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Error bounds in high-order Sobolev norms for POD expansions of parameterized transient temperatures

Estimations d'erreur d'ordre élevé pour la décomposition POD appliquée à l'équation de la chaleur parametrisée

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ABSTRACT

In this work, we analyze the convergence of the POD expansion for the solution to the heat conduction parameterized with respect to the thermal conductivity coefficient. We obtain error bounds for the POD approximation in high-order norms in space that assure an exponential rate of convergence, uniformly with respect to the parameter whenever it remains within a compact set of positive numbers. We present some numerical tests that confirm this theoretical accuracy.

RÉSUMÉ

On considère le problème de la conduction thermique paramétrée par rapport au coefficient de conductivité thermique, et on s'intéresse à la décomposition de sa solution par la méthode POD. Nous analysons la convergence de la solution tensorielle. Nous obtenons des bornes d'erreur pour l'approximation POD dans les normes de Sobolev d'ordre élevé, qui assurent un taux exponentiel de convergence, uniformément par rapport au paramètre si celui-ci reste dans un ensemble compact de nombres positifs. Enfin, nous présentons quelques tests numériques qui confirment nos résultats théoriques.

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La POD est une technique très connue pour son efficacité dans la construction d'approximations tensorielles de données, particulièrement quand celles-ci sont les solutions de problèmes physiques linéaires. (On peut consulter à ce sujet [5,8,10,11].) Dans [1], on démontre que la décomposition par POD de la solution de l'équation de la chaleur paramétrée par rapport à la diffusivité thermique converge en norme L^2 en espace-temps avec un taux exponentiel. Dans cette, note nous apportons une extension de ce résultat pour des normes de Sobolev d'ordre élevé. Nous considérons la solution $T = T(\gamma, z)$ de l'équation de la chaleur (5), où $\gamma \in G = [\gamma_{\min}, \gamma_{\max}] \subset (0, +\infty)$ est la diffusivité thermique et $z = (x, t)$ appartient au cylindre $Q = \Omega \times (0, \tau)$. $\Omega \subset \mathbb{R}^n$ est un domaine borné. Si le domaine Ω est de classe C^s, nous prouvons que, pour des données initiales et aux limites compatibles, T est une fonction analytique de G à valeurs dans $\mathcal{H} := L^2((0, \tau), V^s(\Omega))$, où l'espace $V^s(\Omega)$ est défini par (7) (Proposition 5.1). Pour démontrer l'analyticité, on développe T comme série de Fourier des fonctions propres du laplacien sur Ω et on démontre que la convergence est uniforme par rapport à $\gamma \in G$. Ce résultat est basé sur l'équivalence dans $V^s(\Omega)$ de la norme $\| \cdot \|_{V^s(\Omega)}$ définie par (8) et la norme usuelle dans $H^s(\Omega)$ (Lemme 5.1). L'analyticité de T comme fonction de γ est la clé pour démontrer la convergence exponentielle du développement POD tronqué, car celui-ci possède la propriété d'être la meilleure approximation en norme $L^2(G, \mathcal{H})$ de T par rapport à tous les sous-espaces de $\mathcal{H}$ à dimension donnée (4). L'erreur en norme $L^2(G, \mathcal{H})$ entre T et la série POD tronquée à M termes, T_M, est donc bornée par celle de l'approximation de T par des polynômes de Chebyshev de degré M de la variable γ. Les estimations classiques d'erreur d'interpolation de fonctions analytiques (Lemme 5.3) donnent alors le taux exponentiel de convergence (Théorème 5.2) :

$$\|T - T_M\|_{L^2(G\times Q)} \le C_\rho \, \rho^{-M}, \qquad \forall \rho : 1 < \rho < \rho_*, \tag{1}$$

où $C_\rho > 0$ est une constante qui dépend de ρ et

$$\rho_* = \frac{(\sqrt{\sigma}+1)^2}{\sigma - 1}, \quad \text{with } \sigma = \frac{\gamma_{\max}}{\gamma_{\min}}.$$

Nous présentons quelques tests numériques pour l'équation de la chaleur 2D confirmant le taux théorique de convergence exponentielle (1), indépendamment de s, en normes $L^2((0, \tau), V^s(\Omega))$, pour $s = 0, 1, 2$ (voir Fig. 1).

