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Partial differential equations

Boundedness of classical solutions for a chemotaxis model with consumption of chemoattractant

Les solutions classiques d'un modèle de chimiotaxie avec consommation de chimioattracteurs sont bornées

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ABSTRACT

In this paper, we study the chemotaxis system:

$$\begin{cases} u_t = \nabla \cdot (\xi \nabla u - \chi u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 1$, with smooth boundary. Here, ξ and χ are some positive constants.
We prove that the classical solutions to the above system are uniformly in-time-bounded provided that:

$$\|v_0\|_{L^\infty(\Omega)} < \begin{cases} \frac{1}{\chi}\sqrt{\frac{\xi}{2(n+1)}}\left[\pi + 2\arctan\left(\frac{(1-\xi)}{2}\sqrt{\frac{2(n+1)}{\xi}}\right)\right], & \text{if } 0 < \xi < 1, \\ \frac{\pi}{\chi\sqrt{2(n+1)}}, & \text{if } \xi = 1, \\ \frac{1}{\chi}\sqrt{\frac{\xi}{2(n+1)}}\left[\pi - 2\arctan\left(\frac{(\xi-1)}{2}\sqrt{\frac{2(n+1)}{\xi}}\right)\right], & \text{if } \xi > 1. \end{cases}$$

In the case of $\xi = 1$, the recent results show that the classical solutions are global and bounded provided that $0 < \|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)\chi}$. Because of $\frac{1}{6(n+1)\chi} < \frac{\pi}{\chi\sqrt{2(n+1)}}$, or more precisely, $\lim_{n\to\infty} \frac{\frac{\pi}{\chi\sqrt{2(n+1)}}}{\frac{1}{6(n+1)\chi}} = +\infty$, our results extend the recent results.

RÉSUMÉ

Dans cette Note, nous étudions le système de chimiotaxie suivant :

$$\begin{cases} u_t = \nabla \cdot (\xi \nabla u - \chi u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \end{cases}$$

sous des conditions de Neumann homogènes au bord, supposé lisse, d'un domaine borné $\Omega \subset \mathbb{R}^n$, $n \geq 1$. Ici, ξ et χ sont des constantes positives.

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Nous montrons que les solutions classiques du système ci-dessus sont uniformément bornées en temps, pourvu que :

$$\|v_0\|_{L^\infty(\Omega)} < \begin{cases} \frac{1}{\chi}\sqrt{\frac{\xi}{2(n+1)}}\left[\pi + 2\arctan\left(\frac{(1-\xi)}{2}\sqrt{\frac{2(n+1)}{\xi}}\right)\right], & \text{si } 0<\xi<1, \\ \frac{\pi}{\chi\sqrt{2(n+1)}}, & \text{si } \xi = 1, \\ \frac{1}{\chi}\sqrt{\frac{\xi}{2(n+1)}}\left[\pi - 2\arctan\left(\frac{(\xi-1)}{2}\sqrt{\frac{2(n+1)}{\xi}}\right)\right], & \text{si } \xi>1. \end{cases}$$

Dans le cas $\xi = 1$, des résultats récents montrent que les solutions classiques sont globales et bornées dès que $0 < \|v_0\|_{L^\infty(\Omega)} \le \frac{1}{6(n+1)\chi}$. Comme $\frac{1}{6(n+1)\chi} < \frac{\pi}{\chi\sqrt{2(n+1)}}$ ou, plus précisément, $\lim_{n\to\infty} \frac{\frac{\pi}{\chi\sqrt{2(n+1)}}}{\frac{1}{6(n+1)\chi}} = +\infty$, ces résultats se déduisent des nôtres.

1. Introduction

In this paper, we study the following initial boundary value problem:

$$\begin{cases} u_t = \nabla\cdot(\xi\nabla u - \chi u\nabla v), & x\in\Omega,\ t>0, \\ v_t = \Delta v - uv, & x\in\Omega,\ t>0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0,\ v(x,0)=v_0, & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n, n \ge 1$, is a bounded domain with smooth boundary, and ν denotes the unit outward normal vector to $\partial\Omega$. Also, ξ and χ are some positive constants and u_0 and v_0 are non-negative initial functions. Here, $u = u(x,t)$ denotes the bacteria density and $v = v(x,t)$ is the concentration of oxygen.

