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Dynamical systems

Combinatorial models for spaces of cubic polynomials

Modèles combinatoires pour les espaces de polynômes cubiques

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ABSTRACT

W. Thurston constructed a combinatorial model of the Mandelbrot set $\mathcal{M}_2$ such that there is a continuous and monotone projection of $\mathcal{M}_2$ to this model. We propose the following related model for the space $\mathcal{MD}_3$ of critically marked cubic polynomials with connected Julia set and all cycles repelling. If $(P, c_1, c_2) \in \mathcal{MD}_3$, then every point z in the Julia set of the polynomial P defines a unique maximal finite set A_z of angles on the circle corresponding to the rays, whose impressions form a continuum containing z. Let $G(z)$ denote the convex hull of A_z. The convex sets $G(z)$ partition the closed unit disk. For $(P, c_1, c_2) \in \mathcal{MD}_3$ let c_1^* be the *co-critical point of* c_1. We tag the marked dendritic polynomial (P, c_1, c_2) with the set $G(c_1^*) \times G(P(c_2)) \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Tags are pairwise disjoint; denote by $\mathcal{MD}_3^{\text{comb}}$ their collection, equipped with the quotient topology. We show that tagging defines a continuous map from $\mathcal{MD}_3$ to $\mathcal{MD}_3^{\text{comb}}$ so that $\mathcal{MD}_3^{\text{comb}}$ serves as a model for $\mathcal{MD}_3$.

RÉSUMÉ

W. Thurston a construit un modèle combinatoire de l'ensemble de Mandelbrot $\mathcal{M}_2$ tel qu'il y ait une projection monotone et continue de $\mathcal{M}_2$ sur ce modèle. En relation avec ceci, nous proposons le modèle lié suivant pour l'espace $\mathcal{MD}_3$ des polynômes cubiques à points critiques marqués, avec ensemble de Julia connexe et tous les cycles répulsifs. Si $(P, c_1, c_2) \in \mathcal{MD}_3$, alors chaque point z dans l'ensemble de Julia du polynôme P définit un unique ensemble fini maximal A_z d'angles sur le cercle correspondant aux rayons, dont les impressions forment un continuum contenant z. Soit $G(z)$ l'enveloppe convexe de A_z. Les ensembles convexes $G(z)$ définissent une partition du disque unité fermé. Pour $(P, c_1, c_2) \in \mathcal{MD}_3$, soit c_1^* le *point co-critique de* c_1. Nous balisons le polynôme dendritique marqué (P, c_1, c_2) avec l'ensemble $G(c_1^*) \times G(P(c_2)) \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Les balises sont deux à deux disjointes ; désignons par $\mathcal{MD}_3^{\text{comb}}$ leur collection, équipée de la topologie quotient. Nous

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montrons que le balisage définit une application continue de $\mathcal{MD}_3$ dans $\mathcal{MD}_3^{\text{comb}}$ de sorte que $\mathcal{MD}_3^{\text{comb}}$ est un modèle pour $\mathcal{MD}_3$.

1. Introduction

Let $\mathbb{D}$ be the open disk $\{z \in \mathbb{C} \mid |z| < 1\}$ in the plane, $\overline{\mathbb{D}}$ be its closure, and $\mathbb{S}$ be its boundary circle. Let P be a polynomial of degree d with connected Julia set $J(P)$. We write Φ_P for the conformal isomorphism between $\mathbb{C} \setminus \overline{\mathbb{D}}$ and the complement U of the filled Julia set $K(P)$ asymptotic to the identity at infinity. By a theorem of Carathéodory, if $J(P)$ is locally connected, then Φ_P can be extended to a continuous map $\overline{\Phi}_P : \mathbb{C} \setminus \mathbb{D} \to \overline{U}$, under which $\mathbb{S}$ maps onto $J(P)$. Define the *lamination generated by* P as the equivalence relation $\sim_P$ on $\mathbb{S}$ identifying points of $\mathbb{S}$ if and only if $\overline{\Phi}_P$ sends them to the same point of $J(P)$.