1. Introduction

The Proper Orthogonal Decomposition (POD) is a method for deriving low-rank approximations of parametric functions, which has been successfully used in different fields (see [5,8,10,11]).

We focus this work on quantifying the accuracy of the truncated POD expansion in approximating the solution to the parameterized heat equation with respect to the thermal conductivity coefficient. This is a continuation of work [1], where it is proved that the POD expansion converges in space–time L^2 norm with an exponential rate, with respect to the number of retained modes, and uniformly in the parameter if it remains within a compact set of positive numbers. In the present work, we extend this convergence result to higher-order norms in space, equivalent to the $H^s(\Omega)$-norm for any $s \ge 0$, provided that initial and source data are smooth enough and compatible. We also present some numerical tests for the 2D heat equation for $s = 0, 1$ and 2, and the obtained results confirm the theoretical expectations.

In this context, where the parameter domain is finite dimensional, the POD approximation error is equivalent to the Kolmogorov width of the solution manifold. This quantity is used as a benchmark for the accuracy of the approximations obtained by reduced-order methods. For example, for the reduced-basis method generated by the greedy algorithm, convergence results are established in [3] and [4], which show convergence rates similar to the Kolmogorov width of the solution manifold. Basic references on the application of reduced basis method to parameterized PDEs can be found in [7] and [12].

In this work, the POD approximation error is estimated by an appropriate approximation of the heat solution by Chebyshev polynomials in the parametric variable. This approximation requires the analytic dependence of this function (with values in high-order Sobolev spaces) with respect to the thermal diffusivity coefficient, that we prove by means of a Fourier analysis. This analyticity property also is at the basis of the proof in [4] of the rate of convergence of the approximation of the solution to parameterized elliptic equations with infinitely many parameters, and also to the parameterized heat equation. However, there are several differences between both results (for heat equation): the analysis within [4] apply to the estimation of the n-width (in mean quadratic error measured in low-order parabolic norms) of the variety of solutions to the parameterized heat equation. The n-width is bounded by means of the error of the Taylor and Legendre polynomial expansion. Here we estimate the rate of convergence of the POD expansion in mean quadratic error measured in high-order parabolic norms (that also yield the n-width, but with respect to this norm). We bound the error by means of the error of Chebyshev's polynomial expansion. The convergence rate turns out to be somewhat better (see Remark 2).

The paper is structured as follows: in Section 2 we introduce the POD method that we apply in Section 3 to obtain the POD approximation of solutions to the heat equation. Then we analyze the convergence rate of this approximation, firstly in space–time L^2 norm in Section 4 and later in higher-order norms in space in Section 5. Finally, we present numerical tests in Section 6.

2. The POD expansion on Hilbert spaces

Let $G \subset \mathbb{R}^d$ be a bounded domain, with $d \geq 1$ an integer, and let H be a Hilbert space endowed with the inner product $(\cdot,\cdot)_H$. Assume that T is a given function in the Lebesgue space $L^2(G,H)$, that we want approximate in a low-dimensional variety.

The POD method consists in choosing an orthonormal set of M functions in H, $B_M = \{v_i\}_{i=1}^M$, such that $V_M = \text{span}\{v_1,\dots,v_M\}$ has the following best-approximation property: V_M minimizes the mean distance square between T and any subspace of H of dimension M, W_M. That is, V_M is the subspace where the infimum

$$\delta_M(T)_H^2 := \inf_{\dim(W_M)=M} \int_G \min_{w\in W_M} \|w - T(\mu)\|_H^2 \, d\mu, \tag{2}$$

is attained.

The components of the POD basis, B_M, are characterized from the eigenvectors of the POD operator $A : L^2(G) \mapsto L^2(G)$ such that $A\varphi(\mu) = \int_G (T(\mu), T(\delta))_H \varphi(\delta)\, d\delta$.

