Partial differential equations/Calculus of variations

Interaction energy between vortices of vector fields on Riemannian surfaces

Énergie d’interaction entre les tourbillons des champs de vecteurs sur une surface riemannienne

Radu Ignat\textsuperscript{a}, Robert L. Jerrard\textsuperscript{b}

\textsuperscript{a} Institut de Mathématiques de Toulouse, Université Paul-Sabatier, 31062 Toulouse, France
\textsuperscript{b} Department of Mathematics, University of Toronto, Toronto, Ontario, Canada

\textbf{A R T I C L E   I N F O}

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\textbf{A B S T R A C T}

We study a variational Ginzburg–Landau-type model depending on a small parameter $\varepsilon > 0$ for (tangent) vector fields on a 2-dimensional Riemannian surface. As $\varepsilon \to 0$, the vector fields tend to be of unit length and will have singular points of a (non-zero) index, called vortices. Our main result determines the interaction energy between these vortices as a $\Gamma$-limit (at the second order) as $\varepsilon \to 0$.

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\textbf{R É S U M É}

Nous étudions un modèle variationnel de type Ginzburg–Landau (dépendant d’un petit paramètre $\varepsilon > 0$) pour des champs de vecteurs (tangents) sur une surface riemannienne. Lorsque $\varepsilon \to 0$, ces champs de vecteurs auront des points singuliers d’indice non nul, appelés tourbillons. Notre résultat détermine l’énergie d’interaction entre les tourbillons en tant que $\Gamma$-limite (au second ordre) pour $\varepsilon \to 0$.

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\textbf{Version française abrégée}

Soit $(S, g)$ une surface riemannienne orientée compacte, connexe, sans bord, de dimension 2 et de genre $g$. Nous considérons la fonctionnelle de Ginzburg–Landau $E_\varepsilon$ (voir (1)) dépendant d’un petit paramètre $\varepsilon > 0$ pour des champs de vecteurs (tangents) $u : S \to TS$, i.e. $u(x) \in T_x S$ pour tout $x \in S$, où $TS = \bigcup_{x \in S} T_x S$ est le fibré tangent. Lorsque $\varepsilon \to 0$, ces champs tendent à être de module un (i.e. $|u|_g = 1$ sur $S$) et génèrent des points singuliers $a_k$ appelés tourbillons. Les tourbillons $a_k$ sont caractérisés par des indices (ou degrés topologiques) $\delta_k \in \mathbb{Z}$ qui quantifient l’énergie $E_\varepsilon$ autour de $a_k$ (i.e. $\pi \delta_k^2 \log \varepsilon$) au...
premier ordre) et satisfont la relation d’invariance topologique (3); de plus, les tourbillons $a_k$ et leur degrés $d_k$ sont détectés par la vorticité $\omega(u)$.

Notre premier objectif est de déterminer l’énergie d’interaction entre ces tourbillons (appelée énergie renormalisée) donnée par le développement à l’ordre deux de l’énergie $E_\varepsilon$. Ceci repose sur la notion de champ de vecteur canonique harmonique $u^\ast$ (voir Section 2), qui dépend non seulement de la configuration $a = (a_k)$ et $d = (d_k)$, mais aussi d’un vecteur $\Phi \in \mathbb{R}^2$ qui englobe les intégrales de flux de $u^\ast$ (voir (8)). En effet, l’énergie renormalisée $W(a,d,\Phi)$ représente l’énergie de Dirichlet associée à $u^\ast$ en dehors de petites boules centrées en $a_k$ (voir le livre innovateur de Bethuel–Brezis–Hélein [3]). Nous calculons une formule exacte de $W(a,d,\Phi)$ (voir (14)) en utilisant les fonctions de Green en $a_k$ ainsi que la fonction $\psi_0 = (-\Delta)^{-1}(-\kappa + \vec{k})$, où $\kappa$ est la courbure de Gauss sur $S$ et $\vec{k}$ est la moyenne de $\kappa$ sur $S$. L’énergie renormalisée détermine la position optimale des tourbillons $(a_k)$ pour les configurations limites $u^\varepsilon$ des minimiseurs $u_\varepsilon$ de $E_\varepsilon$ lorsque $\varepsilon \to 0$; dans le cas de la sphère unité $S$ munie de la métrique standard, la formule (14) montre que les configurations optimales sont données par les coupes de deux tourbillons $(a_1, a_2)$ diamétralement opposés, de degrés $d_1 = d_2 = 1$.