By Thurston [7], the map P restricted to its locally connected Julia set $J(P)$ is topologically conjugate to a self-mapping $f_{\sim_P}$ of the quotient space $\mathbb{S}/\sim_P = J_{\sim_P}$ induced by $z^d|_{\mathbb{S}} = \sigma_d$; denote this conjugacy by $\Psi_P : J(P) \to J_{\sim_P}$. The mapping $f_{\sim_P}$ is called a *topological polynomial*. The quotient map of $\mathbb{S}$ onto $\mathbb{S}/\sim_P$ is denoted by $\pi_{\sim_P}$. Given a point $z \in J(P)$, we let $G_P(z) = G(z)$ denote the convex hull of the set $\pi_{\sim_P}^{-1}(\Psi_P(z))$. In other words, we represent z by the point $\Psi_P(z)$ of the model topological Julia set $J_{\sim_P}$ and then take all angles associated with $\Psi_P(z)$ in the sense of the lamination $\sim_P$. By [7], for two points z and w, the sets $G(z)$ and $G(w)$ either coincide or are disjoint.

The *geolamination* (from *geodesic* or *geometric lamination*) of P is the collection of chords, each of which is an edge of the convex hull of a $\sim_P$-class. Geolaminations geometrically interpret and "topologize" laminations, reflecting limit transitions among them. Both laminations and their geolaminations can be defined intrinsically (without polynomials). Then some geolaminations will not directly correspond to an equivalence relation on $\mathbb{S}$, but the family of all geolaminations will be closed. This allows one to work with limits of geolaminations and limits of polynomials (which might have non-locally connected Julia sets).

Thurston [7] models polynomials by their geolaminations, and families of quadratic polynomials by families of quadratic geolaminations. He "tags" quadratic geolaminations with their *minors* which form the *quadratic minor geolamination* QML and generate the corresponding lamination $\sim_{\text{QML}}$. The quotient space $\mathbb{S}/\sim_{\text{QML}}$ models the boundary of the Mandelbrot set $\mathcal{M}_2$ (this is the set of all parameters c such that polynomials $z^2 + c$ have connected Julia set; it is also called the *quadratic connected locus*). The induced quotient space of $\overline{\mathbb{D}}$ serves as a model for $\mathcal{M}_2$. Conjecturally, it is homeomorphic to $\mathcal{M}_2$.

Call a polynomial with connected Julia set *dendritic* if all its periodic points are repelling. By [5], for any dendritic polynomial P, even if $J(P)$ is not locally connected, there is a lamination $\sim_P$ such that there exists a *monotone semi-conjugacy* Ψ_P between $P|_{J(P)}$ and the topological polynomial $f_{\sim_P}$. Thus the sets $G_P(z) = \pi_{\sim_P}^{-1}(\Psi_P(z))$ are well defined for every dendritic polynomial P and every point $z \in J(P)$. As we will see, these nice properties of *individual* dendritic polynomials result in nice properties of *families* of cubic dendritic polynomials.

Let $\mathcal{D}_2 \subset \mathcal{M}_2$ be the set of all parameters $c \in \mathcal{M}_2$ such that the polynomial $P_c(z) = z^2 + c$ is dendritic. Set $H_c = G_{P_c}(c)$, and let $\mathcal{H}$ stand for the collection of all sets H_c, $c \in \mathcal{D}_2$. We denote the union $\bigcup_{c \in \mathcal{D}_2} H_c$ by $\mathcal{H}^+$ (in what follows, for any collection $\mathcal{A}$ of sets, we write $\mathcal{A}^+$ for the union of all sets in $\mathcal{A}$). By a part of a major result of [7], for two parameter values c, $c' \in \mathcal{D}_2$, the sets H_c and $H_{c'}$ are either disjoint or equal. Moreover, the mapping $c \mapsto H_c$ from $\mathcal{D}_2$ to $\mathcal{H}$ is upper semi-continuous (if a sequence of dendritic parameters c_n converges to a dendritic parameter c, then $\limsup_{n\to\infty} G_{c_n} \subset G_c$). The set $\mathcal{D}_2$ (or, equivalently, the set of all dendritic quadratic polynomials defined up to a Moebius change of coordinates) projects continuously onto the quotient space of $\mathcal{H}^+$ defined by the partition of $\mathcal{H}^+$ into sets H_c with $c \in \mathcal{D}_2$.