If we denote T_M the truncated POD series expansion of T:

$$T_M(\mu) = \sum_{m\geq 1}^{M} (T(\mu), v_m)_H \, v_m \tag{3}$$

then T_M converges to T in $L^2(G,H)$ (see [11] for details). Note that, in terms of this approximation, the property (2) implies

$$\delta_M(T)_H = \|T - T_M\|_{L^2(G,H)} \leq \|T - S_M\|_{L^2(G,H)}, \tag{4}$$

for all $S_M \in L^2(G, W_M)$ with W_M any subspace of H of dimension M.

3. POD expansion for the heat equation

We apply the POD method to the solution to the heat equation parameterized with respect to the thermal conductivity coefficient. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $(0,\tau)$ a time interval and $Q = \Omega \times (0,\tau)$. We consider the following boundary value problem for the heat equation:

$$\begin{cases} \partial_t T - \gamma\, \Delta T = f & \text{in } Q \\ T = 0 & \text{in } \partial\Omega \times (0,\tau) \\ T(x,0) = T_0(x) & \text{in } \Omega, \end{cases} \tag{5}$$

where $\gamma > 0$ is the thermal conductivity coefficient. Is well known that if $f \in L^2(Q)$ and $T_0 \in L^2(\Omega)$ this problem admits a unique solution $T \in L^2(0,\tau; H_0^1(\Omega))$ and $\partial_t T \in L^2(0,\tau; H^{-1}(\Omega))$. We consider the POD expansion of the function $T = T(\gamma, z)$ with $z = (x,t)$ and its POD approximation (3), T_M. Our purpose is to analyze the rate of the convergence of T_M to T, considering different norms.

4. Convergence analysis in L^2 norm

Using the notation in the section above, we set $H = L^2(Q)$ and so, $L^2(G,H) = L^2(G \times Q)$. The analysis of convergence of T_M to T in $L^2(G \times Q)$ norm is realized in [1]. This work proves that the rate of convergence is exponential with respect to the number of the modes and this convergence is uniform with respect to the diffusion coefficient, whenever it remains in a compact set of positive numbers. More concretely, the following approximation result holds.

Theorem 4.1. *Assume that $G = [\gamma_{\min}, \gamma_{\max}] \subset (0,+\infty)$, $f \in L^2(\mathcal{Q})$ and $T_0 \in L^2(\Omega)$. Then the truncated POD series expansion T_M (given by (3) with $H = L^2(Q)$) of the solution T to problem (5) satisfies the following error estimate*

$$\|T - T_M\|_{L^2(G\times Q)} \leq C_\rho\, \rho^{-M}, \qquad \forall \rho : 1 < \rho < \rho_*,$$

where $C_\rho > 0$ is a constant depending on ρ and

$$\rho_* = \frac{(\sqrt{\sigma}+1)^2}{\sigma - 1}, \quad \text{with } \sigma = \frac{\gamma_{\max}}{\gamma_{\min}}. \tag{6}$$

5. Convergence analysis in higher-order norm

We aim the extension of the convergence result stated in Theorem 4.1 to higher-order Sobolev norms in space. For that, we introduce the following space: for any integer $k \geq 1$ and $s = 2k-1$ or $s = 2k$,

$$V^s(\Omega) = \Big\{ v \in H^s(\Omega) : \quad v = \Delta v = \ldots = \Delta^{k-1} v = 0 \text{ on } \partial\Omega \Big\}. \tag{7}$$

For a unified notation, we denote $V^0(\Omega) = L^2(\Omega)$. We also define

$$\|v\|_{V^s(\Omega)} = \begin{cases} \|\nabla \Delta^{k-1} v\|_{L^2(\Omega)} & \text{if } s = 2k-1 \\ \|\Delta^k v\|_{L^2(\Omega)} & \text{if } s = 2k \end{cases} \tag{8}$$

Lemma 5.1. *$V^s(\Omega)$ is a closed subspace of $H^s(\Omega)$. If Ω is of class C^s, then $\|\cdot\|_{V^s(\Omega)}$ is a norm equivalent to the usual norm in $H^s(\Omega)$.*