Notre résultat principal consiste à établir la $\Gamma$-convergence de $E_\varepsilon$ à l’ordre deux. Plus précisément, nous montrons la compacité des vorticités $\omega(u_\varepsilon)$ et des intégrales de flux $\Phi(u_\varepsilon)$ pour des champs de vecteurs $u_\varepsilon$ d’énergie d’ordre $|\log \varepsilon|$. Ensuite, nous établisons les bornes inférieures et supérieures de $E_\varepsilon$ à l’ordre deux qui font apparaître l’énergie renormalisée. Les preuves des résultats annoncés dans cette note font partie de notre article [9].

1. Introduction

Let $(S, g)$ be a closed (i.e. compact, connected without boundary) oriented 2-dimensional Riemannian manifold of genus $g$. We will focus on (tangent) vector fields

$$u : S \rightarrow TS , \quad \text{i.e. } u(x) \in TS \text{ for every } x \in S$$

where $TS = \bigcup_{x \in S} T_x S$ is the tangent bundle of $S$. It is well known that there are no smooth vector fields $\mathcal{X}(S)$ (or more generally, of Sobolev regularity $\mathcal{X}^{1,2}(S)$) of unit length $|u|_g = 1$ on $S$ (unless $g = 1$). In fact, vector fields of unit length have in general singular points with a (non-zero) index. Our aim is to determine the interaction energy between these singular points in a variational model of Ginzburg–Landau type depending on a small parameter $\varepsilon > 0$, where the penalty $|u|_g = 1$ in $S$ is relaxed.

Model. For vector fields $u : S \rightarrow TS$, we define the energy functional

$$E_\varepsilon(u) = \int_S e_\varepsilon(u) \text{vol}_g , \quad e_\varepsilon(u) := \frac{1}{2} |Du|^2_g + \frac{1}{4\varepsilon^2} F(|u|^2_g) ,$$

where $|Du|^2_g := |D_t u|^2_g + |D_\tau u|^2_g$ in $S$, $\text{vol}_g$ is the volume 2-form on $(S, g)$ and $D_v$ denotes covariant differentiation (with respect to the Levi-Civita connection) of $u$ in direction $v$ and $\{\tau_1, \tau_2\}$ is any local orthonormal basis of $TS$. The potential $F : \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous function with $F(1) = 0$ and there exists some $c > 0$ such that $F(s^2) \geq c(1-s^2)$ for every $s \geq 0$; in particular, 1 is the unique zero of $F$. The parameter $\varepsilon > 0$ is small penalizing $|u|_g \neq 1$ in $S$; the goal is to analyze the asymptotic behavior of $E_\varepsilon$ in the framework of $\Gamma$-convergence (at first and second order) in the limit $\varepsilon \to 0$. This is a "toy" problem for some physical models arising for thin shells in micromagnetics and in nematic liquid crystals (see, e.g., [4,5]).