We propose a related model for the space $\mathcal{MD}_3$ of *marked dendritic* cubic polynomials (P, c_1, c_2) with connected Julia set (c_1, c_2 are the critical points of P). Define the *co-critical* point associated with a critical point τ of P as the only point τ^* such that $P(\tau^*) = P(\tau)$, $\tau^* \neq \tau$ unless P has a unique critical point, in which case $\tau = \tau^*$. Then, with every marked dendritic cubic polynomial (P, c_1, c_2), we associate the corresponding *mixed tag* $\text{Tag}(P, c_1, c_2) = G(c_1^*) \times G(P(c_2)) \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. This defines the mixed tag $\text{Tag}(P, c_1, c_2)$ for *all marked dendritic cubic polynomials*. Our choice of tags is based on the following two requirements. Firstly, the tag of $\text{Tag}(P, c_1, c_2)$ must determine $\sim_P$. Secondly, different tags must be disjoint. It is easy to see that the post-critical tag $G(P(c_1)) \times G(P(c_2))$ does not determine $G(c_1)$ and $G(c_2)$. Hence it does not determine $\sim_P$ either. Co-critical tags $G(c_1^*) \times G(c_2^*)$ do not satisfy our requirements either since these tags may intersect without being the same (this happens, e.g., for unicritical polynomials). For this reason, we use mixed tags.

Theorem 1.1. *Mixed tags of elements in $\mathcal{MD}_3$ are disjoint or coincide so that sets $\text{Tag}(P, c_1, c_2)$ form a partition of the set $\text{Tag}(\mathcal{MD}_3)^+ \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$ and generate the corresponding quotient space of $\text{Tag}(\mathcal{MD}_3)^+$ denoted by $\mathcal{MD}_3^{\text{comb}}$. Then $\mathcal{MD}_3^{\text{comb}}$ is a separable metric space and the map $\text{Tag} : \mathcal{MD}_3 \to \mathcal{MD}_3^{\text{comb}}$ is continuous.*

There are few papers studying the parameterization of geolaminations of higher degree. One of them is due to D. Schleicher [6], who extended Thurston's results to geolaminations of any degree with one critical set. We have also heard of an old preprint of D. Ahmadi and M. Rees, in which they study cubic laminations.

The results of this paper are based upon [2], which in fact applies to laminations of any degree. An extended version of the present paper can be found in [3].

2. Main ideas of the proof

Let us begin with the notions and tools developed for polynomials of any degree. If G is the convex hull of some set $G' \subset \mathbb{S}$, then we write $\sigma_d(G)$ for the convex hull of the set $\sigma_d(G')$.

Definition 2.1 *(Geolaminations).* Two distinct chords of $\overline{\mathbb{D}}$ are said to be *linked* if they intersect in $\mathbb{D}$. A *geolamination* $\mathcal{L}$ is a collection of pairwise unlinked chords in $\overline{\mathbb{D}}$, called *leaves* of $\mathcal{L}$, such that the union $\mathcal{L}^+$ of all leaves is compact, and every point in $\mathbb{S}$ is a *degenerate* leaf of $\mathcal{L}$. A *gap* of $\mathcal{L}$ is the closure of a component of $\mathbb{D} \setminus \mathcal{L}^+$. A *chord* of $\mathcal{L}$ is a chord of $\overline{\mathbb{D}}$ that is either a leaf of $\mathcal{L}$ or is disjoint from $\mathcal{L}^+$ in $\mathbb{D}$.