If the second equation of problem (1.1) is replaced with $v_t = \Delta v - v + u$, then this problem is the classical chemotaxis system that was proposed by Keller and Segel in 1970 [3]. For this model, it is known that for $n = 1$, all solutions are global and bounded [7]. Also, for $n = 2$, the same result is true provided that $\|u_0\|_{L^1(\Omega)} < 4\pi$ [6], whereas for $\|u_0\|_{L^1(\Omega)} > 4\pi$, blow up occurs either in finite or infinite time [2]. For $n \ge 3$, under some suitable conditions on initial data and $\|u_0\|_{L^1(\Omega)} > 0$, there exist radial solutions that become unbounded in finite time [11]. While if for each $q > \frac{n}{2}$ and $p > n$, there exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, $\|u_0\|_{L^q(\Omega)} < \epsilon$ and $\|\nabla v_0\|_{L^p(\Omega)} < \epsilon$, then the classical solutions become global and bounded [10].

Problem (1.1) with $\xi = 1$ is studied by Tao and Winkler in bounded convex domains with smooth boundary [9]. In the two-dimensional case, they proved that the classical solutions for this problem are global and bounded and satisfy the following convergence properties:

$$\begin{cases} u(.,t) \longrightarrow \bar{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0(x) & \text{in } L^\infty(\Omega) \quad \text{as } t\longrightarrow\infty, \\ v(.,t) \longrightarrow 0 & \text{in } \quad L^\infty(\Omega) \quad \text{as} \quad t\longrightarrow\infty. \end{cases} \tag{1.2}$$

Besides, in the three-dimensional case, they showed that bounded weak solutions exist for arbitrarily large initial data, and these solutions satisfy the convergence properties (1.2). Also, for $n \ge 3$, the classical solutions for this problem are global and bounded provided that $0 < \|v_0\|_{L^\infty(\Omega)} \le \frac{1}{6(n+1)\chi}$, and satisfy the convergence properties (1.2) [8,13]. In the presence of a logistic source, Zheng and Mu [14] studied problem (1.1) when the first equation is replaced with $u_t = \nabla\cdot(\delta\nabla u - \chi(v)u\nabla v) + f(u)$, where f is the logistic function and χ measures the chemotactic sensitivity. They proved that if χ satisfies $\chi(s) \le \frac{\chi_0}{(1+\alpha s)^k}$ with $\chi_0 > 0$, $\alpha > 0$ and $k > 0$, then the classical solutions for this problem are global and bounded provided that $\|v_0\|_{L^\infty(\Omega)} \le \frac{1}{6(n+1)\chi_0}$. Recently, in the presence of a logistic source $f(s) = as - bs^2$ for $s \ge 0$ with $a \in \mathbb{R}$ and $b > 0$, Lankeit and Wang proved that classical solutions are global and bounded when b is sufficiently large, whereas the weak solutions exist for every $b > 0$ [4]. In the case of logistic source, see also [1].

When the chemotaxis system is with rotational flux terms, the first equation (1.1) is written as $u_t = \nabla\cdot(\nabla u - uS(x,u,v)\nabla v)$, where $S \in C^2(\bar{\Omega}\times[0,\infty)^2;\mathbb{R}^{n\times n})$ is a matrix-valued function. If $|S(x,u,v)| \le \frac{S_0(v)}{(1+u)^\theta}$, where S_0 is some non-decreasing function and $\theta > 0$, then for $n = 1$ and $\theta \ge 0$, the classical solutions are global and bounded [12]. Also, for $n \ge 2$ and $\theta = 0$, the same result is true provided that $S_0(\|v_0\|_{L^\infty(\Omega)})\|v_0\|_{L^\infty(\Omega)} < \frac{2}{\sqrt{3n(11n+2)}}$, whereas, for $n \ge 2$ and $\theta > 0$, the classical solutions are global and bounded without any restriction on the initial data [12]. The results obtained in [12] extend the recent results obtained in [5], which assert that solutions are global and bounded in two dimensions with $\theta = 0$ and $\|v_0\|_{L^\infty(\Omega)}$ sufficiently small.