Connection 1-form. On an open subset $O \subset S$, a moving frame is a pair of smooth, properly oriented, orthonormal vector fields $\tau_k \in \mathcal{X}(O)$, $k = 1, 2$, i.e. $(\tau_1, \tau_2) = \delta_{kl}$, $l = 1, 2$, and $\text{vol}_g(\tau_1, \tau_2) = 1$ in $O$, where $(\cdot, \cdot)_g$ is the scalar product on $TS$. (We will use the same notation $(\cdot, \cdot)_g$ for the inner product associated with $k$-forms, $k = 0, 1, 2$.) Defining $i : TS \rightarrow TS$ such that $i$ is an isometry of $TS$ to itself for every $x \in S$ satisfying

$$i^2 w = -w , \quad (i w, v)_g = -(w, i v)_g = \text{vol}_g(w, v) ,$$

then every smooth vector field $\tau \in \mathcal{X}(O)$ of unit length provides a moving frame $\{\tau_1, \tau_2\} := \{\tau, i \tau\}$ on $O$. Moreover, if $\{\tau_1, \tau_2\}$ is any moving frame in $O$, then $\tau_2 = i\tau_1$. Given a moving frame $\{\tau_1, \tau_2\}$ on an open subset $O \subset S$, the connection 1-form $A$ associated with $\{\tau_1, \tau_2\}$ is defined for every smooth vector field $v \in \mathcal{X}(O)$:

$$A(v) := (D_v \tau_2, \tau_1)_g = - (D_v \tau_1, \tau_2)_g \quad \text{in } O.$$
\[
\int_\Sigma \kappa \operatorname{vol}_g = 2\pi \chi(S),
\]
where \(\chi(S)\) is the Euler characteristic, related to the genus \(g\) of \(S\) by \(\chi(S) = 2 - 2g\).

Vortices. We will identify vortices of a vector field \(u\) with small geodesic balls centered at some points around which \(u\) has a (non-zero) index. To be more precise, we introduce the Sobolev space \(X^{1,p}(S)\) of vector fields \(u : S \to TS\) such that \(|u|_g\) and \(|Du|_g\) belong to \(L^p(S)\) (with respect to the volume \(2\)-form), \(p \geq 1\). Given \(u \in X^{1,p}(S) \cap L^q(S)\) such that \(\frac{1}{p} + \frac{1}{q} = 1\), \(p, q \in [1, \infty]\), we define the 1-form \(j(u)\) by

\[
j(u) = (Du, iu)g.
\]

In particular, \(j(u)\) is a well-defined 1-form in \(L^1(S)\) if \(u \in X^{1,1}(S)\) with \(|u|_g = 1\) almost everywhere in \(S\); the same is true if \(u \in X^{1,p}(S)\) for \(p \geq \frac{4}{3}\). To introduce the notion of index, we assume that \(O\) is a simply connected open subset of \(S\) and \(u \in X^{1,2}(N)\) is a vector field in a neighborhood \(N\) of \(\partial O\) such that \(|u|_g \geq \frac{1}{2}\) a.e. in \(N\); then the index (or winding number) of \(u\) along \(\partial O\) is defined by

\[
\operatorname{deg}(u; \partial O) := \frac{1}{2\pi} \left( \int_{\partial O} j(u) \frac{1}{|u|^2} + \int_0^\infty \kappa \operatorname{vol}_g \right)
\]

(see [7], Chapter 6.1). In particular, if \(u\) is defined in \(O \cup N\) and has unit length on \(\partial O\), then one has

\[
\int_0^\infty \omega(u) = 2\pi \operatorname{deg}(u; \partial O)
\]

where \(\omega(u)\) is the vorticity associated with the vector field \(u\):

\[
\omega(u) := d(j(u)) + \kappa \operatorname{vol}_g.
\]

Sometimes we can identify the index of \(u\) at a point \(P \in S\) with the index of \(u\) along a curve around \(P\). Note that every smooth vector field \(u \in X(0)\) (or more generally, \(u \in X^{1,2}(O)\) of unit length in \(O\) has \(\operatorname{deg}(u; \partial O) = 0\); moreover, a vortex with non-zero index will carry infinite energy \(E_\varepsilon\) as \(\varepsilon \to 0\).