In the dynamical context, we use Definition 2.2, slightly different from Thurston's [7].

Definition 2.2 *(Invariant geolaminations [1]).* A *chord (leaf)* of a geolamination $\mathcal{L}$ is *critical* if its two distinct endpoints are mapped to the same point under σ_d. A geolamination $\mathcal{L}$ is σ_d*-invariant* if for any $\ell \in \mathcal{L}$ there exists $\ell^* \in \mathcal{L}$ such that $\sigma_d(\ell^*) = \ell$, and, if ℓ is non-critical and non-degenerate, then $\sigma_d(\ell) \in \mathcal{L}$ and there exist d **pairwise disjoint** leaves $\ell_1 = \ell$, ℓ_2, $\dots$, ℓ_d in $\mathcal{L}$ with $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i = 1, \dots, d$.

In the sequel, by invariant laminations we always understand the above-defined notion. Below we use the positive (counter-clockwise) circular order on $\mathbb{S}$.

Definition 2.3 *(Critical quadrilaterals).* A *critical quadrilateral* is a circularly ordered quadruple of points a, b, c, d in $\mathbb{S}$, not necessarily different, such that $a \neq c$ are mapped to the same point under σ_d, and the same is true for the pair $b \neq d$. We refer to chords and polygons just by listing their vertices. Also, we identify quadrilaterals $abcd$, $bcda$, $cdab$ and $dabc$, and call the chords ac and bd *diagonals* of the critical quadrilateral $abcd$.

A critical chord xy can be viewed as a critical quadrilateral $xxyy$. A triangle abc with critical edges can be viewed as a critical quadrilateral $abbc$, or $bcca$, or $caab$.

Definition 2.4 *(Strong linkage).* Let A and B be two quadrilaterals. Say that A and B are *strongly linked* if the vertices of A and B can be numbered so that

$$a_0 \leqslant b_0 \leqslant a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant a_3 \leqslant b_3 \leqslant a_0$$

with respect to the circular order of points on $\mathbb{S}$, where a_i, $0 \leqslant i \leqslant 3$, are vertices of A and b_i, $0 \leqslant i \leqslant 3$, are vertices of B.

We now consider geolaminations with sufficiently many critical quadrilaterals.

Definition 2.5 *(Qc-portraits).* An ordered $(d-1)$-tuple QCP of critical quadrilaterals $Q_1, \dots, Q_{d-1}$ is called a *quadratically critical portrait (qc-portrait)* if there is a geolamination $\mathcal{L}$ such that every Q_i is a gap or a leaf of $\mathcal{L}$ and any collection $\ell_i \subset Q_i$, $1 \leqslant i \leqslant d-1$ of diagonals of Q_i's contains no loops (call such collections *full*). The pair $(\mathcal{L}, \mathrm{QCP})$ is then called a *geolamination with qc-portrait*. The space of all qc-portraits is denoted by $\mathcal{QCP}_d$; the space of all geolaminations with qc-portraits is denoted by $\mathbb{L}\mathcal{QCP}_d$. Here both spaces are equipped with the topology induced by the Hausdorff metric on sets QCP^+ (in the case of $\mathcal{QCP}_d$) or $(\mathcal{L}^+, \mathrm{QCP}^+)$ (in the case of $\mathbb{L}\mathcal{QCP}_d$).

