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Partial differential equations

On the sub poly-harmonic property for solutions to $(-\Delta)^p u < 0$ in $\mathbb{R}^n$

De la sous-poly-harmonicité des solutions de $(-\Delta)^p u < 0$ dans $\mathbb{R}^n$

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ABSTRACT

In this note, we mainly study the relation between the sign of $(-\Delta)^p u$ and $(-\Delta)^{p-i} u$ in $\mathbb{R}^n$ with $p \geqslant 2$ and $n \geqslant 2$ for $1 \leqslant i \leqslant p-1$. Given the differential inequality $(-\Delta)^p u < 0$, first we provide several sufficient conditions so that $(-\Delta)^{p-1} u < 0$ holds. Then we provide conditions such that $(-\Delta)^i u < 0$ for all $i = 1, 2, \ldots, p-1$, which is known as the sub poly-harmonic property for u. In the last part of the note, we revisit the super poly-harmonic property for solutions to $(-\Delta)^p u = e^{2pu}$ and $(-\Delta)^p u = u^q$ with $q > 0$ in $\mathbb{R}^n$.

RÉSUMÉ

Dans cette Note, nous étudions principalement la relation entre le signe de $(-\Delta)^p u$ et $(-\Delta)^{p-i} u$ dans $\mathbb{R}^n$ pour $1 \le i \le p-1$, avec n, $p \ge 2$. Étant donnée l'inégalité différentielle $(-\Delta)^p < 0$, nous montrons, dans un premier temps, plusieurs conditions suffisantes afin que l'inégalité $(-\Delta)^{p-1} u < 0$ soit satisfaite. Puis, sous une hypothèse de croissance, nous montrons que $(-\Delta)^i u < 0$ pour tout $i = 1, 2, \ldots, p-1$, c'est-à-dire que u satisfait la propriété de sous-poly-harmonicité. Dans la dernière partie de la Note, nous considérons la sur-poly-harmonicité des solutions de l'équation $(-\Delta)^p u = e^{2pu}$ et $(-\Delta)^p u = u^q$, avec $q > 0$, dans $\mathbb{R}^n$.

1. Introduction

Given $p \geqslant 2$ and $n \geqslant 2$, of interest in this note is the following interesting differential inequality

$$(-\Delta)^p u < 0 \tag{1.1}$$

in $\mathbb{R}^n$. Given a real number $q > 0$, the motivation of studying such inequalities goes back to the following equation

$$(-\Delta)^p u + u^{-q} = 0, \quad u > 0 \tag{1.2}$$

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with a negative exponent in $\mathbb{R}^n$ that can be seen as a counterpart of the equation

$$(-\Delta)^p u = u^q, \quad u > 0 \tag{1.3}$$

with a positive exponent in the whole space $\mathbb{R}^n$. In the literature, equations of the form (1.3) have already been captured so much attention since they have a root in conformal geometry as well as in blow-up theory that appears everywhere in elliptic PDEs. A particular but important case of (1.3) is when $q = (n+2p)/(n-2p) > 0$. In this scenario, it turns out that a precise formula for solutions to

$$(-\Delta)^p u = u^{\frac{n+2p}{n-2p}}, \quad u > 0, \tag{1.4}$$

in $\mathbb{R}^n$ plays an important role in many aspects of mathematics such as analysis, geometry, and PDE.

In PDE, among others, the problem of classification of positive solutions to (1.4) has been in the center of the field for the last two decades. In this research direction, the first remarkable result was obtained by Xu in [17] for biharmonic equations and then by Wei and Xu in [16] for poly-harmonic equations. They proved, among other things, that classical solutions to (1.4) obey the so-called *super poly-harmonic* property in the following sense

$$(-\Delta)^i u > 0$$

for all $i = 1, 2, \dots, p-1$ provided $u(x) = o(|x|^2)$; see [16, Theorem 1.1]. In this particular problem, the super poly-harmonic property above turns out to be extremely important as it helps us to make use of the maximum principle, a key ingredient in the "traditional" method of moving planes [1], to attack high-order equations, which usually is not available. It turns out that the super poly-harmonic property also holds for solutions to (1.4) on the half space $\mathbb{R}^n_+$ with different boundary conditions such as the Navier boundary condition [4]; see also [12,11] and the references therein for different contexts.