We will prove a \(\Gamma\)-convergence result (at the second order) of \(E_\varepsilon\) as \(\varepsilon \to 0\). In particular, at the level of minimizers \(u_\varepsilon\) of \(E_\varepsilon\), we show that \(u_\varepsilon\) converges in \(X^{1,1}(S)\) (for a subsequence) to a canonical harmonic vector field \(u^*\) of unit length that is smooth\(^3\) away from \(n = |\chi(S)|\) distinct singular points \(a_1, \ldots, a_n\), each singular point \(a_k\) carrying the same index \(d_k = \operatorname{sign} \chi(S) \in \{\pm 1\}\) so that

\[
\sum_{k=1}^n d_k = \chi(S).
\]

The vorticity \(\omega(u^*)\) detects the singular points \(\{a_k\}_{k=1}^n\) of \(u^*\):

\[
\omega(u^*) = 2\pi \sum_{k=1}^n d_k \delta_{a_k}
\]

in \(S\),

\[
\delta_{a_k}
\]

where \(\delta_{a_k}\) is the Dirac measure (as a 2-form) at \(a_k\). The expansion of the minimal energy \(E_\varepsilon\) at the second order is given by

\[
E_\varepsilon(u_\varepsilon) = n\pi \log \frac{1}{\varepsilon} + \lim_{r \to 0} \left( \int_{S \setminus \bigcup_{k=1}^n B_r(a_k)} \frac{1}{2} |Du^*|^2_g \operatorname{vol}_g + n\pi \log r \right) + n\gamma_F + o(\varepsilon), \quad \varepsilon \to 0,
\]

where \(\gamma_F > 0\) is a constant depending only on the potential \(F\) and \(B_r(a_k)\) is the geodesic ball centered at \(a_k\) of radius \(r\). The second term in the above RHS is called the renormalized energy between the vortices \(a_1, \ldots, a_n\) and governs the optimal location of these singular points as in the Euclidean case (see the seminal book of Bethuel–Brézis–Hélein [3], which is essential for our note). In particular, if \(S\) is the unit sphere in \(\mathbb{R}^3\) endowed with the standard metric \(g\), then \(n = 2\) and \(a_1\) and \(a_2\) are two diametrically opposed points on \(S\).

Outline of the note. The note is divided as follows. Section 2 is devoted to characterize canonical harmonic vector fields of unit length. In Section 3, we determine the renormalized energy between singular points of canonical harmonic vector fields. The main \(\Gamma\)-convergence result is stated in the last section. The proofs of these results are part of our forthcoming article [9].

\footnote{Note that if \(\{\tau_1, \tau_2\}\) is a moving frame on an open set \(O \subset S\), then the connection 1-form \(A\) associated with the moving frame is given by \(A = -j(\tau_1)\) on \(O\). In particular, \(d(j(u)) = -\kappa \operatorname{vol}_g\) in \(O\) for every smooth \(u \in X(0)\) of unit length.}

\footnote{In the case of a surface \((S, g)\) with genus 1 (i.e. homeomorphic with the flat torus), then \(n = 0\) and \(u^*\) is smooth in \(S\).}

\footnote{In fact, \(\operatorname{deg}(u^*; \gamma) = d_k\) for every closed simple curve \(\gamma\) around \(a_k\) and lying near \(a_k\).}
2. Canonical harmonic vector fields of unit length

We will say that a canonical harmonic vector field of unit length having the singular points $a_1, \ldots, a_n \in \mathbb{S}$ of index $d_1, \ldots, d_n \in \mathbb{Z}$ for some $n \geq 1$ is a vector field $u^* \in \mathcal{X}^{1,1}(S)$ such that $|u^*|_g = 1$ in $S$. (4) holds and

$$d^* j(u^*) = 0 \quad \text{in } S.$$  \hfill (5)

Here, $d^*$ is the adjoint of the exterior derivative $d$, i.e. $d^* j(u^*)$ is the unique 0-form on $S$ such that

$$\int_S (d^* j(u^*), \zeta)_g \text{ vol}_g = \int_S (j(u^*), d\zeta)_g \text{ vol}_g \quad \text{for every smooth 0-form } \zeta.$$  

