, \mathcal{A})\right)^{\natural} \xrightarrow{\mu_{\phi}} \mathbf{C}_{\bullet}(\mathcal{A}_{\Gamma})^{\natural}_{[\phi]}.$$

This gives an isomorphism $\mathbf{HC}_{\bullet}(\mathcal{A})_{[\phi]} \simeq H_{\bullet}(C^{\sigma}(\overline{\Gamma}_{\phi}, \mathscr{C}))$, under which the periodicity operator of $\mathbf{HC}_{\bullet}(\mathcal{A})_{[\phi]}$ is the cap product $e_{\phi} \frown - : H_{\bullet}(\mathrm{Tot}(C^{\sigma}(\overline{\Gamma}_{\phi}, \mathscr{C}))) \to H_{\bullet-2}(\mathrm{Tot}(C^{\sigma}(\overline{\Gamma}_{\phi}, \mathscr{C})))$.

In the same way as in Part I, the bi-paracyclic Alexander–Whitney map enables us to construct a differential graded bilinear map $\triangleright : C^{\bullet}(\overline{\Gamma}_{\phi}, k) \times \mathrm{Tot}_{\bullet}\left(\mathbf{C}^{\phi}(\Gamma_{\phi}, \mathcal{A})\right)^{\natural} \longrightarrow \mathrm{Tot}_{\bullet}\left(\mathbf{C}^{\phi}(\Gamma_{\phi}, \mathcal{A})\right)^{\natural}$). For general infinite order actions, we then have the following result.

Theorem 1.4. *Let $\phi \in \Gamma$ have infinite order.*

(1) Suppose we are given a quasi-isomorphism of parachain complexes $\alpha : C^{\phi}_{\bullet}(\mathcal{A}) \to \mathscr{C}_{\bullet}$, where $\mathscr{C}$ is a ϕ-parachain complex. Then we have spectral sequence $E^2_{p,q} = H_p(\overline{\Gamma}_{\phi}, H_q(\mathscr{C})) \Longrightarrow \mathrm{HC}_{p+q}(\mathcal{A}_{\Gamma})_{[\phi]}$.
(2) The bilinear map $\triangleright$ and the quasi-isomorphisms (1) give rise to an associative action of the cohomology ring $H^{\bullet}(\overline{\Gamma}_{\phi}, \mathbb{C})$ on $\mathbf{HC}_{\bullet}(\mathcal{A}_{\Gamma})_{[\phi]}$. The periodicity operator is given by the action of the Euler class $e_{\phi} \in H^2(\overline{\Gamma}_{\phi}, \mathbb{C})$. In particular, $\mathbf{HP}_{\bullet}(\mathcal{A}_{\Gamma})_{[\phi]} = 0$ whenever e_{ϕ} is nilpotent in $H^{\bullet}(\Gamma_{\phi}, \mathbb{C})$.

2. Equivariant cohomology and mixed equivariant homology

From now on we assume that Γ acts by diffeomorphisms on a manifold M. Let $\Omega(M) = (\Omega^{\bullet}(M), d)$ be the de Rham complex of differential forms on M. Recall that the *equivariant cohomology* $H^{\bullet}_{\Gamma}(M)$ is the cohomology of the total complex of Bott's cochain bicomplex $C_{\Gamma}(M) = (C^{\bullet,\bullet}_{\Gamma}(M), \partial, d)$, where $C^{p,q}_{\Gamma}(M) := C^p(\Gamma, \Omega^q(M))$ consists of Γ-equivariant maps $\omega : \Gamma^{p+1} \to \Omega^q(M)$. In other words, $H^{\bullet}_{\Gamma}(M)$ is the cohomology of the cochain complex $(\mathrm{Tot}_{\bullet}(C_{\Gamma}(M)), d^{\dagger})$, where $\mathrm{Tot}_m(C_{\Gamma}(M)) = \oplus_{p+q=m} C^{p,q}_{\Gamma}(M)$ and $d^{\dagger} = \partial + (-1)^p d$ on $C^{p,q}_{\Gamma}(M)$. It is isomorphic to the cohomology of the homotopy quotient $E\Gamma \times_{\Gamma} M$.