Recently, Chen, Li, and Ou introduced in [3] a new form of the "traditional" method of moving planes, called the method of moving planes in the integral form. Loosely speaking, instead of using certain local properties of the underlying differential operators such as maximum principles, this new method makes use of global properties of the differential operators as well as certain integral estimates. Using this new method, they were able to classify solutions to the following integral equation

$$u(x) = \int_{\mathbb{R}^n} |x-y|^{2p-n} u(y)^{\frac{n+2p}{n-2p}} \, dy \tag{1.5}$$

with $p < n/2$, which is the integral form associated with (1.4). This "new" method of moving planes has been rapidly used by many mathematicians to establish various symmetry and non-existence results as well as a priori estimates for various integral equations and systems. Also in [3], the authors made use of the super poly-harmonic property established in [16] to prove the equivalence of (1.4) and (1.5). Since then, the equivalence between differential equations and their corresponding integral equations has been considered for different problems in different context; see [2,4,7].

Backing to Eq. (1.2), equations with negative exponents of the form (1.2) have recently attracted the interest of many mathematicians. In this research field, we refer the interested reader to [15,10,13,5,6] for vast interesting aspects of solutions for several equations of the form (1.2). In this note, as a counterpart of (1.4) as well as (1.3), we mainly consider (1.2). More precisely, instead of studying Eq. (1.2), we are interested in differential inequality (1.1). We shall establish that solutions to (1.1) obey the *sub poly-harmonic* property, which can be considered as the opposite of the super poly-harmonic property, namely

$$(-\Delta)^i u < 0$$

for all $i = 1, 2, \dots, p-1$.

In order to achieve the goal, we first establish a simple result to relate the sign of $(-\Delta)^p$ and $(-\Delta)^{p-1}$, which can also be considered as a slight improvement of [11, Theorem 1.1]. The following general result is the heart of the note.

Theorem 1. *Given $p \geqslant 2$ and $n \geqslant 2$, let u be a C^{2p}-solution the differential inequality $(-\Delta)^p u < 0$ in $\mathbb{R}^n$. We assume further that*

$$\limsup_{|x| \to +\infty} \frac{(-1)^{p-1} u(x)}{|x|^{2p-2}} \leqslant 0.$$

Then we necessarily have $(-\Delta)^{p-1} u < 0$ everywhere in $\mathbb{R}^n$.

Clearly, the conclusion of Theorem 1 provides us with a simpler result, which can be restated in the following way.

Corollary 1. *Given $p \geqslant 2$ and $n \geqslant 2$, let u be a C^{2p}-solution the differential inequality $(-\Delta)^p u < 0$ in $\mathbb{R}^n$. We assume further that*

(a) *either $u(x) = o(|x|^{p+1})$ at infinity when p is odd*
(b) *or u is*
  (b1) *either bounded from below*
  (b2) *or $u(x) = o(|x|^p)$ at infinity*
  when p is even.

Then we necessarily have $(-\Delta)^{p-1} u < 0$ everywhere in $\mathbb{R}^n$.

Next, we discuss several applications of Theorem 1 above. The main aim of the following part is to obtain the sub poly-harmonic property for solutions to (1.1) in different contexts. First, we consider the case when solutions to (1.1) have a lower bound such as the positivity of solutions. Note that this restriction naturally arises when we deal with positive solutions to several PDEs with polynomial nonlinearities.

Theorem 2. *Given $p \geqslant 2$ and $n \geqslant 2$, for each lower-bounded, C^{2p}-solution u of the differential inequality $(-\Delta)^p u < 0$ in $\mathbb{R}^n$ with $u(x) = o(|x|^4)$ at infinity, there holds $(-\Delta)^i u < 0$ everywhere in $\mathbb{R}^n$ and for all $i = 1, 2, \dots, p-1$.*

We note that we can relax the growth condition $u(x) = o(|x|^4)$ at infinity in Theorem 2 by assuming that the solution u is also bounded from above, hence is bounded in $\mathbb{R}^n$; see Corollary 7.

Next we consider the case when solutions to (1.1) is not bounded from below; hence we require "further" behavior of solutions at infinity. We shall prove the following theorem.