If $u^*$ satisfies (4), then the Gauss–Bonnet theorem combined with (2) implies that necessarily (3) holds. We will see that condition (3) is also sufficient. Indeed, if (3) holds, we will construct solutions to (4) and (5), as follows: let $\psi = \psi(a, d)$ be the unique 2-form on $S$ solving:

$$-\Delta \psi = -\kappa \text{ vol}_g + 2\pi \sum_{k=1}^n d_k \delta_{a_k} \quad \text{in } S, \quad \int_S \psi = 0,$$  \hfill (6)

with the sign convention that $-\Delta = dd^* + d^* d$. The idea is to find $u^*$ such that $j(u^*) - d^* \psi$ is an harmonic 1-form, i.e. $\text{Harm}^1(S) = \{\text{integrable 1-forms } \eta \quad \text{on } S : d\eta = d^* \eta = 0 \text{ as distributions}\}$. The dimension of the space $\text{Harm}^1(S)$ is twice the genus (i.e. $2g$) of $(S, g)$ and we fix an orthonormal basis $\eta_1, \ldots, \eta_{2g}$ of $\text{Harm}^1(S)$ such that $\int_S (\eta_k, \eta_l)_g \text{ vol}_g = \delta_{kl}$ for $k, l = 1, \ldots, 2g$. Therefore, it is expected that

$$j(u^*) = d^* \psi + \sum_{k=1}^{2g} \Phi_k \eta_k \quad \text{in } S$$  \hfill (7)

for some constant vector $\Phi = (\Phi_1, \ldots, \Phi_{2g}) \in \mathbb{R}^{2g}$. These constants are called flux integrals as they can be recovered by

$$\Phi_k = \int_S (j(u^*), \eta_k)_g \text{ vol}_g, \quad \text{for } k = 1, \ldots, 2g.$$  \hfill (8)

Note that (7) combined with (6) automatically yields (4) and (5). One important point is to characterize for which values of $\Phi$ the RHS of (7) arises as $j(u^*)$ for some vector field $u^*$ of unit length in $S$. For that condition, we need to recall the following classical fact (see for example [8]): there exist $2g$ simple closed geodesics $\gamma_\ell$ on $S, \ell = 1, \ldots, 2g$, such that for any closed Lipschitz curve $\gamma$ on $S$, one can find integers $c_1, \ldots, c_{2g}$ such that $\gamma$ is homologous to $\sum_{\ell=1}^{2g} c_\ell \gamma_\ell$, i.e. there exists an integrable function $f : S \to \mathbb{Z}$ such that $\int_{\gamma} \zeta - \sum_{\ell=1}^{2g} c_\ell \int_{\gamma_\ell} \zeta = \int_S f \text{ d}\zeta$ for all smooth 1-forms $\zeta$. Having chosen the geodesic curves $\{\gamma_\ell\}_{\ell=1}^{2g}$ and the harmonic 1-forms $\{\eta_k\}_{k=1}^{2g}$, we fix the notation

$$\alpha_{\ell k} := \int_{\gamma_\ell} \eta_k, \quad k, \ell = 1, \ldots, 2g.$$  \hfill (9)

**Theorem 1.** Let $n \geq 1$ and $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ satisfy (3). Then for every $a = (a_1, \ldots, a_n) \in S^n$, there exists

$$\zeta_\ell = \zeta_\ell(a; d) \in \mathbb{R}/2\pi\mathbb{Z}, \quad \ell = 1, \ldots, 2g$$

such that if a vector field $u^* \in \mathcal{X}^{1,1}(S)$ of unit length solves (4) and (5), then $j(u^*)$ has the form (7) for constants $\Phi_1, \ldots, \Phi_{2g}$ such that