In the present paper, we will study problem (1.1) and prove that this problem admits a unique classical solution, which is global and bounded provided that:

$$\|v_0\|_{L^\infty(\Omega)} < \begin{cases} \frac{1}{\chi}\sqrt{\frac{\xi}{2(n+1)}}\left[\pi + 2\arctan\left(\frac{(1-\xi)}{2}\sqrt{\frac{2(n+1)}{\xi}}\right)\right], & \text{if } 0<\xi<1, \\ \frac{\pi}{\chi\sqrt{2(n+1)}}, & \text{if } \xi = 1, \\ \frac{1}{\chi}\sqrt{\frac{\xi}{2(n+1)}}\left[\pi - 2\arctan\left(\frac{(\xi-1)}{2}\sqrt{\frac{2(n+1)}{\xi}}\right)\right], & \text{if } \xi > 1. \end{cases}$$

This result extends the results obtained in [8], which assert that the classical solutions for this problem with $\xi = 1$ are global and bounded provided that $0 < \|v_0\|_{L^\infty(\Omega)} \le \frac{1}{6(n+1)\chi}$.

2. Global existence

The following lemma, which is the standard well-posedness and classical solvability, is proven in [8, Lemma 2.1].

Lemma 2.1. *Let the non-negative functions u_0 and v_0 satisfy $(u_0, v_0) \in (W^{1,q}(\Omega))^2$ for some $q > n$. Then problem (1.1) has a unique local in time classical solution*

$$(u, v) \in \left(C\left([0, T_{\max}); W^{1,q}(\Omega)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{\max})\right)\right)^2,$$

where $T_{\max}$ denotes the maximal existence time. Moreover, u and v satisfy the following inequalities:

$$u \ge 0, \quad 0 \le v \le \|v_0\|_{L^\infty(\Omega)} \quad \textit{in} \quad \Omega \times (0, T_{\max}). \tag{2.1}$$

In addition, if there exists a constant $c > 0$ such that

$$\|(u(t), v(t))\|_{L^\infty(\Omega)} \le c,$$

then $T_{\max} = +\infty$. Also, the total mass of u satisfies the following identity:

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \textit{for all} \quad t \in (0, T_{\max}).$$

Our key idea is stated in the following lemma.

Lemma 2.2. *Let φ be a twice-differentiable increasing function that is defined for $0 \le s \le \|v_0\|_{L^\infty(\Omega)}$ and bounded from below by c, where c is some positive constant. Also, assume that $p > 1$ and $\frac{1}{p}\varphi'' - \chi\varphi' \ge 0$. Then, the solution to (1.1) satisfies the following estimate:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \varphi(v)\,\mathrm{d}x \le p\int_\Omega \Phi(v) u^{p-1}|\nabla u||\nabla v|\,\mathrm{d}x \tag{2.2}$$

with $\Phi(v) = \left|\chi(p-1)\varphi(v) - (\xi+1)\varphi'(v)\right| - 2\sqrt{\xi(p-1)\varphi(v)\left(\frac{1}{p}\varphi''(v) - \chi\varphi'(v)\right)}$.

Proof. We use from (1.1) and integration by parts to obtain

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x &= \int_\Omega u^{p-1}\varphi(v)\,u_t\,\mathrm{d}x + \frac{1}{p}\int_\Omega u^p\,\varphi'(v)v_t\,\mathrm{d}x \\
&= -\xi(p-1)\int_\Omega u^{p-2}\varphi(v)\,|\nabla u|^2\,\mathrm{d}x - (\xi+1)\int_\Omega u^{p-1}\,\varphi'(v)\nabla u\cdot\nabla v\,\mathrm{d}x \\
&\quad + \chi(p-1)\int_\Omega u^{p-1}\varphi(v)\;\nabla u\cdot\nabla v\,\mathrm{d}x + \chi\int_\Omega u^p\varphi'(v)|\nabla v|^2\,\mathrm{d}x \\
&\quad - \frac{1}{p}\int_\Omega u^p\varphi''(v)|\nabla v|^2\,\mathrm{d}x - \frac{1}{p}\int_\Omega v\varphi'(v)u^{p+1}\mathrm{d}x.
\end{aligned}$$