Theorem 3. *Given $p \geqslant 2$ and $n \geqslant 2$, for each C^{2p}-solution u to the differential inequality $(-\Delta)^p u < 0$ in $\mathbb{R}^n$ with $u(x) = o(|x|^2)$ at infinity, there holds $(-\Delta)^i u < 0$ everywhere in $\mathbb{R}^n$ and for all $i = 1, 2, \dots, p-1$.*

The main reason to study such sub poly-harmonic property for solutions to (1.1) goes back to [5] and [18], where the authors studied the equivalence of the differential equation $\Delta^2 u + u^{-q} = 0$ with $q > 1$ in $\mathbb{R}^3$ and its corresponding integral equation $u(x) = \int_{\mathbb{R}^3} |x-y| u(y)^{-q} dy$. As now it becomes well-known that these two equations are not equivalent, not only for general q but also in the critical case $q = 7$; see [9]. Note that, in the critical case, solutions to the differential equation $\Delta^2 u + u^{-7} = 0$ in $\mathbb{R}^3$ verify the corresponding integral equation $u(x) = \int_{\mathbb{R}^3} |x-y| u(y)^{-7} dy$ provided these solutions have exactly linear growth uniformly at infinity; see [5]. However, it was proved in [6] that the differential equation $\Delta^2 u + u^{-7} = 0$ in $\mathbb{R}^3$ not only admit solutions with exactly linear growth at infinity but also solutions with exactly quadratic growth at infinity.

2. Proof of Theorems 1, 2, and 3

In this section, we prove Theorem 1, namely the differential inequality $(-\Delta)^{p-1} u < 0$ holds everywhere in $\mathbb{R}^n$. For the sake of simplicity, for each point x_0 and each $r \geqslant 0$, we denote by $\overline{u}(x_0, r)$ the average of u over the ball $B(x_0, r)$ which is given as follows

$$\overline{u}(x_0, r) = \begin{cases} \dfrac{1}{|\partial B(x_0, r)|} \displaystyle\int_{\partial B(x_0, r)} u \, d\sigma & \text{if } r > 0 \\ u(x_0) & \text{if } r = 0. \end{cases}$$

Sometimes, for simplicity, we write $\overline{u}(r)$ instead of $\overline{u}(x_0, r)$ if the center x_0 is clear, and no confusion occurs.

2.1. Proof of Theorem 1

To prove $(-\Delta)^{p-1} u < 0$ everywhere in $\mathbb{R}^n$, we suppose on the contrary that $\sup_{\mathbb{R}^n} (-\Delta)^{p-1} u \geqslant 0$. Depending of the precise sign of $(-\Delta)^{p-1} u$, we have the following two possible cases.

2.1.1. Estimate of $(-\Delta)^{p-1} u$: the case $(-\Delta)^{p-1} u \leqslant 0$

Suppose that $\sup_{x \in \mathbb{R}^n} (-\Delta)^{p-1} u(x) = 0$. We claim that $(-\Delta)^{p-1} u < 0$ everywhere in $\mathbb{R}^n$. In fact, suppose not, then there must exist some point $x_0 \in \mathbb{R}^n$ such that $(-\Delta)^{p-1} u(x_0) = 0$. In this scenario, we know that x_0 is a maximum point of $(-\Delta)^{p-1} u$; hence at x_0 we must have $-\Delta(-\Delta)^{p-1} u(x_0) \geqslant 0$ which contradicts with our assumption $(-\Delta)^p u < 0$ everywhere in $\mathbb{R}^n$.

2.1.2. Estimate of $(-\Delta)^{p-1}u$: the case $(-\Delta)^{p-1}u > 0$

There exists some point $x_0 \in \mathbb{R}^n$ such that $(-\Delta)^{p-1}u(x_0) > 0$. Up to a translation, we may assume $x_0 \equiv 0$ and write $\overline{u}(r)$ for the average of u over $\partial B(0,r)$ instead of $\overline{u}(x_0, r)$. Denote

$$c_0 = (-\Delta)^{p-1}u(0) = (-\Delta)^{p-1}\overline{u}(0) > 0$$

and

$$v(r) = (-\Delta)^{p-1}\overline{u}(r).$$

It follows from the given differential inequality that

$$\Delta v = r^{1-n}(r^{n-1}v')' > 0.$$

From this we obtain

$$v(r) \geqslant v(0) = c_0$$

for all $r \geqslant 0$. Set

$$w(r) = (-1)^{p-1}\overline{u}(r)$$

and take

$$\Phi(r) = \sum_{0 \leqslant k \leqslant p-1} a_k r^{2k}$$

in such a way that

$$\begin{cases} \Delta^k \Phi(0) = \Delta^k w(0) & \text{for all } 0 \leqslant k \leqslant p-2, \\ \Delta^{p-1}\Phi(r) = c_0 & \text{for all } r \geqslant 0. \end{cases}$$