$$\sum_{k=1}^{2g} \alpha_{\ell k} \Phi_k + \zeta_\ell(a, d) \in 2\pi\mathbb{Z}, \quad \ell = 1, \ldots, 2g.$$  \hfill (10)

where $(\alpha_{\ell k})$ were defined in (9). Conversely, given any $\Phi_1, \ldots, \Phi_{2g}$ satisfying (10), there exists a vector field $u^* \in \mathcal{X}^{1,1}(S)$ of unit length solving (4) and (5) and such that $j(u^*)$ satisfies (7). In addition, the following hold:
1) \( \xi_\ell (\cdot; d) \) depends continuously on \( a \in S^n \) for every \( \ell = 1, \ldots, 2g \). More generally, if

\[
\mu^1 := 2\pi \sum_{l=1}^{n} d_1^l \delta_{d^l} \rightarrow \mu^0 := 2\pi \sum_{l=1}^{n} d_0^l \delta_{d^0} \quad \text{in} \ W^{-1,1} \quad \text{as} \ t \downarrow 0,
\]

\( \{d^l_0\} \) are integers with (3) and \( \sum_{l=1}^{n} \{d^l_1\} \) is uniformly bounded in \( t \), then \( \xi_\ell (a^l, d^l) \rightarrow \xi_\ell (a^0, d^0) \) as \( t \downarrow 0 \).

2) Any \( u^* \) solving (4) and (5) belongs to \( \lambda_{1, p} (S) \) for all \( 1 \leq p < 2 \), and is smooth away from \( \{a_k\}_{k=1}^n \).

3) If \( u^*, \tilde{u}^* \) both satisfy (7) for the same \( (a, d) \) and the same \( \{\Phi_k\}_{k=1}^{2g} \), then \( \tilde{u}^* = e^{\phi} u^* \) for some \( \beta \in \mathbb{R} \).

The constants \( \{\xi_\ell (a; d)\}_{\ell=1}^{2g} \) are determined as follows. For every \( \ell = 1, \ldots, 2g \), let \( \lambda_\ell \) be some smooth simple closed curve such that \( \lambda_\ell \) is homologous to \( \gamma_\ell \) (the geodesics fixed in (9)) so that \( \{a_k\}_{k=1}^n \) is disjoint from \( \lambda_\ell \); for example, \( \lambda_\ell \) is either \( \gamma_\ell \) or, if \( \gamma_\ell \) intersects some \( a_k \), a small perturbation thereof. We now define \( \xi_\ell (a, d) \) to be the element of \( \mathbb{R} / 2\pi \mathbb{Z} \) such that

\[
\xi_\ell (a, d) := \int_{\lambda_\ell} (d^* \psi + A) \mod 2\pi, \quad \ell = 1, \ldots, 2g,
\]

where \( \psi = \psi (a, d) \) is the 2-form given by (6) and \( A \) is the connection 1-form associated with any moving frame defined in a neighborhood of \( \lambda_\ell \). The integral in (11) is independent, modulo \( 2\pi \mathbb{Z} \), of the choice of moving frame and of the curve \( \lambda_\ell \) homologous to \( \gamma_\ell \). In examples in which it can be explicitly computed, in general \( \xi_\ell (a, d) \neq 0 \) \( \mod 2\pi \) for \( \ell = 1, \ldots, 2g \).

3. Renormalized energy

For any \( n \geq 1 \), we consider \( n \) distinct points \( a = (a_1, \ldots, a_n) \in S^n \). Let \( d = (d_1, \ldots, d_n) \in \mathbb{Z}^n \) satisfying (3), \( \{\xi_\ell (a; d)\}_{\ell=1}^{2g} \) be given in Theorem 1 and \( \Phi \in \mathbb{R}^{2g} \) be a constant vector belonging to the set:

\[
L (a, d) := \{ \Phi = (\Phi_1, \ldots, \Phi_{2g}) \in \mathbb{R}^{2g} : \sum_{k=1}^{2g} \alpha_{\ell k} \Phi_k + \xi_\ell (a, d) \in 2\pi \mathbb{Z}, \ell = 1, \ldots, 2g \}.
\]