Proof. We use the regularity of the weak solution for the problem: *find v, solution to $-\Delta v = f$ in Ω with $v = 0$ on $\partial\Omega$* stated in [2], p. 181, and reads if Ω is of class C^s and $f \in H^{s-2}(\Omega)$, then $v \in H^s(\Omega)$ and

$$\|v\|_{H^s(\Omega)} \leq C \, \|f\|_{H^{s-2}(\Omega)}. \tag{9}$$

For $k = 1$: $V^1(\Omega) = H_0^1(\Omega)$, the $\|\cdot\|_{V^1(\Omega)}$ norm and the $H^1(\Omega)$ norm are equivalent. Next, $V^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$, where the result holds, applying (9) with $s = 2$. Let us suppose that the result holds for any $k \geq 2$. We prove it now for $k+1$: let $v \in V^s(\Omega)$. Then $f = \Delta v \in H^{s-2}(\Omega)$ and v verifies (9). Note that $\Delta^n f = \Delta^{n+1} v$ for $0 \leq n \leq k-1$, thus $f = \Delta f = \ldots = \Delta^{k-1} f = 0$ on $\partial\Omega$ and $f \in V^{s-2}(\Omega)$. We now use the induction hypothesis: for $s = 2k+1$, $s-2 = 2k-1$ and f verifies $\|f\|_{H^{s-2}(\Omega)} \leq C \, \|\nabla \Delta^{k-1} f\|_{L^2(\Omega)} = C \, \|\nabla \Delta^k v\|_{L^2(\Omega)} = C \, \|v\|_{V^s(\Omega)}$. Also, for $s = 2k+2$, $s-2 = 2k$ and f verifies $\|f\|_{H^{s-2}(\Omega)} \leq C \, \|\Delta^k f\|_{L^2(\Omega)} = C \, \|\Delta^{k+1} v\|_{L^2(\Omega)} = C \, \|v\|_{V^s(\Omega)}$. Combining these estimates with (9), we obtain

$$\|v\|_{H^s(\Omega)} \leq C \, \|v\|_{V^s(\Omega)}, \quad \forall\, v \in V^s(\Omega). \qquad \square$$

We study the convergence of the POD expansion in the space $\mathcal{H} := L^2((0,\tau), V^s(\Omega))$. The following result holds.

Theorem 5.2. *Assume that Ω is of class C^s, for some integer $s \geq 1$. Let $G = [\gamma_{\min}, \gamma_{\max}] \subset (0, +\infty)$, $f \in L^2((0,\tau), V^{s-1}(\Omega))$ and $T_0 \in V^{s-1}(\Omega)$. Then the truncated POD series expansion T_M (given by (3) with $H = \mathcal{H}$) of the solution T to the problem (5) satisfies the following error estimate*

$$\|T - T_M\|_{L^2(G,\mathcal{H})} \leq C_\rho \; \rho^{-M}, \; \forall\, \rho : 1 < \rho < \rho_*, \tag{10}$$

where $C_\rho > 0$ is a constant depending on ρ and ρ_* given in (6).

Remark 1. Note that when $s = 1$ in Theorem 5.2, under the same hypotheses about the data as in Theorem 4.1, the estimate of the POD error obtained in (10) applies to $L^2(G, L^2(0,\tau; H_0^1(\Omega)))$ instead of $L^2(G \times Q)$, as in Theorem 4.1.

The proof of Theorem 5.2 follows from the analyticity of the function T upon γ, which we prove below.

Proposition 5.1. *The function $\gamma \in (0, +\infty) \mapsto T(\gamma) \in \mathcal{H}$ is analytic.*

Proof. We consider the Dirichlet eigenvalue problem for the Laplace operator:

$$\begin{cases} -\Delta e_n &= \alpha_n \, e_n \quad \text{in } \Omega, \\ \quad e_n &= 0 \qquad\quad \text{on } \partial\Omega. \end{cases}$$

The spectrum of this operator is a non-decreasing sequence of positive numbers, $\{\alpha_n\}_{n\geq 1}$ such that $\lim\limits_{n\to\infty} \alpha_n = +\infty$. Also, the sequence $\{e_n\}_{n\geq 1}$, with e_n a eigenfunction associated with α_n, is an orthonormal basis of $L^2(\Omega)$.

