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Besov continuity of pseudo-differential operators on compact Lie groups revisited

Continuité de Besov des opérateurs pseudo-différentiels sur les groupes de Lie compacts revisitée

Duván Cardona

Mathematics Department, Universidad de los Andes, Carrera 1 No. 18a 10, Bogotá, Colombia

A R T I C L E I N F O

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A B S T R A C T

In this note we present some results on the action of global pseudo-differential operators on Besov spaces on compact Lie groups.

R É S U M É

Dans cette note, nous présentons quelques résultats sur l'action des opérateurs pseudo-différentiels globaux sur les espaces de Besov des groupes de Lie compacts.

Version française abrégée

Dans cet article, nous présentons plusieurs résultats sur la continuité dans les espaces de Besov des opérateurs pseudo-différentiels sur les groupes de Lie compacts. Plus précisé, nous présentons un analogue pour sur les espaces de Besov des groupes de Lie compacts de quelques théorèmes classiques sur la continuité des multiplicateurs de Fourier et des opérateurs pseudo-différentiels sur les espaces $L^p(\mathbb{R}^n)$. Les résultats que nous généralisons ici sont les suivants : le théorème de Hörmander–Mihlin [12], la condition de Hörmander sur la continuité L^p–L^q des multiplicateurs de Fourier [12] et un résultat de Fefferman [8] sur la continuité L^p des opérateurs pseudo-différentiels associés classes de Hörmander sur $\mathbb{R}^n$. Une des motivations pour ces travaux est que chacun des résultats mentionnés ci-dessus a été précédemment généralisé au cadre des espaces L^p sur les groupes de Lie compacts – voir [1], [7], [24] et [10].

1. Introduction

In this note, we investigate the mapping properties of pseudo-differential operators on compact Lie groups acting on their corresponding Besov spaces. In order to illustrate our results, we recall some fundamental results in harmonic analysis

E-mail addresses: d.cardona@uniandes.edu.co, duvanc306@gmail.com.

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on the mapping properties of multipliers and pseudo-differential operators on L^p-spaces. It was proved by Marcinkiewicz in the following multiplier theorem [14]: if $(\sigma(\xi))_{\xi\in\mathbb{Z}}$ is a sequence satisfying

$$\sup_{\xi\in\mathbb{Z}}|\sigma(\xi)|<\infty, \qquad \sup_{j\in\mathbb{N}_0}\sum_{2^j\leq|\xi|^2<2^{j+1}}|\sigma(\xi+1)-\sigma(\xi)|<\infty, \tag{1}$$

then the periodic Fourier multiplier T defined by $Tf:=\mathscr{F}^{-1}[\sigma(\xi)\mathscr{F}(f)]$ extends to a bounded operator on $L^p(\mathbb{T})$, $1<p<\infty$. Here $\mathbb{T}:=[0,1]$ is the one-dimensional torus and $\mathscr{F}$ is the Fourier transform on the torus. Later, Mihlin in [15] established the Fourier multiplier theorem, which states that every function $\sigma(\cdot)$ of class $C^\kappa(\mathbb{R}^n\setminus\{0\})$, $\kappa=[\frac{n}{2}]+1$, satisfying

$$|\partial_\xi^\alpha\sigma(\xi)|\leq C_\alpha|\xi|^{-|\alpha|},\ \ |\alpha|\leq\kappa, \tag{2}$$

generates a Fourier multiplier T defined by $Tf:=\mathscr{F}^{-1}[\sigma(\xi)\mathscr{F}(f)]$, which extends to a bounded operator on $L^p(\mathbb{R}^n)$, $1<p<\infty$. Here $\mathscr{F}$ is the Euclidean Fourier transform. Hörmander improved the Mihlin theorem in [12] by imposing the following condition:

$$\sup_{r>0}\|\sigma(r\cdot)\|_{H^s_{loc}(\mathbb{R}^n)}<\infty, \tag{3}$$

where H^s_{loc} is the standard local Sobolev space of order s and, in this case, $s>\frac{n}{2}$ is fixed. The case of L^p–L^q-multiplier theorems also is a classical problem. In $\mathbb{R}^n$, Hörmander in [12] shows that if σ is a function on $\mathbb{R}^n$ satisfying