Because of $\varphi' \ge 0$, we have

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x \le -\xi(p-1)\int_\Omega u^{p-2}\varphi(v)\,|\nabla u|^2\,\mathrm{d}x - \int_\Omega \Big[\frac{1}{p}\varphi''(v)-\chi\varphi'(v)\Big]u^p|\nabla v|^2\,\mathrm{d}x$$
$$+\int_\Omega \big|\chi(p-1)\varphi(v)-(\xi+1)\varphi'(v)\big|u^{p-1}|\nabla u||\nabla v|\,\mathrm{d}x. \tag{2.3}$$

The main idea of our proof is based on the identity $-(a^2+b^2)=-(a-b)^2-2ab$. Note that we can apply the above identity to the first and second terms on the right-hand side of (2.3). Thus, we add and subtract the term:

$$2\int_\Omega \sqrt{\xi(p-1)\varphi(v)\Big(\frac{1}{p}\varphi''(v)-\chi\varphi'(v)\Big)}u^{p-1}|\nabla u||\nabla v|\,\mathrm{d}x$$

on the right-hand side of (2.3) to obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x \le -\xi(p-1)\int_\Omega u^{p-2}\varphi(v)\,|\nabla u|^2\,\mathrm{d}x - \int_\Omega \Big(\frac{1}{p}\varphi''(v)-\chi\varphi'(v)\Big)u^p|\nabla v|^2\,\mathrm{d}x$$
$$+2\int_\Omega \sqrt{\xi(p-1)\varphi(v)\Big(\frac{1}{p}\varphi''(v)-\chi\varphi'(v)\Big)}u^{p-1}|\nabla u||\nabla v|\,\mathrm{d}x$$
$$+\int_\Omega \big|\chi(p-1)\varphi(v)-(\xi+1)\varphi'(v)\big|u^{p-1}|\nabla u||\nabla v|\,\mathrm{d}x$$
$$-2\int_\Omega \sqrt{\xi(p-1)\varphi(v)\Big(\frac{1}{p}\varphi''(v)-\chi\varphi'(v)\Big)}u^{p-1}|\nabla u||\nabla v|\,\mathrm{d}x$$
$$=-\int_\Omega\left(\sqrt{\xi(p-1)\varphi(v)}\,u^{\frac{p}{2}-1}|\nabla u|-\sqrt{\frac{1}{p}\varphi''(v)-\chi\varphi'(v)}\,u^{\frac{p}{2}}|\nabla v|\right)^2\mathrm{d}x$$
$$+\int_\Omega\left[\Big|\chi(p-1)\varphi(v)-(\xi+1)\varphi'(v)\Big|-2\sqrt{\xi(p-1)\varphi(v)\Big(\frac{1}{p}\varphi''(v)-\chi\varphi'(v)\Big)}\right]u^{p-1}|\nabla u||\nabla v|\,\mathrm{d}x.$$

This inequality implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x \le p\int_\Omega \Phi(v)u^{p-1}|\nabla u||\nabla v|\,\mathrm{d}x$$

with $\Phi(v)=\big|\chi(p-1)\varphi(v)-(\xi+1)\varphi'(v)\big|-2\sqrt{\xi(p-1)\varphi(v)\big(\frac{1}{p}\varphi''(v)-\chi\varphi'(v)\big)}$. Thus, the desired result is obtained. □

In the following lemma, we provide a smooth function φ that fulfills our requirements.