Note that $a_{p-1} > 0$ since $c_0 > 0$. Then it is easy to see that

$$\Delta^{p-1}(w - \Phi) = v - c_0 \geqslant 0$$

everywhere in $\mathbb{R}^n$. Then by a well-known comparison principle for polyharmonic operators [8, Proposition A.2], we deduce that

$$w - \Phi \geqslant 0$$

in $\mathbb{R}^n$ and this is enough for us to obtain a contradiction since Φ is equivalent to Ar^{2p-2} at infinity for some constant $A > 0$.

Remark 1. Note that in the preceding proof, we have used a comparison principle for polyharmonic operators. However, this case can also be proved by using integration by parts. Indeed, by mathematical induction, it is not hard to see that the following key estimate holds

$$(-1)^i(-\Delta)^{p-i}\overline{u}(r) \leqslant (-1)^i(-\Delta)^{p-i}\overline{u}(0) + \sum_{l=1}^{i-1} \frac{(-1)^{i-l}(-\Delta)^{p-i+l}\overline{u}(0)r^{2l}}{\prod_{k=1}^{l}(2k)\prod_{k=1}^{l}[n+2(k-1)]} \tag{2.1}$$

for each $i \leqslant p$ and for any r. In the special case $i = p$, we obtain from (2.1) the following

$$(-1)^p\overline{u}(r) \leqslant (-1)^p\overline{u}(0) + \sum_{l=1}^{p-1} \frac{(-1)^{p-l}(-\Delta)^{l}\overline{u}(0)r^{2l}}{\prod_{k=1}^{l}(2k)\prod_{k=1}^{l}[n+2(k-1)]}. \tag{2.2}$$

It is important to note that, up to this point, we have not fully used the sign of $(-\Delta)^{p-1}u$, except the one place where we chose the center x_0, which was presumed to be the origin, of spherical averages. Now, under the contradiction assumption $(-\Delta)^{p-1}u(0) > 0$, we notice that the coefficient of the leading term on the right-hand side of (2.2) always has a negative sign by our contradiction assumption; hence, there holds

$$(-1)^{p-1}\overline{u}(r) \geqslant \frac{(-\Delta)^{p-1}\overline{u}(0)}{2\prod_{k=1}^{p-1}(2k)\prod_{k=1}^{p-1}[n+2(k-1)]} r^{2(p-1)} \tag{2.3}$$

for $r \gg 1$ large enough. Since the coefficient of the term $r^{2(p-1)}$ on the right-hand side of (2.3) is positive, there exists a sequence of points $\{x_i\}_{i\geqslant 1}$ in $\mathbb{R}^n$ such that $|x_i| \to +\infty$ and we have

$$\frac{(-1)^{p-1}u(x_i)}{|x_i|^{2(p-1)}} \geqslant \frac{(-\Delta)^{p-1}\overline{u}(0)}{2\prod_{k=1}^{p-1}(2k)\prod_{k=1}^{p-1}[n+2(k-1)]} > 0$$

for all i. From this, we can obtain a contradiction.

2.2. Several applications of Theorem 1

Clearly Corollary 1 is just an application of Theorem 1. It is interesting to note that Theorem 1 also implies the following simple result.

Corollary 2. *Given $p \geqslant 2$ and $n \geqslant 2$. Then for any C^{2p}-solution u of the differential inequality $(-\Delta)^p u < 0$ in $\mathbb{R}^n$ with $u(x) = o(|x|^p)$ at infinity, we necessarily have $(-\Delta)^{p-1} u < 0$ everywhere in $\mathbb{R}^n$.*

The conclusion in Corollary 2 is obvious since $2p - 2 \geqslant p$ whenever $p \geqslant 2$. From a PDE point of view, we may apply Corollary 1(b) to obtain the sub poly-harmonic property for solutions to a class of biharmonic equations in $\mathbb{R}^n$, which generalizes [5, Lemma 2.2] from $\mathbb{R}^3$ to $\mathbb{R}^n$; see also [11, Lemma 4.2].