The *even/odd equivariant cohomology* $H^{\mathrm{ev/odd}}_{\Gamma}(M)$ is the cohomology of the complex $C^{\mathrm{ev/odd}}_{\Gamma}(M) = (C^{\mathrm{ev/odd}}_{\Gamma}(M), d^{\dagger})$, where $C^{\mathrm{ev/odd}}_{\Gamma}(M) = \prod_{p+q \text{ even/odd}} C^{p,q}_{\Gamma}(M)$. This is a natural receptacle for the construction of equivariant characteristic classes (*cf.* [2]). In particular, given any Γ-equivariant vector bundle E over M, we have a well-defined equivariant Chern character $\mathrm{Ch}_{\Gamma}(E) \in H^{\mathrm{ev}}_{\Gamma}(M)$ (see [2,10]).

We can define the equivariant homology $H^{\Gamma}_{\bullet}(M)$ of the Γ-manifold M by using a dual version of Bott's bicomplex. For our purpose, we actually need to construct a "mixed complex" version of equivariant homology. More precisely, we introduce the *equivariant mixed bicomplex* $C(\Gamma, M) := (C_{\bullet,\bullet}(\Gamma, M), \partial, 0, 0, d)$, where $C_{p,q}(\Gamma, M) = C_p(\Gamma) \otimes_{\Gamma} \Omega^q(M)$. Its total mixed complex is $\mathrm{Tot}(C(\Gamma, M)) = (\mathrm{Tot}_{\bullet}(C(\Gamma, M), \partial, (-1)^p d)$.

Definition 2.1. The cyclic homology of the mixed complex $\mathrm{Tot}(C(\Gamma, M))$ is called the *mixed equivariant homology* of the Γ-manifold M and is denoted by $H^{\Gamma}_{\bullet}(M)^{\natural}$. Its periodic cyclic homology is called the *even/odd mixed equivariant homology* of M and is denoted by $H^{\Gamma}_{\mathrm{ev/odd}}(M)^{\sharp}$.

The mixed equivariant homology is the natural receptacle of the cap product between equivariant cohomology and group homology. Namely, the usual cap product $\frown: C^{p,q}_{\Gamma}(M) \times C_m(\Gamma, \mathbb{C}) \to C_{m-p,q}(\Gamma, M)$ is compatible with the differentials ∂ and d, and so it gives rise to a cap product $\frown: H^{\mathrm{ev/odd}}_{\Gamma}(M) \times H_{\mathrm{ev/odd}}(\Gamma, \mathbb{C}) \to H^{\Gamma}_{\mathrm{ev/odd}}(M)^{\sharp}$. In particular, caping equivariant characteristic classes with group homology provides us with a geometric construction of mixed equivariant homology classes.

3. The cyclic homology of $C^{\infty}(M) \rtimes \Gamma$

In this section, we assume that Γ is a group acting by diffeomorphisms on a compact manifold M. We get an action on the Fréchet algebra $\mathcal{A} := C^{\infty}(M)$. We shall now explain how to use the results of the previous sections for constructing explicit quasi-isomorphisms for the cyclic and periodic homologies of the crossed-product algebra $\mathcal{A}_{\Gamma} = \mathcal{A} \rtimes \Gamma$. Given $\phi \in \Gamma$, we denote by M^{ϕ} its fixed-point set in M. We shall say that the action of ϕ on M is *clean* when, for every $x_0 \in M^{\phi}$, the fixed-point set M^{ϕ} is a submanifold of M near x_0, and we have $T_{x_0}M^{\phi} = \ker(\phi'(x_0) - 1)$ and $T_{x_0}M = T_{x_0}M^{\phi} \oplus \mathrm{ran}(\phi'(x_0) - 1)$. These conditions are satisfied when ϕ preserves a metric or more generally an affine connection. In particular, they are always satisfied when ϕ has finite order.

Suppose that ϕ acts cleanly on M. For $a = 0, 1, \ldots, \dim M$, set $M^{\phi}_a := \{x \in M^{\phi};\ \dim\ker(\phi'(x) - 1) = a\}$. Each subset M^{ϕ}_a is a submanifold of M, and so we have a stratification $M^{\phi} = \bigsqcup M^{\phi}_a$. This enables us to define the de Rham complex