$$\sup_{r>0}r\cdot[\mu\{\xi\in\mathbb{R}^n:|\sigma(\xi)|>r\}]^{\frac{1}{p}-\frac{1}{q}}<\infty,\ \ 1<p\leq2\leq q<\infty, \tag{4}$$

then the Fourier multiplier T associated with σ extends to a bounded operator from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. The general case of pseudo-differential operators is very complicated. For a function σ on $\mathbb{R}^{2n}$, the corresponding pseudo-differential operator T_σ is defined on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ by

$$T_\sigma f(x)=\int_{\mathbb{R}^n}e^{i2\pi\langle x,\xi\rangle}\sigma(x,\xi)(\mathscr{F}f)(\xi)d\xi,\ \ (\mathscr{F}f)(\xi):=\widehat{f}(\xi)=\int_{\mathbb{R}^n}e^{-i2\pi\langle x,\xi\rangle}f(x)dx, \tag{5}$$

where $\mathscr{F}f$ is the Fourier transform of $f\in\mathscr{S}(\mathbb{R}^n)$. A multiplier is, roughly speaking, a special case of pseudo-differential operator where the corresponding symbol depends only on ξ. If we denote by $S^m_{\rho,\delta}(\mathbb{R}^{2n})$, $0\leq\rho,\delta\leq1$, to the class of functions σ on $\mathbb{R}^{2n}$ satisfying

$$|\partial_x^\beta\partial_\xi^\alpha\sigma(x,\xi)|\leq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|},\ \ \alpha,\beta\in\mathbb{N}^n, \tag{6}$$

then we can announce a classical theorem due to Fefferman (cf. [8]) known to be one of the more sharp theorems on the boundedness of pseudo-differential operators: for general $1<p<\infty$ and $m\leq-m_p=-n(1-\rho)|\frac{1}{p}-\frac{1}{2}|$, a pseudo-differential operator with symbol $\sigma\in S^{-m_p}_{\rho,\delta}(\mathbb{R}^{2n})$ is L^p-bounded. An important and recent problem is the generalization of the theorems above in the setting of compact Lie groups, when the Fourier multipliers and the pseudo-differential operators are considered in different function spaces. It is important to mention that this problem has been solved for L^p-spaces in the recent works of Akylzhanov and Ruzhansky [1], Delgado and Ruzhansky [7], Ruzhansky and Wirth [24], and Fischer [10]. A version of the Fefferman theorem for the periodic case of the torus has been studied in [6].

The results of this paper present a version of the results mentioned above in the setting of Besov spaces on compact Lie groups. For properties in Besov spaces of pseudo-differential operators associated with Hörmander classes on $\mathbb{R}^n$, we refer to Bordaud [3] and Gibbons [11].

2. Besov spaces, pseudo-differential operators on compact Lie groups and L^p-boundedness

In this section, we present some basics on the theory of pseudo-differential operators on compact Lie groups. The preliminaries of the theory of global pseudo-differential operators can be found in [18] and [20]. Let us assume that G is a compact Lie group and denote by $\widehat{G}$ its unitary dual, i.e. the set of equivalence classes of all strongly continuous irreducible unitary representations of G. If A is a linear operator from $C^\infty(G)$ into $C^\infty(G)$ and $\xi:G\to U(H_\xi)$ denotes an irreducible unitary representation, we can associate with A a matrix-valued symbol $\sigma_A(x,\xi)\in\mathbb{C}^{d_\xi\times d_\xi}$ (this correspondence gives rise to the global calculus of Ruzhansky and Turunen [18]) satisfying

$$Af(x):=\sum_{[\xi]\in\widehat{G}}d_\xi\text{Tr}[\xi(x)\sigma_A(x,\xi)(\mathscr{F}f)(\xi)],\ \ (\mathscr{F}f)(\xi):=\widehat{f}(\xi)=\int_G f(x)\xi(x)^*dx\in\mathbb{C}^{d_\xi\times d_\xi}. \tag{7}$$

Here, for each class $[\xi]$ we pick just one representative $\xi\in[\xi]$, $d_\xi=\dim(H_\xi)$ and $(\mathscr{F}f)(\xi)$ is the Fourier transform at ξ. We end this section with the following formulation of Besov spaces on compact Lie groups, characterized in terms of representations in [17] and [16].