The sequence $\{e_n\}_{n\geq 1}$ is also an orthogonal basis of $V^s(\Omega)$, with

$$(e_n, e_m)_{V^s(\Omega)} = \alpha_n^s \, \delta_{nm}, \quad \|e_n\|^2_{V^s(\Omega)} = \alpha_n^s. \tag{11}$$

We consider the Fourier decomposition of the data of problem (5) T_0 and f:

$$T_0(x) = \sum_{n\geq 1} a_n \, e_n(x), \quad f(x,t) = \sum_{n\geq 1} f_n(t) \, e_n(x),$$

with $a_n = (T_0, e_n)_{L^2(\Omega)}$ and $f_n(t) = (f(\cdot,t), e_n)_{L^2(\Omega)}$. Due to the assumed regularity of T_0 and f, these series are convergent in $V^{s-1}(\Omega)$ and $L^2((0,\tau), V^{s-1}(\Omega))$, respectively, and from (11) it holds

$$\|T_0\|^2_{V^{s-1}(\Omega)} = \sum_{n\geq 1} |a_n|^2\, \alpha_n^{s-1}, \tag{12}$$

$$\|f\|^2_{L^2((0,\tau),V^{s-1}(\Omega))} = \sum_{n\geq 1} \|f_n\|^2_{L^2(0,\tau)}\, \alpha_n^{s-1}. \tag{13}$$

As a consequence, the solution to problem (5) can be expressed as

$$T(\gamma,x,t) = \sum_{n\geq 1} T_n(\gamma,t)\, e_n(x), \text{ with } T_n(\gamma,t) = a_n\, \mathrm{e}^{-\gamma\,\alpha_n\, t} + \int_0^t f_n(s)\, \mathrm{e}^{-\gamma\alpha_n(t-s)}\, \mathrm{d}s. \tag{14}$$

We analyze the convergence of the series (14). For that, we check out the two terms coming from the two sums in T_n. On the one hand,

$$\sum_{n\geq 1} \|a_n\, \mathrm{e}^{-\gamma\,\alpha_n\, t}\, e_n\|^2_{\mathcal{H}} = \sum_{n\geq 1} |a_n|^2\, \|\mathrm{e}^{-\gamma\,\alpha_n\, t}\|^2_{L^2(0,\tau)}\, \|e_n\|^2_{V^s(\Omega)}.$$

Using the fact that $\|\mathrm{e}^{-\gamma\,\alpha_n\, t}\|^2_{L^2(0,\tau)} = \int_0^\tau \mathrm{e}^{-2\gamma\,\alpha_n\, t}\, \mathrm{d}t = \frac{1}{2\,\gamma\,\alpha_n}(1-\mathrm{e}^{-2\,\gamma\,\alpha_n\,\tau}) < \frac{1}{2\,\gamma\,\alpha_n}$ and (11), we have

$$\sum_{n\geq 1} \|a_n\, \mathrm{e}^{-\gamma\,\alpha_n\, t}\, e_n\|^2_{\mathcal{H}} \leq \frac{1}{2\,\gamma} \sum_{n\geq 1} |a_n|^2\, \alpha_n^{s-1} = \frac{1}{2\,\gamma}\, \|T_0\|^2_{V^{s-1}(\Omega)}, \tag{15}$$

taking into account (12).