$\Omega(M^\phi)=(\Omega^\bullet(M^\phi),d)$ as the direct sum of the de Rham complexes $\Omega(M_a^\phi)$. Note also that each component M_a^ϕ is preserved by the action of the centralizer Γ_ϕ. We also define the equivariant bicomplex $C_{\Gamma_\phi}(M^\phi)$ and the equivariant mixed bicomplex $C(\Gamma_\phi,M^\phi)$ as the direct sums of the bicomplexes $C_{\Gamma_\phi}(M_a^\phi)$ and $C(\Gamma_\phi,M_a^\phi)$, respectively. This enables us to define the equivariant cohomology $H^\bullet_{\Gamma_\phi}(M^\phi)$ and the mixed equivariant homology $H_\bullet^{\Gamma_\phi}(M^\phi)^\natural$. We have a map of parachain complexes $\alpha^\phi:\mathbf{C}_\bullet^\phi(\mathcal{A})\to\Omega^\bullet(M^\phi)$ given by $\alpha^\phi(f^0\otimes\cdots\otimes f^m)=\frac{1}{m!}\sum_a(f^0df^1\wedge\cdots\wedge df^m)_{|M_a^\phi}$, $f^j\in\mathcal{A}$. It is known to be a quasi-isomorphism ([4,5]). For $\phi=1$, this result agrees with that of Connes [6]. We thus can input this quasi-isomorphism into the framework of Section 1 to get explicit quasi-isomorphisms as follows.

Suppose that ϕ has finite order. For $\phi=1$ Connes [7,8] constructed an explicit quasi-isomorphism from $C_\Gamma^{\mathrm{ev/odd}}(M)$ to the periodic cyclic cochain complex of the homogeneous component $\mathbf{C}(\mathcal{A}_\Gamma)_{[1]}$. In general, as the mixed equivariant complex $C(\Gamma_\phi,M^\phi)$ is just the mixed bicomplex $C^\flat(\Gamma_\phi,\mathscr{C})$ for $\mathscr{C}=\Omega(M^\phi)$, Theorem 1.1 immediately gives the following result.

Theorem 3.1. *Let $\phi\in\Gamma$ have finite order. Then the following are quasi-isomorphisms,*

$$\operatorname{Tot}_\bullet\left(C(\Gamma_\phi,M^\phi)\right)^\natural \xleftarrow{(\varepsilon\nu_\phi)\otimes\alpha^\phi} \operatorname{Tot}_\bullet\left(\mathbf{C}^\phi(\Gamma_\phi,\mathcal{A})\right)^\natural \underset{\mathrm{AW}^\natural}{\overset{\text{ш}^\natural}{\rightleftarrows}} \operatorname{Diag}_\bullet\left(\mathbf{C}^\phi(\Gamma_\phi,\mathcal{A})\right)^\natural \xrightarrow{\mu_\phi} \mathbf{C}_\bullet(\mathcal{A}_\Gamma)^\natural_{[\phi]}. \tag{2}$$

There are similar quasi-isomorphisms between the respective periodic cyclic complexes. We thus obtain isomorphisms $\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}\simeq H_\bullet^{\Gamma_\phi}(M^\phi)^\natural$ and $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}\simeq H^{\Gamma_\phi}_{\mathrm{ev/odd}}(M^\phi)^\sharp$.

Remark 3.2. Brylinski–Nistor [5] (see also Crainic [9]) expressed $\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ and $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ in terms of the equivariant homology of M^ϕ. We obtain explicit quasi-isomorphisms with the equivariant homology complex by combining the quasi-isomorphisms (2) with the Poincaré duality for the de Rham complex $\Omega(M^\phi)$. In particular, this enables us to recover the aforementioned results of [5]. When Γ_ϕ is finite, by Remark 1.2 we have quasi-isomorphisms that allow us to express $\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ and $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ in terms of the Γ_ϕ-invariant de Rham cohomology $H^\bullet(M^\phi)^{\Gamma_\phi}$. In particular, when Γ is finite we recover the description of $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)$ given by Baum–Connes [1].

Let $\eta^\phi:H^{\Gamma_\phi}_{\mathrm{ev/odd}}(M^\phi)^\sharp\to\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ be the isomorphism defined by the quasi-isomorphisms (2). Composing it with the cap product from Section 2 provides us with the following corollary.

Corollary 3.3. *Let $\phi\in\Gamma$ have finite order. Then we have a graded bilinear graded map,*

$$\eta^\phi(-\frown-):H^{\mathrm{ev/odd}}_{\Gamma_\phi}(M^\phi)\times H_{\mathrm{ev/odd}}(\Gamma_\phi,\mathbb{C})\longrightarrow\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}.$$

In particular, equivariant characteristic classes naturally give rise to classes in $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$.