If an invariant geolamination $\mathcal{L}$ has a gap whose boundary maps forward k-to-1 with $\infty > k > 1$ as a covering, then there is no qc-portrait for $\mathcal{L}$. In the term "quadratically critical portrait", the word "quadratic" refers to the analogy between a critical quadrilateral and a simple (quadratic) critical point. Quadrilaterals Q_i and Q_j from a qc-portrait for $\mathcal{L}$ cannot share a diagonal, as otherwise the two coinciding diagonals form a degenerate loop. The "no-loop" condition in Definition 2.5 guarantees that a qc-portrait captures all critical objects of $\mathcal{L}$: if, say, $\mathcal{L}$ is a degree-4 lamination with a triangle Δ formed by critical leaves and another critical leaf $\bar{c}$, then any qc-portrait of $\mathcal{L}$ must contain $\bar{c}$; the collection of all edges of Δ is not a qc-portrait exactly because the edges form a loop. A condition equivalent to the "no-loop" condition can be stated as follows: for each component E of the closed disk minus the union of sets in the qc-portrait, the map σ_d maps the boundary of E forward in the one-to-one fashion, except for (possibly existing) critical edges of the boundary of E.

If all the sets Q_i of a qc-portrait are gaps or leaves of a geolamination $\mathcal{L}$, then a part of $\mathcal{L}$ can be recovered uniquely by taking pullbacks of Q_i's that are disjoint from Q_i's. Thus, parameterizing geolaminations is closely related to parameterizing qc-portraits.

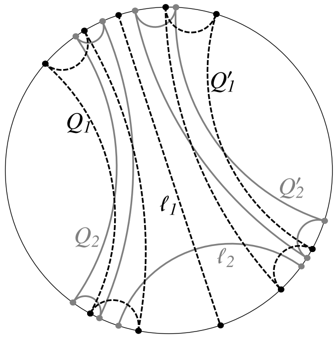

Fig. 1. This picture shows an accordion $\ell_1 \cup \ell_2$ for two linked cubic geolaminations. The first geolamination is sketched in light grays lines, while the second one in dotted boldface lines.

Lemma 2.6. *The spaces $\mathcal{QCP}_d$ and $\mathbb{L}\mathcal{QCP}_d$ are compact.*

The following lemma explains the importance of full collections of diagonals.

Lemma 2.7. *If $\mathcal{C}$ is a full collection of diagonals and W is a complementary component of $\overline{\mathbb{D}} \setminus \mathcal{C}^+$, then the circle arcs from the boundary of W add up to the total length $\frac{1}{d}$ and the restriction of σ_d to the union of these circle arcs is an orientation-preserving homeomorphism, except for the endpoints. Thus, a pair of linked chords disjoint from $\mathcal{C}^+$ is mapped to a pair of linked chords preserving orientation on their endpoints.*

If $\mathcal{L}$ has a simple loop of critical leaves, then it does not matter how to choose critical quadrilaterals in the polygon bounded by this loop. This motivates the following definition.

Definition 2.8. A *critical cluster* of $\mathcal{L}$ is a convex subset of $\overline{\mathbb{D}}$ whose boundary is a union of critical leaves (e.g., a critical leaf is itself a critical cluster).

To avoid confusion, we will use the notation $\mathcal{L}^q$ (with subscripts) for geolaminations with qc-portraits.

Definition 2.9 *(Linked geolaminations).* Let $\mathcal{L}_1^q$, $\mathcal{L}_2^q$ be geolaminations with qc-portraits $\mathrm{QCP}_1 = (Q_1^i)_{i=1}^{d-1}$, $\mathrm{QCP}_2 = (Q_2^i)_{i=1}^{d-1}$ (the sets Q_1^i, Q_2^i, $1 \leqslant i \leqslant d-1$ are called *associated*). Let k, $0 \leqslant k \leqslant d-1$ be such that:

(1) for every $i, 1 \leqslant i \leqslant k$, the quadrilaterals Q_1^i and Q_2^i are strongly linked;
(2) for each $j > k$ the sets Q_1^j and Q_2^j are contained in a common critical cluster of $\mathcal{L}_1$ and $\mathcal{L}_2$ (in what follows these common clusters will be called *special clusters*).

Then qc-portraits $\mathrm{QCP}_1, \mathrm{QCP}_2$, and geolaminations with qc-portraits $(\mathcal{L}_1^q, \mathrm{QCP}_1)$ and $(\mathcal{L}_2^q, \mathrm{QCP}_2)$, are called *linked*.