Corollary 3. *For each positive solution u of $\Delta^2 u + u^{-q} = 0$ in $\mathbb{R}^n$ with $q > 1$, there holds $\Delta u > 0$.*

In the same fashion of Corollary 3 above, we also obtain the super poly-harmonic property for solutions to a class of triharmonic equations in $\mathbb{R}^n$.

Corollary 4. *For each positive solution u of $\Delta^3 u = u^{-q}$ in $\mathbb{R}^n$ with $q > 0$, if $u(x) = o(|x|^4)$ at infinity, then there hold $\Delta^2 u < 0$ and $\Delta u > 0$.*

2.3. Proof of Theorems 2 and 3

In this subsection, we further discuss applications of Theorem 1. First, we apply Theorem 1 to prove Theorem 2.

Proof of Theorem 2. Using the hypothesis $u(x) = o(|x|^4)$ and the fact that $2p - 2 \geqslant 4$ whenever $p \geqslant 3$, by backward induction, we get $(-\Delta)^i u < 0$ for $2 \leqslant i \leqslant p - 1$. In particular, we arrive at $\Delta^2 u < 0$. To conclude $-\Delta u < 0$, we make use of the fact that u is bounded from below; see Corollary 1(b). □

As an immediate consequence of Theorem 2, we obtain the following corollary.

Corollary 5. *Given $p \geqslant 2$ and $n \geqslant 2$, for each upper-bounded, C^{2p}-solution u of the differential inequality $(-\Delta)^p u > 0$ in $\mathbb{R}^n$ with $u(x) = o(|x|^4)$ at infinity, there holds $(-\Delta)^i u > 0$ everywhere in $\mathbb{R}^n$ and for all $i = 1, 2, \dots, p - 1$.*

In Section 3, we make use of Corollary 5 to realize the super poly-harmonic property for solutions to $(-\Delta)^p u = (2p - 1)! e^{2pu}$ in $\mathbb{R}^{2p}$ and compare this result with [16, Lemma 2.1].

Finally, making use of Corollary 2, we can prove Theorem 3 as follows.

Proof of Theorem 3. The condition $u(x) = o(|x|^2)$ at infinity implies that $u(x) = o(|x|^p)$ at infinity for any $p \geqslant 2$. The proof follows by using Corollary 2 repeatedly. □

3. Some remarks

3.1. Super poly-harmonic property for solutions to $(-\Delta)^p u = e^{2pu}$ and $(-\Delta)^p u = u^q$ in $\mathbb{R}^n$ revisited

From PDE point of view, for solutions to $(-\Delta)^p u = (2p-1)! e^{2pu}$ in $\mathbb{R}^{2p}$, we may apply Corollary 5 to obtain the following corollary, which has a same fashion of [16, Lemma 2.1].

Corollary 6. *Let u be a solution to $(-\Delta)^p u = (2p - 1)! e^{2pu}$ on $\mathbb{R}^{2p}$ with finite energy condition $\int_{\mathbb{R}^{2p}} e^{2pu} < +\infty$. If u satisfies the growth condition $u(x) = o(|x|^4)$ at infinity, then we have $(-\Delta)^i u > 0$ everywhere in $\mathbb{R}^{2p}$ and for all $i = 1, 2, \dots, p - 1$.*

Proof. Since u solves the PDE, we obtain $(-\Delta)^p u > 0$. In order to apply Corollary 5, we observe that the finite energy condition tells us that u is bounded from above; see [14, Corollary 14]. From this our proof follows. □

Compared with [16, Lemma 2.1], we note that the result of Corollary 6 is stronger than that of [16] since it requires $u(x) = o(|x|^2)$ at infinity in [16]. This little difference comes from the fact that the boundedness from above for the solution u did not used in the proof of [16, Lemma 2.1]; for instant, see [16, Eqn. (2.10)].

Notice that in [14, Theorem 1], the conclusion of the boundedness from above for u does not rely on the asymptotic behavior $u(x) = o(|x|^2)$; however, in order to classify solutions to the equation, we require $u(x) = o(|x|^2)$; see [14, Theorem 2].