The definition of the isomorphism η^ϕ involves the bi-paracyclic versions of the shuffle and Alexander–Whiteny maps. As it turns out, we actually obtain a very simple formula when we pair η^ϕ with cochains arising from equivariant currents. To see this, let $\Omega^\Gamma(M)=(\Omega^\Gamma_\bullet(M),d)$ be the cochain complex of equivariant currents, where $\Omega^\Gamma_m(M)$, $m\geq 0$, consists of maps $C:\Gamma\to\Omega_m(M)$ that are Γ-equivariant in the sense that $C(\psi_1^{-1}\psi_0\psi_1)=(\psi_1)_*[C(\psi_0)]$ for all $\psi_j\in\Gamma$. (Here $\Omega_m(M)$ is the space of m-dimensional currents.) Any equivariant current $C\in\Omega^\Gamma_m(M)$ defines a cochain $\varphi_C\in C^m(\mathcal{A}_\Gamma)$ by $\varphi_C(f^0u_{\psi_0},\ldots,f^mu_{\psi_m})=\frac{1}{m!}\langle C(\psi),f^0d\hat{f}^1\wedge\cdots\wedge d\hat{f}^m\rangle$, where we have set $\psi=\psi_0\cdots\psi_m$ and $\hat{f}^j=f^j\circ(\psi_0\cdots\psi_{j-1})^{-1}$. This provides us with a map of mixed complexes from $(\Omega^\Gamma_\bullet(M),d,0)$ to $(C^\bullet(\mathcal{A}_\Gamma),B,b)$. Therefore, we obtain cochain maps between their respective cyclic and periodic cochain complexes. Note that the periodic cyclic complex of $\Omega^\Gamma(M)$ is just $(\Omega^\Gamma_{\mathrm{ev/odd}}(M),d)$. The transverse fundamental class cocycle of Connes [7] and the CM cocycle of an equivariant Dirac spectral triple [14] are examples of cocycles arising from equivariant currents. In what follows, given any equivariant chain $\omega=(\omega_{p,q})$, $\omega_{p,q}\in C_{p,q}(\Gamma,M)$, we denote by ω_0 its component in $C_{0,\bullet}(\Gamma,M)\simeq\Omega^\bullet(M)$.

Proposition 3.1. *Let $\phi\in\Gamma$ have finite order. Then, for any closed equivariant current $C\in\Omega^\Gamma_{\mathrm{ev/odd}}(M)$ and any equivariant cycle $\omega\in C_{\mathrm{ev/odd}}(\Gamma_\phi,M^\phi)$, we have $\langle\varphi_C,\eta^\phi(\omega)\rangle=\langle C(\phi),\tilde{\omega}_0\rangle$, where $\tilde{\omega}_0\in\Omega^{\mathrm{ev}}(M)$ is such that $\tilde{\omega}_{0|M^\phi}=\omega_0$.*

Let E be a Γ_ϕ-equivariant vector bundle over a submanifold component M_a^ϕ. Given any connection ∇^E on E, the equivariant Chern character of E is represented by a cocycle $\mathrm{Ch}_{\Gamma_\phi}(\nabla^E)\in C^{\mathrm{ev}}_{\Gamma_\phi}(M_a^\phi)$ (see [2,10]). The space $C_0(\Gamma_\phi,\mathbb{C})\simeq\mathbb{C}$ is spanned by the cycle $1:=1\otimes_{\Gamma_\phi}1$. It can be checked that $(\mathrm{Ch}_{\Gamma_\phi}(\nabla^E)\frown 1)_0=\mathrm{Ch}(\nabla^E)$, where $\mathrm{Ch}(\nabla^E)$ is the Chern form of ∇^E. Thus, for any closed equivariant current $C\in\Omega^\Gamma_{\mathrm{ev}}(M)$ such that $\operatorname{supp}C(\phi)\subset M_a^\phi$, we have $\langle\varphi_C,\eta^\phi(\mathrm{Ch}_{\Gamma_\phi}(\nabla^E)\frown 1)\rangle=\langle C(\phi),\mathrm{Ch}(\nabla^E)\rangle$. More generally, let $\xi=\sum_\ell\lambda_\ell(\psi_0^\ell,\ldots,\psi_{2q}^\ell)\otimes_{\Gamma_\phi}1$ be a cycle in $C_{2q}(\Gamma_\phi,\mathbb{C})$, $q\geq 1$. We then have