We define the renormalized energy between the vortices \( a \) of indices \( d \) by

\[
W (a, d, \Phi) := \lim_{r \to 0} \left( \int_{S \setminus \bigcup_{k=1}^n B_r (a_k)} \frac{1}{2} \|Du^*\|^2_g \vol_g + \pi \log r \sum_{k=1}^n d_k^2 \right),
\]

where \( u^* = u^* (a, d, \Phi) \) is the unique (up to a multiplicative complex number) canonical harmonic vector field given in Theorem 1 and \( B_r (a_k) \) is the geodesic ball centered at \( a_k \) of radius \( r \). Our arguments show that the above limit indeed exists. As in the Euclidean case (see [3]), we can compute the renormalized energy by using the Green’s function. For that, let \( G (x, y) \) be the unique function on \( S \times S \) such that for every \( y \in S \):

\[
-\Delta_y G (\cdot, y) \vol_g = \delta_y - \frac{\vol_g}{\vol_g (S)} \text{ distributionally in } S, \quad \int_S G (x, y) \vol_g (x) = 0,
\]

with \( \vol_g (S) := \int_S \vol_g \). Then \( G \) may be represented in the form (see [2], Chapter 4.2):

\[
G (x, y) = G_0 (x, y) + H (x, y), \quad \text{with } H \in C^1 (S \times S),
\]

where \( G_0 \) is smooth away from the diagonal, with

\[
G_0 (x, y) = -\frac{1}{2\pi} \log (\text{dist} (x, y)) \quad \text{if the geodesic distance } \text{dist} (x, y) < \frac{1}{2} (\text{injectivity radius of } S).
\]

The 2-form \( \psi = \psi (a, d) \) defined in (6) can be written as:

\[
\psi = 2\pi \sum_{k=1}^n d_k G (\cdot, a_k) \vol_g + \psi_0 \vol_g \quad \text{in } S.
\]
where $\psi_0 \in C^\infty(S)$ has zero average on $S$ and solves
\[
-\Delta \psi_0 = -\kappa + \bar{k}, \quad \text{for } \bar{k} = \frac{1}{\text{Vol}(S)} \int_S \kappa \text{vol}_g = \frac{2\pi \chi(S)}{\text{Vol}(S)}.
\] (13)

In other words, the 2-form $x \mapsto \psi(x) + d_k \log(\text{dist}(x,a_k)) \text{vol}_g$ is $C^1$ in a neighborhood of $a_k$ for every $1 \leq k \leq n$. We have the following expression of the renormalized energy defined in (12).

**Proposition 2.** Given $n \geq 1$ distinct points $a_1, \ldots, a_n \in S$, integers $d_1, \ldots, d_n$ with (3) and $\Phi \in \mathcal{L}(a,d)$, then
\[
W(a,d,\Phi) = 4\pi^2 \sum_{l \neq k} d_l d_k G(a_l,a_k) + 2\pi \sum_{k=1}^n \left( \sum_{l \neq k} d_l^2 H(a_l,a_k) + d_k \psi_0(a_k) \right) + \frac{1}{2} |\Phi|^2 + \int_S \frac{|d\psi_0|^2}{2} \text{vol}_g,
\] (14)

where $\psi_0$ is defined in (13).

For the unit sphere $S$ in $\mathbb{R}^3$ endowed with the standard metric (in particular, $\psi_0$ vanishes in $S$, $g = 0$ and $\Phi = 0$), if $n = 2$ and $d_1 = d_2 = 1$, then the second term in the RHS of (14) is independent of $a_k$ (as $x \mapsto H(x,x)$ is constant, see [4]); thus, minimizing $W$ is equivalent by minimizing the Green’s function $G(a_1,a_2)$ over the set of pairs $(a_1,a_2)$ in $S \times S$, namely, the minimizing pairs are diametrically opposed.