Lemma 2.3. *Assume that $\varphi(s)=e^{z(s)}$, where $z(s)$ for $0\le s\le\|v_0\|_{L^\infty(\Omega)}$ is defined as:*

$$z(s)=\int_0^s\left[-\frac{B}{2A}+\frac{\sqrt{4AD-B^2}}{2A}\tan\left(\frac{\sqrt{4AD-B^2}}{2}\Big(\frac{1}{C}\tau+\frac{K}{A}\Big)\right)\right]\mathrm{d}\tau$$

with

$$A=\Big((\xi+1)^2-\frac{4\xi(p-1)}{p}\Big),\ B=2\chi(p-1)(\xi-1),\ C=\frac{4\xi(p-1)}{p},\ \ D=\chi^2(p-1)^2,$$
$$K=\frac{2A}{\sqrt{4AD-B^2}}\arctan\Big(\frac{B}{\sqrt{4AD-B^2}}\Big) \tag{2.4}$$

and $p>1$. Also, assume that $\|v_0\|_{L^\infty(\Omega)}$ satisfies the following condition:

$$\|v_0\|_{L^\infty(\Omega)}<\frac{1}{\chi}\sqrt{\frac{\xi}{p}}\left[\pi-2\arctan\Big(\frac{(\xi-1)}{2}\sqrt{\frac{p}{\xi}}\Big)\right]. \tag{2.5}$$

Then, the function z is well defined and φ is an increasing function as well as $\varphi''-p\chi\varphi'\ge 0$.

Proof. At first, we show that the function z is well defined. In order to show this, we consider the values A, B and D and compute

$$\begin{aligned} 4AD - B^2 &= 4\Big((\xi+1)^2 - \frac{4\xi(p-1)}{p}\Big)\chi^2(p-1)^2 - 4\chi^2(p-1)^2(\xi-1)^2 \\ &= 4\chi^2(p-1)^2\Big((\xi+1)^2 - \frac{4\xi(p-1)}{p} - (\xi-1)^2\Big) \\ &= 4\chi^2(p-1)^2\Big(4\xi - \frac{4\xi(p-1)}{p}\Big) = \frac{16\chi^2(p-1)^2\xi}{p} > 0 \end{aligned} \tag{2.6}$$

and

$$K = \frac{2A}{\sqrt{4AD-B^2}}\arctan\Big(\frac{B}{\sqrt{4AD-B^2}}\Big) = \frac{\Big(p(\xi-1)^2+4\xi\Big)}{2\chi(p-1)\sqrt{p\xi}}\arctan\Big(\frac{(\xi-1)}{2}\sqrt{\frac{p}{\xi}}\Big).$$

We also have

$$\frac{\sqrt{4AD-B^2}}{2}\Big(\frac{1}{C}s + \frac{K}{A}\Big) = \frac{\sqrt{4AD-B^2}}{2C}s + \arctan\Big(\frac{B}{\sqrt{4AD-B^2}}\Big) = \frac{\chi\sqrt{p}}{2\sqrt{\xi}}s + \arctan\Big(\frac{(\xi-1)}{2}\sqrt{\frac{p}{\xi}}\Big). \tag{2.7}$$

Now, the condition (2.5) along with $0 \leqslant s \leqslant \|v_0\|_{L^\infty(\Omega)}$ implies that

$$0 \leqslant \frac{\sqrt{4AD-B^2}}{2}\Big(\frac{1}{C}s + \frac{K}{A}\Big) < \frac{\pi}{2}.$$

Thus the function z is well defined. From the definition of z, for $0 \le s \le \|v_0\|_{L^\infty(\Omega)}$, we can write

$$z'(s) = -\frac{B}{2A} + \frac{\sqrt{4AD-B^2}}{2A}\tan\left(\frac{\sqrt{4AD-B^2}}{2}\Big(\frac{1}{C}s + \frac{K}{A}\Big)\right).$$

Hence,

$$\tan\left(\frac{\sqrt{4AD-B^2}}{2}\Big(\frac{1}{C}s + \frac{K}{A}\Big)\right) = \frac{2Az'(s)+B}{\sqrt{4AD-B^2}}. \tag{2.8}$$