$\langle\varphi_C, \eta^\phi(\mathrm{Ch}_{\Gamma_\phi}(\nabla^E) \frown \xi)\rangle = \sum_\ell \langle C(\phi), \mathrm{CS}\left((\psi_0^\ell)_*\nabla^E, \ldots, (\psi_{2q}^\ell)_*\nabla^E\right)\rangle$, where $\mathrm{CS}((\psi_0)_*\nabla^E, \ldots, (\psi_{2q})_*\nabla^E)$ is the Chern–Simons form of the connections $((\psi_0)_*\nabla^E, \ldots, (\psi_{2q})_*\nabla^E)$ as defined in [10].

Suppose now that ϕ has infinite order and acts cleanly on M. As $\Omega(M^\phi)$ is a ϕ-invariant mixed complex, we may form the triangular S-module $C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi))$ as in Section 1 and Part I. Its total S-module is $(\mathrm{Tot}_\bullet(C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi))), d^\dagger, u_\phi \frown -)$, where $\mathrm{Tot}_m(C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi))) = \oplus_{p+q=m} C_p(\overline{\Gamma}_\phi) \otimes_{\overline{\Gamma}_\phi} \Omega^q(M^\phi)$ and $d^\dagger = \partial + (-1)^p d(u_\phi \frown -)$ on $C_p(\overline{\Gamma}_\phi) \otimes_{\overline{\Gamma}_\phi} \Omega^q(M^\phi)$. Applying Theorem 1.3 then gives the following result.

Theorem 3.4. *Let $\phi \in \Gamma$ have infinite order and act cleanly on M. The following are quasi-isomorphisms,*

$$\mathrm{Tot}_\bullet(C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi))) \xleftarrow{\theta(1\otimes\alpha)} \mathrm{Tot}_\bullet\left(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \underset{\mathrm{AW}^\natural}{\overset{\sqcup\!\sqcup^\natural}{\rightleftarrows}} \mathrm{Diag}_\bullet\left(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \xrightarrow{\mu_\phi} C_\bullet(\mathcal{A}_\Gamma)^\natural_{[\phi]}. \tag{3}$$

This gives an isomorphism $\mathbf{HC}_\bullet(\mathcal{A})_{[\phi]} \simeq H_\bullet\left(\mathrm{Tot}(C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi)))\right)$, under which the periodicity operator of $\mathbf{HC}_\bullet(\mathcal{A})_{[\phi]}$ is the cap product $e_\phi \frown - : H_\bullet\left(\mathrm{Tot}(C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi)))\right) \to H_{\bullet-2}\left(\mathrm{Tot}(C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi)))\right)$.

Remark 3.5. The quasi-isomorphisms (3) and the filtration by columns of $\mathrm{Tot}_\bullet(C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi)))$ give rise to a spectral sequence $E^2_{p,q} = H_p(\overline{\Gamma}_\phi, \Omega^q(M^\phi)) \Longrightarrow \mathbf{HC}_{p+q}(\mathcal{A}_\Gamma)$, where the E^2-differential is given by $(-1)^p d(u_\phi \frown -) : H_p(\overline{\Gamma}_\phi, \Omega^q(M^\phi)) \to H_{p-2}(\overline{\Gamma}_\phi, \Omega^{q+1}(M^\phi))$. Crainic [9] obtained such a spectral sequence, and inferred from this that $\mathbf{HC}_m(\mathcal{A})_{[\phi]} \simeq \oplus_{p+q=m} H_p(\overline{\Gamma}_\phi, \Omega^q(M^\phi))$ (see [9, Corollary 4.15]). What we really have is the isomorphism $\mathbf{HC}_\bullet(\mathcal{A})_{[\phi]} \simeq H_\bullet\left(\mathrm{Tot}(C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi)))\right)$ given by Theorem 3.4.

Remark 3.6. All the results of this section have analogues for group actions on smooth varieties by combining the results of Part I with the twisted Hochschild–Kostant–Rosenberg Theorem of [3]. When Γ_ϕ is finite, we get explicit quasi-isomorphisms that enable us to recover the description of cyclic and periodic homology in terms of (algebraic) orbifold cohomology in [3]. More generally, when ϕ has finite order, the results are expressed in terms of a mixed equivariant homology for smooth varieties. Furthermore, the framework of Section 1 enables us to extend those results to the I-adic completions considered in [3].

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