By another hand, denoting $F_n(\gamma,t) = \int_0^t f_n(s)\, \mathrm{e}^{-\gamma\alpha_n(t-s)}\, \mathrm{d}s$,

$$\sum_{n\geq 1} \|F_n(\gamma)\, e_n\|^2_{\mathcal{H}} = \sum_{n\geq 1} \|F_n(\gamma)\|^2_{L^2(0,\tau)} \|e_n\|^2_{V^s(\Omega)}.$$

As $|F_n(\gamma,t)|^2 \leq \|f_n\|^2_{L^2(0,\tau)} \int_0^t \mathrm{e}^{-2\gamma\alpha_n(t-s)}\, \mathrm{d}s = \frac{1}{2\gamma\alpha_n} \|f_n\|^2_{L^2(0,\tau)}\, (1-\mathrm{e}^{-2\gamma\alpha_n t})$, then $\|F_n(\gamma)\|^2_{L^2(0,\tau)} \leq \frac{\tau}{2\gamma\alpha_n} \|f_n\|^2_{L^2(0,\tau)}$. Thus,

$$\sum_{n\geq 1} \|F_n(\gamma)\, e_n\|^2_{\mathcal{H}} \leq \frac{\tau}{2\gamma} \sum_{n\geq 1} \|f_n\|^2_{L^2(0,\tau)}\, \alpha_n^{s-1} = \frac{\tau}{2\gamma} \|f\|^2_{L^2((0,\tau),V^{s-1}(\Omega))}, \tag{16}$$

taking into account (11) and (13).

From the estimates (15) and (16) we deduce that the series (14) is uniformly convergent in any compact sub-sets of $(0,+\infty)$. Also, each of the terms in the series is an analytic function in $(0,+\infty)$ (see [1] for details), thus the limit is also an analytic function in $(0,+\infty)$. □

To prove Theorem 5.2, we use the following preliminary result (cf. [9]).

Lemma 5.3. *Let $\mathcal{E}_\rho = \{z \in \mathbb{C}$ such that $|z-1|+|z+1| \leq \rho+\rho^{-1}\}$ for some $\rho > 1$. Let $F : \mathcal{E}_\rho \to H$ be an analytic and bounded function in $\mathcal{E}_\rho$, with H a Hilbert space. For a given integer number $M \geq 0$, let F_M be the truncated Chebyshev polynomial series expansion of F of degree M with coefficients in H. Then it holds*

$$\max_{z\in[-1,1]} \|F(z) - F_M(z)\|_H \leq C_\rho\, \rho^{-M}, \tag{17}$$

with $C_\rho = \frac{2}{\rho-1} \|F\|_{L^\infty(\mathcal{E}_\rho)}$.

Lemma 5.4. *There exists a polynomial with degree smaller than or equal to M, $S_M : G \to \mathcal{H}$ such that:*

$$\max_{\gamma\in G} \|T(\gamma) - S_M(\gamma)\|_{\mathcal{H}} \leq C_\rho\, \rho^{-M},\ \forall \rho : 1 < \rho < \rho_*, \tag{18}$$

with ρ^* given in (6).

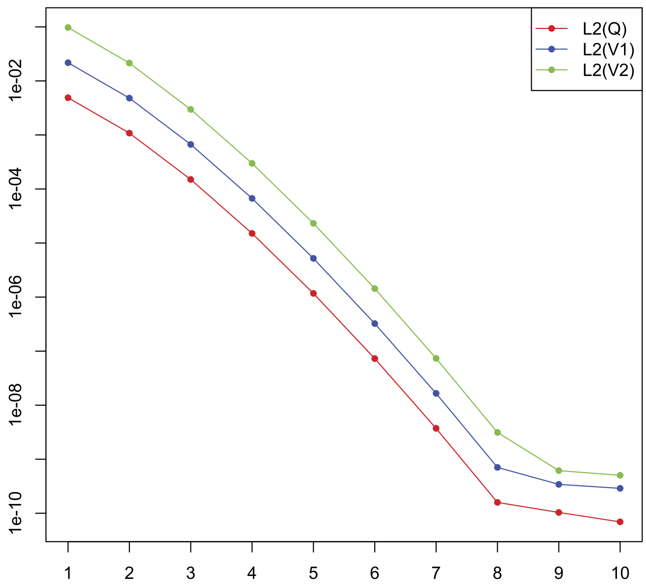

Fig. 1. POD error in $L^2(G, H)$ norm with $H = L^2(Q)$, $H = L^2(0, 1; V^1(0, 1))$ and $H = L^2(0, 1; V^2(0, 1))$. x-Axis: number of modes. y-Axis: error in $L^2(G, H)$ norm, in logarithmic coordinates.