Definition 2.1. Let $r \in \mathbb{R}$, $0 \le q < \infty$ and $0 < p \le \infty$. If f is a measurable function on G, we say that $f \in B^r_{p,q}(G)$ if the function f satisfies

$$\|f\|_{B^r_{p,q}} := \left(\sum_{m=0}^{\infty} 2^{mrq} \| \sum_{2^m \le \langle \xi \rangle < 2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\widehat{f}(\xi)] \|^q_{L^p(G)} \right)^{\frac{1}{q}} < \infty. \tag{8}$$

If $q = \infty$, $B^r_{p,\infty}(G)$ consists of those functions f satisfying

$$\|f\|_{B^r_{p,\infty}} := \sup_{m \in \mathbb{N}} 2^{mr} \| \sum_{2^m \le \langle \xi \rangle < 2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\widehat{f}(\xi)] \|_{L^p(G)} < \infty. \tag{9}$$

Applications of the matrix-valued calculus of Ruzhansky and Turunen can be found in [19,21,23]. The work of Fischer [9] includes a summary of essential properties on the Hörmander classes including a version of the Calderón–Vaillancourt theorem. The problem of classification of global operators in L^p-spaces has been treated recently. The main results can be summarized in the following theorem.

Theorem 2.1. *If G is a compact Lie group and n is its dimension, $\varkappa$ is the less integer larger than $\frac{n}{2}$ and $l := [n/p] + 1$, under one of the following conditions:*

1. $\|\partial_x^\beta \mathbb{D}^\alpha \sigma_A(x,\xi)\|_{\mathrm{op}} \le C_\alpha \langle \xi \rangle^{-|\alpha|}$, *for all* $|\alpha| \le \varkappa$, $|\beta| \le l$ *and* $[\xi] \in \widehat{G}$,
2. $\|\mathbb{D}_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\|_{\mathrm{op}} \le C_{\alpha,\beta} \langle \xi \rangle^{-m-\rho|\alpha|+\delta|\beta|}$, $|\alpha| \le \varkappa$, $|\beta| \le l$, $m \ge \varkappa(1-\rho)|\frac{1}{p} - \frac{1}{2}| + \delta l$,
3. $\|\mathbb{D}_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\|_{\mathrm{op}} \le C_{\alpha,\beta} \langle \xi \rangle^{-\nu-\rho|\alpha|+\delta|\beta|}$, $\alpha \in \mathbb{N}^n$, $|\beta| \le l$, $0 \le \nu < \frac{n}{2}(1-\rho)$, $|\frac{1}{p} - \frac{1}{2}| \le \frac{\nu}{n}(1-\rho)^{-1}$,
4. $\|\sigma_A\|_{\Sigma_s} := \sup_{[\xi]\in\widehat{G}} [\|\sigma_A(\xi)\|_{\mathrm{op}} + \|\sigma_A(\xi)\eta(r^{-2}\mathcal{L}_G)\|_{\dot{H}^s(\widehat{G})}] < \infty$,

the global operator $A \equiv T_a$ is bounded on $L^p(G)$, $1 < p < \infty$.

It is important to mention that the last condition above is a discrete analogue of the classical Hörmander–Mihlin theorem proved in [12]. *Also, the condition*

5. $\sup_{s>0} s[\mu\{[\xi] \in \widehat{G} : \|\partial_x^\beta \sigma_A(x,\xi)\|_{\mathrm{op}} > s\}]^{\frac{1}{p_1} - \frac{1}{p_2}} < \infty$, $1 < p_1 \le 2 \le p_2 < \infty$, $|\beta| \le l$,

implies the L^{p_1}–L^{p_2}-boundedness of A. The proof of the assertions above can be found in the papers [2,7,10,22,24]. *Finally, it is important to mention that, as a consequence of the Sharp Garding inequality for global operators, it was proved in [19] that under the condition*

6. $A \in \Psi^1(G)$ with $\sigma_A(x,\xi) \ge 0$,

the linear operator A extends to a bounded operator from $L^2(G)$ to $L^2(G)$.

In the next section, we will use the last results on L^p-estimates to prove our results on Besov boundedness for global pseudo-differential operators. In this sense, the following results can be thought of as analogues of the L^p-results above in the framework of Besov spaces. The main tool is the following result (see Cardona [4]).