In what follows, we fix linked geolaminations with qc-portraits $(\mathcal{L}_1^q, \mathrm{QCP}_1)$ and $(\mathcal{L}_2^q, \mathrm{QCP}_2)$.

Definition 2.10 *(Accordions).* If a leaf $\ell_1 \in \mathcal{L}_1^q$ is not contained in a special cluster, then the union $A_{\mathcal{L}_2^q}(\ell_1)$ of ℓ_1 and all leaves of $\mathcal{L}_2^q$ linked with ℓ_1 is called an *accordion*. The union $A_{\ell_2}(\ell_1)$ of ℓ_1 and all leaves from the orbit of a leaf $\ell_2 \in \mathcal{L}_2^q$ that are linked with ℓ_1 is also called an *accordion* (see Fig. 1).

Lemma 2.11 is used in studying accordions of linked geolaminations with qc-portraits.

Lemma 2.11 *(Smart criticality). If $\ell_1 \in \mathcal{L}_1^q$ is not contained in a special cluster, then every critical set of* QCP$_2$ *has a diagonal unlinked with ℓ_1 or coinciding with ℓ_1. Denote this full collection of diagonals by $\mathcal{E}$. Then $A = A_{\mathcal{L}_2^q}(\ell_1)$ is contained in the closure of a component of $\mathbb{D} \setminus \mathcal{E}^+$ and $\sigma_d|_{A \cap \mathbb{S}}$ is (non-strictly) monotone.*

To prove Lemma 2.11, observe that by the assumption, critical chords from special clusters are unlinked with ℓ_1. Otherwise, take a pair of associated critical quadrilaterals $A \in \mathcal{L}_1, B \in \mathcal{L}_2$ with non-strictly alternating on $\mathbb{S}$ vertices $a_0 \leqslant b_0 \leqslant a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant a_3 \leqslant b_3 \leqslant a_0$, and observe that ℓ_1 is contained, say, in the circle arc $[a_0, a_1]$, and hence is unlinked with the diagonal $b_1 b_3$ of B.

To treat sets X formed by linked leaves of *two* linked geolaminations with qc-portraits, we vary our choice of the full collection of diagonals for successive images of X on each step, so that the orbit of X avoids *that particular* full collection of diagonals on *that particular* step (thus *smart* criticality). Therefore, similarly to the case of *one* geolamination, any power of the map is order preserving on X (see Lemma 2.7). This serves as the basis for Theorem 2.12. Suppose that the orbit of a quadrilateral Q is the union of $k \geqslant 1$ components permuted by σ_d. Suppose that either all components are single images of Q or all components are unions of $m > 1$ images of Q such that $\sigma_d^i(Q) \cap \sigma_d^{i+k}(Q) \neq 0$, the σ_d^k-images of the first diagonal of Q form a convex m-gon, the images of the second diagonal of Q form a convex m-gon, and these two polygons have vertices alternating on $\mathbb{S}$. Then we say that Q *gives rise to a periodic cluster*.

Theorem 2.12 *(Dynamics of accordions). Let ℓ_1, ℓ_2 be linked leaves of $\mathcal{L}_1^q$, $\mathcal{L}_2^q$. The set $B = \mathrm{CH}(\ell_1, \ell_2)$ is either wandering or, for some k, the sets $\sigma_3^i(B)$, $0 \leqslant i < k$ are pairwise disjoint and $\sigma_3^k(B)$ gives rise to a periodic cluster unless, for some t, there are two chains of diagonals of* QCP_1 *and of* QCP_2 *connecting two adjacent on the circle endpoints of $\sigma_d^t(\ell_1) \cup \sigma_d^t(\ell_2)$.*

Theorem 2.12 implies Corollary 2.13.