Proof. We consider the affine transformation g that transforms $[-1, 1]$ into G and the function $\tilde{T}(\tilde{\gamma}) = T(g(\tilde{\gamma}))$. From Proposition 5.1, T is analytic in $(0, +\infty)$, and hence $\tilde{T}$ is analytic in a ellipse $\tilde{\mathcal{E}}_\rho$, with $\rho < \rho^*$. So, we can apply Lemma 5.3 and obtain (17) for the function $\tilde{T}$ in $\mathcal{H}$:

$$\max_{\tilde{\gamma}\in[-1,1]} \|\tilde{T}(\tilde{\gamma}) - \tilde{S}_M(\tilde{\gamma})\|_{\mathcal{H}} \le C_\rho\, \rho^{-M},$$

where $\tilde{S}_M$ is the truncated Chebyshev polynomial series expansion of $\tilde{T}$ of degree M. Now, we construct the polynomial $S_M(\gamma) = \tilde{S}_M(g^{-1}(\gamma))$ and it holds

$$\max_{\gamma\in G} \|T(\gamma) - S_M(\gamma)\|_{\mathcal{H}} = \max_{\tilde{\gamma}\in[-1,1]} \|\tilde{T}(\tilde{\gamma}) - \tilde{T}_M(\tilde{\gamma})\|_{\mathcal{H}}.$$

Then, (18) follows from the error estimate above. □

Now we are in a position to conclude about the proof of Theorem 5.2.

Proof of Theorem 5.2. Let S_M be the polynomial constructed in Lemma 5.4. From the best-approximation property of T_M, (4), it holds

$$\|T - T_M\|_{L^2(G,\mathcal{H})} \le \|T - S_M\|_{L^2(G,\mathcal{H})} \le |G|^{1/2} \max_{\gamma\in G} \|T(\gamma) - S_M(\gamma)\|_{\mathcal{H}}.$$

Finally, from (18) it follows (10). □

Remark 2. Note that the convergence rate (6) established by Theorem 5.2 is determined by the Chebyshev polynomial S_M derived in Lemma 5.4. This approximation in estimating the POD error improves the convergence rate obtained in [4] (estimates (3.175) and (3.185)): $\rho_* = \dfrac{\sigma + 1}{\sigma - 1}$, obtained by comparison with the truncated Taylor expansion.

6. Numerical tests

We carry out a numerical investigation in order to illustrate the convergence rates predicted by Theorem 5.2. We consider the problem (5) in the domain $Q = (0, 1)^2 \times (0, 1)$ with data $T_0(x, y) = x(x - 1)y(y - 1)$ and $f(x, y, t) = x(x - 1)y(y - 1)t$. We set the thermal conductivities interval to $G = [1, 100]$. The heat problem is discretized by a semi-implicit Euler in time and P3 Finite Element method in space with time step $\Delta t = 10^{-2}$, mesh size $h = 0.28$, and 22801 degrees of freedom. The calculation of the correlation matrix corresponding to the kernel of operator A, $K(\gamma_i, \gamma_j) = (T(\gamma_i), T(\gamma_j))_{\mathcal{H}}$, is performed using the trapezoid quadrature formula. We have used the LAPACK package to compute the eigenvalues and eigenvectors of matrix K. All computations have been performed with the free software FreeFem++ (cf. [6]). In Fig. 1, we display the

POD error $\|T - T_M\|_{L^2(G,\mathcal{H})}$ (in logarithmic coordinates) in terms of the number of modes in the approximation T_M for the solution T to the problem proposed. We have considered $\mathcal{H} = L^2(0, 1; V^s(0, 1))$ for $s = 0$, 1 and 2. In all cases, we obtain an exponential convergence rate, with approximately the same slope in logarithmic coordinates, according to the theoretical estimate (10), where ρ^* does not depend on s.

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