Then, we have

$$z''(s) = \frac{4AD-B^2}{4AC}\left[1 + \tan^2\left(\frac{\sqrt{4AD-B^2}}{2}\Big(\frac{1}{C}s + \frac{K}{A}\Big)\right)\right].$$

The above equality along with (2.6) says that $z''(s) > 0$ for all $0 \le s \le \|v_0\|_{L^\infty(\Omega)}$. Thus, z' is an increasing function. We now obtain

$$\begin{aligned} z''(s) &= \frac{4AD-B^2}{4AC}\left[1 + \tan^2\left(\frac{\sqrt{4AD-B^2}}{2}\Big(\frac{1}{C}s + \frac{K}{A}\Big)\right)\right] \\ &= \frac{4AD-B^2}{4AC}\left[1 + \left(\frac{2Az'(s)+B}{\sqrt{4AD-B^2}}\right)^2\right] \\ &= \frac{4AD-B^2}{4AC}\left[1 + \frac{4A^2(z'(s))^2 + 4ABz'(s) + B^2}{4AD-B^2}\right] \\ &= \frac{4AD-B^2}{4AC}\left[\frac{4A^2(z'(s))^2 + 4ABz'(s) + 4AD}{4AD-B^2}\right] \\ &= \frac{A}{C}(z'(s))^2 + \frac{B}{C}z'(s) + \frac{D}{C}, \end{aligned} \tag{2.9}$$

where we have used from (2.8) in the second equality. We also have

$$\varphi'(s) = z'(s)\varphi(s), \qquad \varphi''(s) = \Big(z''(s) + (z'(s))^2\Big)\varphi(s).$$

We now show that $\varphi'(s) \ge 0$ for all $0 \le s \le \|v_0\|_{L^\infty(\Omega)}$. In order to prove this, we use from (2.4) and (2.7) to write

$$z'(s) = -\frac{B}{2A} + \frac{\sqrt{4AD - B^2}}{2A} \tan\left(\frac{\sqrt{4AD - B^2}}{2}\left(\frac{1}{C}s + \frac{K}{A}\right)\right)$$
$$= -\frac{\chi p(p-1)(\xi - 1)}{p(\xi - 1)^2 + 4\xi} + \frac{2\chi(p-1)\sqrt{p\xi}}{p(\xi-1)^2 + 4\xi} \tan\left(\frac{\chi\sqrt{p}}{2\sqrt{\xi}}s + \arctan\left(\frac{(\xi - 1)}{2}\sqrt{\frac{p}{\xi}}\right)\right).$$

This equality implies $z'(0) = 0$. Now, from the fact that z' is an increasing function, we conclude that $z'(s) \geq z'(0) = 0$ for all $0 \leq s \leq \|v_0\|_{L^\infty(\Omega)}$. Thus, $\varphi' \geq 0$. Finally, we prove that $\frac{1}{p}\varphi'' - \chi\varphi' \geq 0$. In order to do this, we write

$$\begin{aligned}
\frac{1}{p}\varphi''(s) - \chi\varphi'(s) &= \frac{1}{p}\left(z''(s) + (z'(s))^2\right)\varphi(s) - \chi z'(s)\varphi(s) \\
&= \varphi(s)\left[\frac{1}{p}\left(\frac{A}{C}(z'(s))^2 + \frac{B}{C}z'(s) + \frac{D}{C} + (z'(s))^2\right) - \chi z'(s)\right] \\
&= \frac{1}{p}\varphi(s)\left[\frac{A+C}{C}(z'(s))^2 + \frac{B - p\chi C}{C}z'(s) + \frac{D}{C}\right] \\
&= \frac{1}{Cp}\varphi(s)\left[(A+C)(z'(s))^2 + (B - p\chi C)z'(s) + D\right] \\
&= \frac{1}{Cp}\varphi(s)\left[(\xi+1)^2(z'(s))^2 - 2\chi(p-1)(\xi+1)z'(s) + \chi^2(p-1)^2\right] \\
&= \frac{1}{4\xi(p-1)}\varphi(s)\left((\xi+1)z'(s) - \chi(p-1)\right)^2 \geq 0,
\end{aligned} \tag{2.10}$$

where we have used from (2.9) in the second equality. Thus, we obtain the desired result. □

We now can obtain our main result.