Corollary 2.13. *The set of all leaves of $\mathcal{L}_2^q$ non-disjoint from a leaf ℓ_1 of $\mathcal{L}_1^q$ is at most countable. Thus, if ℓ_2 is an accumulation set of uncountably many leaves of $\mathcal{L}_2^q$, then ℓ_2 is unlinked with any leaf of $\mathcal{L}_1^q$.*

To apply Corollary 2.13, we need the following definition.

Definition 2.14 *(Perfect sublamination).* For a geolamination $\mathcal{L}$, the maximal sublamination $\mathcal{L}^c \subset \mathcal{L}$ of $\mathcal{L}$ without isolated leaves is called the *perfect* sublamination of $\mathcal{L}$. If $\mathcal{L}^c = \mathcal{L}$, then $\mathcal{L}$ is called *perfect*.

Note that for any $\ell \in \mathcal{L}^c$ and any neighborhood U of ℓ, there are uncountably many leaves of $\mathcal{L}^c$ in U.

Theorem 2.15. *We have $(\mathcal{L}_1^q)^c = (\mathcal{L}_2^q)^c$. Moreover, suppose that $\mathcal{L}_1$, $\mathcal{L}_2$ are geolaminations with finite critical sets and there are linked geolaminations with qc-portraits $(\mathcal{L}_1^q, \mathrm{QCP}_1)$, $(\mathcal{L}_2^q, \mathrm{QCP}_2)$ such that $\mathcal{L}_1^q \supset \mathcal{L}_1$ and $\mathcal{L}_2^q \supset \mathcal{L}_2$. Then $\mathcal{L}_1^c = \mathcal{L}_2^c$.*

Indeed, otherwise choose a leaf $\ell_1^c \in (\mathcal{L}_1^q)^c \setminus \mathcal{L}_2^q$. By Corollary 2.13, the leaf ℓ_1^c (except for its endpoints) is contained in the interior of a gap G of $\mathcal{L}_2^q$ (if not, a leaf of $\mathcal{L}_2$ linked with ℓ_1^c would have an uncountable accordion). Since $(\mathcal{L}_1^q)^c$ is perfect, from at least one side all one-sided neighborhoods of ℓ_1^c contain uncountably many leaves of $(\mathcal{L}_1^q)^c$. Hence G is uncountable with uncountably many leaves of $(\mathcal{L}_1^q)^c$ connecting points of $G \cap \mathbb{S}$. Interiors of images of G are disjoint from the critical sets of $\mathcal{L}_2^q$, since these critical sets are finite. Hence eventually G maps to a *Siegel* gap, i.e. a gap on which the appropriate iterate of σ_d is semi-conjugate to an irrational rotation. This forces images of leaves of $(\mathcal{L}_1^q)^c$ inside G to intersect, a contradiction. The second part of Theorem 2.15 follows easily.

From now on **we consider only the cubic case** (i.e. $d = 3$). Call a geolamination *dendritic* if all its gaps are finite and pairwise disjoint. It is known that dendritic geolaminations are perfect. If $\mathcal{L}$ is a cubic dendritic geolamination, then it has either two disjoint critical sets of degree two each, or one critical set of degree three. For a critical set Q of $\mathcal{L}$ its *co-critical set* Q^* is defined as follows: if Q is of degree three, set $Q^* = Q$, otherwise Q^* is the convex hull of all points in $\mathbb{S} \setminus Q$ that map to $\sigma_3(Q)$ under σ_3. By a *marked cubic dendritic geolamination* we mean a triple $(\mathcal{L}, Q_1, Q_2)$ where Q_1 and Q_2 are critical sets of $\mathcal{L}$ and $Q_1 \neq Q_2$ if possible; the family of all of them is denoted by $\mathbb{L}\mathcal{MD}_3$. We now introduce a labeling of such pairs of sets (Q_1, Q_2).