We now turn our attention to solutions to $(-\Delta)^p u = u^q$ with $q > 0$. Note that it was proved in [16] that positive solutions to $(-\Delta)^p u = u^q$ with $q > 1$ in the whole space $\mathbb{R}^n$ have super poly-harmonic property. The interesting point of this conclusion is that no information of u at infinity is assumed. For this reason, we put here another result where the little-oh assumption in Corollary 1 is replaced by the boundedness.

Corollary 7. *Given $p \geqslant 2$, $n \geqslant 2$, and a bounded, C^{2p}-function u. If $(-\Delta)^p u$ has a sign in $\mathbb{R}^n$, then all $(-\Delta)^i u$ have the same sign as that of $(-\Delta)^p u$ in $\mathbb{R}^n$ where $i = 1, 2, \dots, p-1$.*

To apply Corollary 7 to solutions to $(-\Delta)^p u = u^q$ with $q > 1$ to get the same result as in [16], it has to have the boundedness property for solutions to the PDE. However, it turns out that, at least when the equation is sub-critical, that is when $n > 2p$ and for $1 < q < (n+2p)/(n-2p)$, positive solutions to $(-\Delta)^p u = u^q$ are indeed bounded from above; hence they are bounded. (This follows from a non-existence result when $1 < q < (n+2p)/(n-2p)$ in [16, Theorem 1.4].) Therefore, our result from Corollary 7 agrees with that in [16] in the sub-critical case. In other cases, Corollary 7 says nothing; hence to fully recover the result in [16], it is necessary to use further information from the equation itself, not from the differential inequality; see [16, Theorem 1.4].

3.2. On the asymptotic behavior $u(x)|x|^{2-2p}$ at infinity

In this section, we discuss the crucial assumption

$$\limsup_{|x|\to+\infty} \frac{(-1)^{p-1}u(x)}{|x|^{2p-2}} \leqslant 0 \tag{3.1}$$

appearing in Theorem 1. We shall construct an example to show that both the assumption (3.1) and the inequality $(-\Delta)^{p-1}u < 0$ fail; hence we cannot improve (3.1) in general.

Indeed, for $n \geqslant 3$ and any radial function $\varphi > 0$ such that $\varphi \in L^1(\mathbb{R}^n)$, then there exists some radial function w such that

$$\Delta w = \varphi$$

in $\mathbb{R}^n$ and $w = O(r^{2-n})$ at infinity. Then there exists some radial solution u_0 to the equation

$$(-\Delta)^{p-1} u_0 = w$$

in $\mathbb{R}^n$. Clearly $u_0(r) = o(r^{2p-2})$ as $r \to +\infty$ since $\lim_{r\to+\infty} w = 0$. Take

$$u(r) = u_0(r) + \sum_{0\leqslant k\leqslant p-1} a_k r^{2k},$$

we have always

$$(-\Delta)^p u = (-\Delta)^p u_0 = -\varphi < 0$$

in $\mathbb{R}^n$ and

$$(-\Delta)^{p-1} u = (-\Delta)^{p-1} u_0 + (-1)^{p-1} a_{p-1} \prod_{i=1}^{p-1} 2i(n-1+2i).$$

(Here we have used $\Delta^{p-1}(r^{2p-2}) = \prod_{i=1}^{p-1} 2i(n-1+2i)$ once.) Furthermore, we have

$$\limsup_{|x|\to+\infty} \frac{(-1)^{p-1}u(x)}{|x|^{2p-2}} = (-1)^{p-1} a_{p-1}.$$

Hence, choosing suitable $a_{p-1} \neq 0$ so that $(-\Delta)^{p-1}u > 0$, we get the desired counterexample.

It is now natural to ask whether or not the condition (3.1) is also necessary for validity of the differential inequality $(-\Delta)^{p-1}u < 0$ given the differential inequality $(-\Delta)^p u < 0$. At the moment, we do not have the answer; however, it is not hard to observe from (2.3) that

$$\liminf_{|x|\to+\infty} \frac{(-1)^{p-1}u(x)}{|x|^{2p-2+\varepsilon}} \geqslant 0 \tag{3.2}$$

for any but fixed $\varepsilon > 0$ is necessary.

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