4. $\Gamma$-convergence

Given the potential $F$ in Section 1, we compute the energy $E_\varepsilon$ of the radial profile of a vortex of index 1 inside a geodesic ball of radius $R > 0$:
\[
I_F(R,\varepsilon) := \inf \left\{ \pi \int_0^R \left[ f'(r)^2 + \frac{f(r)^2}{r^2} + \frac{1}{2\varepsilon^2} F(f(r)^2) \right] r \, dr : \text{f(0) = 0, f(R) = 1} \right\}.
\]

Then $I_F(R,\varepsilon) = I_F(\lambda R,\lambda \varepsilon) = I_F(1,\frac{\varepsilon}{R})$ for every $\lambda > 0$, and the following limit exists (see [3]):
\[
\gamma_F := \lim_{\varepsilon \to 0} (I_F(t) + \pi \log t).
\]

We state our main result as follows.

**Theorem 3.** The following $\Gamma$-convergence result holds.

1) (Compactness) Let $(u_k)_{k \geq 0}$ be a family of vector fields in $X^{1,2}(S)$ satisfying $E_\varepsilon(u_k) \leq N \pi |\log \varepsilon| + C$ for some integer $N \geq 0$ and a constant $C > 0$. We define
\[
\Phi(u_k) := \left( \int_S (j(u_k), \eta_1) \text{vol}_g, \ldots, \int_S (j(u_k), \eta_2 \theta) \text{vol}_g \right) \in \mathbb{R}^{2\theta}.
\]

Then there exists a sequence $\varepsilon \downarrow 0$ such that
\[
\omega(u_k) \rightharpoonup 2\pi \sum_{k=1}^n d_k \delta_{a_k} \text{ in } W^{-1,1}, \text{ as } \varepsilon \to 0,
\] (15)

where $\{a_k\}_{k=1}^n$ are distinct points in $S$ and $\{d_k\}_{k=1}^n$ are nonzero integers satisfying (3) and $\sum_{k=1}^n |d_k| \leq N$. Moreover, if $\sum_{k=1}^n |d_k| = N$, then $n = N$, $|d_k| = 1$ for every $k = 1, \ldots, n$ (in particular, $n = \chi(S)$ modulo 2) and there exists $\Phi \in \mathcal{L}(a,d)$ such that $\Phi(u_k) \to \Phi$ for a further sequence $\varepsilon \to 0$.

2) ($\Gamma$-liminf) Assume that the vector fields $u_\varepsilon \in \mathcal{X}^{1,2}(S)$ satisfy (15) for $n$ distinct points $\{a_k\}_{k=1}^n \in S^n$ and $|d_k| = 1, k = 1, \ldots, n$ that satisfy (3) and $\Phi(u_\varepsilon) \to \Phi$ for some $\Phi \in \mathcal{L}(a,d)$. Then
\[
\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \geq W(a, d, \Phi) + n \gamma_F.
\]

3) ($\Gamma$-limsup) For every $n$ distinct points $a_1, \ldots, a_n \in S$ and $d_1, \ldots, d_n \in \{\pm 1\}$ satisfying (3) and every $\Phi \in \mathcal{L}(a,d)$, there exists a sequence of vector fields $u_\varepsilon$ on $S$ such that (15) holds, $\Phi(u_\varepsilon) \to \Phi$ and
\[
E_\varepsilon(u_\varepsilon) \to \frac{n \pi |\log \varepsilon|}{\varepsilon} \to W(a, d, \Phi) + n \gamma_F \text{ as } \varepsilon \to 0.
\]

This theorem is the generalization of the $\Gamma$-convergence result for $E_\varepsilon$ in the Euclidean case (see [6,11,13,1]) and it is based on topological methods for energy concentration (vortex ball construction, vorticity estimates etc.) as introduced in [10,12].
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References