Lemma 2.4. *Assume that $\|v_0\|_{L^\infty(\Omega)}$ satisfies the condition (2.5). Then, there exists a constant $c > 0$ such that the first component of solution to problem (1.1) satisfies*

$$\|u(.,t)\|_{L^{2(n+1)}(\Omega)} \leq c, \quad \text{for all } t \in (0, T_{\max}). \tag{2.11}$$

Proof. We consider the function φ which is defined in Lemma 2.3 and use (2.10). Thus, we can write

$$\begin{aligned}
\Phi(v) &= \left|\chi(p-1)\varphi(v) - (\xi+1)\varphi'(v)\right| - 2\sqrt{\xi(p-1)\varphi(v)\left(\frac{1}{p}\varphi''(v) - \chi\varphi'(v)\right)} \\
&= \left|\chi(p-1) - (\xi+1)z'(v)\right|\varphi(v) - 2\sqrt{\frac{1}{4}\varphi^2(v)\left((\xi+1)z'(v) - \chi(p-1)\right)^2} \\
&= \varphi(v)\left(\left|\chi(p-1) - (\xi+1)z'(v)\right| - \left|(\xi+1)z'(v) - \chi(p-1)\right|\right) = 0.
\end{aligned}$$

The above equality along with the inequality (2.2) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x \leq 0.$$

By integration of the last inequality from 0 to t, we obtain $\int_\Omega u^p\varphi(v)\,\mathrm{d}x \leq c$ with $c = \int_\Omega u_0^p\varphi(v_0)\,\mathrm{d}x$. Since φ is an increasing function, thus $\varphi(s) \geq \varphi(0) = e^{z(0)} = 1$ for all $0 \leq s \leq \|v_0\|_{L^\infty(\Omega)}$. We set $p = 2(n+1)$ and use the fact that $\varphi \geq 1$ and obtain the desired result (2.11). □

Lemma 2.5. *Assume that the condition (2.5) holds. Then there exists a constant $c > 0$ such that the first component of solution to problem (1.1) satisfies*

$$\|u(.,t)\|_{L^\infty(\Omega)} \leq c \quad \text{for all } t \in (0, T_{\max}).$$

This Lemma is proven in [8, Lemma 3.2].

Theorem 2.6. *Assume that the non-negative functions u_0 and v_0 satisfy $(u_0, v_0) \in (W^{1,q}(\Omega))^2$ for some $q > n$. Also, assume that the condition (2.5) holds. Then the solution of (u, v) to problem (1.1) is global and bounded.*

Proof. By considering the extensibility criterion provided by Lemma 2.1, the proof is a consequence of (2.1) and Lemma 2.5. □

Remark 2.7. The condition (2.5) can be written as follows:

$$\|v_0\|_{L^\infty(\Omega)} < \begin{cases} \frac{1}{\chi}\sqrt{\frac{\xi}{2(n+1)}}\left[\pi + 2\arctan\left(\frac{(1-\xi)}{2}\sqrt{\frac{2(n+1)}{\xi}}\right)\right], & \text{if } 0 < \xi < 1, \\ \frac{\pi}{\chi\sqrt{2(n+1)}}, & \text{if } \xi = 1, \\ \frac{1}{\chi}\sqrt{\frac{\xi}{2(n+1)}}\left[\pi - 2\arctan\left(\frac{(\xi-1)}{2}\sqrt{\frac{2(n+1)}{\xi}}\right)\right], & \text{if } \xi > 1. \end{cases}$$

Because of $\frac{1}{6(n+1)\chi} < \frac{\pi}{\chi\sqrt{2(n+1)}}$, or more precisely, $\lim_{n\to\infty} \frac{\frac{\pi}{\chi\sqrt{2(n+1)}}}{\frac{1}{6(n+1)\chi}} = +\infty$, we can conclude that the above condition extends the condition $0 < \|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)\chi}$, which is obtained by Tao for $\xi = 1$ in [8].

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