Definition 2.16 *(Mixed tags).* The *mixed tag* of $(\mathcal{L}, Q_1, Q_2)$ is the set $\mathrm{Tag}(\mathcal{L}, Q_1, Q_2) = Q_1^* \times \sigma_3(Q_2)$.

Lemma 2.17 is based on simple geometric considerations and Theorem 2.15.

Lemma 2.17. *The mixed tags of two distinct elements of $\mathbb{L}\mathcal{MD}_3$ are non-disjoint if and only if these elements of $\mathbb{L}\mathcal{MD}_3$ coincide (see Fig. 2).*

To prove Lemma 2.17, assume that $(\mathcal{L}, C_1, C_2) \in \mathbb{L}\mathcal{MD}_3$ and $(\mathcal{T}, D_1, D_2) \in \mathbb{L}\mathcal{MD}_3$ have non-disjoint mixed tags. Then C_1^* and D_1^* have either a common vertex or two linked edges. The definition of the co-critical set implies then that either C_1 and D_1 have a common critical diagonal, or they contain strongly linked critical quadrilaterals with opposite edges being edges of C_1, D_1. Using this and analyzing the fact that $\sigma_3(C_2)$ and $\sigma_3(D_2)$ are non-disjoint, one can see that the sets C_2, D_2 also either share a critical diagonal, or contain strongly linked quadrilaterals with opposite edges being edges of C_2, D_2. If we insert the just-found shared critical diagonals or strongly linked quadrilaterals in critical sets of our geolaminations and pull the inserted objects back, we will construct two cubic geolaminations with qc-portraits that are linked. By Theorem 2.15, this implies that $(\mathcal{L}, C_1, C_2) = (\mathcal{T}, D_1, D_2)$.

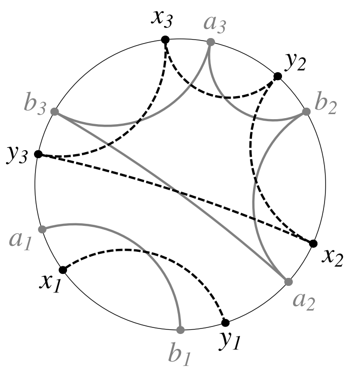

Fig. 2. This picture illustrates Lemma 2.17. The leaf a_1b_1 is a co-critical set of a critical quadrilateral $a_2b_2a_3b_3$. The leaf x_1y_1 is a co-critical set of a critical quadrilateral $x_2y_2x_3y_3$. Again, the two geolaminations are sketched in light gray lines and dotted boldface lines.

Standard topological arguments, based on properties of dendritic geolaminations and results of [4], imply now Theorem 1.1.

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References

[1] A. Blokh, D. Mimbs, L. Oversteegen, K. Valkenburg, Laminations in the language of leaves, Trans. Amer. Math. Soc. 365 (2013) 5367–5391.
[2] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, Laminational models for spaces of polynomials of any degree, preprint, arXiv:1401.5123, 2014, second version 2016.
[3] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, Models for spaces of dendritic polynomials, preprint, arXiv:1701.08825, 2017.
[4] L. Goldberg, J. Milnor, Fixed points of polynomial maps. Part II. Fixed point portraits, Ann. Sci. Éc. Norm. Supér. (4) 26 (1993) 51–98.
[5] J. Kiwi, $\mathbb{R}$eal laminations and the topological dynamics of complex polynomials, Adv. Math. 184 (2004) 207–267.
[6] D. Schleicher, On fibers and local connectivity of Mandelbrot and multibrot sets, in: M. Lapidus, M. van Frankenhuysen (Eds.), Fractal Geometry and Applications: a Jubilee of Benoit Mandelbrot, in: Proc. Symp. Pure Math., vol. 72, American Mathematical Society, Providence, RI, USA, 2004, pp. 477–517. Part 1.
[7] W. Thurston, The combinatorics of iterated rational maps (1985), in: D. Schleicher, A.K. Peters (Eds.), Complex Dynamics: Families and Friends, 2009, pp. 1–108.