Theorem 2.2. *Let G be a compact Lie group and A be a Fourier multiplier. If A is bounded from $L^{p_1}(G)$ into $L^{p_2}(G)$, then A extends to a bounded operator from $B^r_{p_1,q}(G)$ into $B^r_{p_2,q}(G)$, for all $r \in \mathbb{R}$ and $0 < q \le \infty$. If A is of weak type $(1,1)$, then A extends to a bounded operator from $B^r_{1,q}(G)$, into $B^r_{p,q}(G)$ for all $r \in \mathbb{R}$, $0 < q \le \infty$ and $0 < p < 1$.*

3. Pseudo-differential operators on Besov spaces, the case of compact Lie groups

In this section, we present our main results. On $\mathbb{R}^n$, pseudo-differential operators with symbols in $S^0_{1,0}(\mathbb{R}^{2n})$ are known to be bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$ (see [13]). Similar results on Besov and L^p-boundedness can be obtained for the case of compact Lie groups. Here, $\mathbb{D}_\xi$ is the difference operator introduced by Ruzhansky and Turunen in [24].

Theorem 3.1. *If G is a compact Lie group and n is its dimension, $\varkappa$ is the less integer larger than $\frac{n}{2}$ and $l := [n/p] + 1$, under one of the following conditions:*

1′. $\|\mathbb{D}_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\|_{op} \leq C_\alpha \langle\xi\rangle^{-|\alpha|}$, *for all* $|\alpha| \leq \varkappa$, $|\beta| \leq l$ *and* $[\xi] \in \widehat{G}$,
2′. $\|\mathbb{D}_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\|_{op} \leq C_{\alpha,\beta} \langle\xi\rangle^{-\nu-\rho|\alpha|}$, $\alpha \in \mathbb{N}^n$, $|\beta| \leq l$, $|\frac{1}{p}-\frac{1}{2}| \leq \frac{\nu}{n}(1-\rho)^{-1}$, $0 \leq \nu < \frac{n}{2}(1-\rho)$,
3′. $\|\mathbb{D}_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\|_{op} \leq C_{\alpha,\beta} \langle\xi\rangle^{-m-\rho|\alpha|+\delta|\beta|}$, $|\alpha| \leq \varkappa$, $|\beta| \leq l$, $m \geq \varkappa(1-\rho)|\frac{1}{p}-\frac{1}{2}|+\delta l$,

the corresponding pseudo-differential operator A extends to a bounded operator from $B^r_{p,q}(G)$ *into* $B^r_{p,q}(G)$ *for all* $r \in \mathbb{R}$, $1 < p < \infty$ *and* $0 < q \leq \infty$. *In particular, if* $\sigma_A(x,\xi) \geq 0$ *and* $A \in \Psi^1(G)$, *then A is bounded on* $B^r_{2,q}(G)$ *for all* $r \in \mathbb{R}$ *and* $0 < q \leq \infty$.

The following is an L^1–L^p-boundedness theorem of global Fourier multipliers for $0 < p < 1$.

Theorem 3.2. *If G is a compact Lie group and n is its dimension,* $\varkappa$ *is the less integer larger than* $\frac{n}{2}$, *then under the condition:*

4′. $\|\mathbb{D}^\alpha \sigma(\xi)\|_{op} \leq C_\alpha \langle\xi\rangle^{-|\alpha|}$, *for all* $|\alpha| \leq \varkappa$, *and* $[\xi] \in \widehat{G}$

the corresponding multiplier A is bounded from $L^1(G)$ *into* $L^p(G)$ *for* $0 < p < 1$. *Also, A is bounded from* $B^r_{1,q}(G)$ *into* $B^r_{p,q}(G)$ *for all* $r \in \mathbb{R}$, $0 < p < 1$ *and* $0 < q \leq \infty$.

Also, we have some conditions of type Hörmander and Mihlin–Hörmander (inspired by the recent work of V. Fischer [8] on multipliers on compact Lie groups and by a paper of the author with M. Ruzhansky [5] where Besov spaces and multipliers were considered in the case of graded Lie groups).

Theorem 3.3. *Let G be a compact Lie group and n its dimension. Under the following condition*

5′. $\sup_{s>0} s[\mu\{[\xi] \in \widehat{G} : \|\partial_x^\beta \sigma_A(x,\xi)\|_{op} > s\}]^{\frac{1}{p_1}-\frac{1}{p_2}} < \infty$, $1 < p_1 \leq 2 \leq p_2 < \infty$, $|\beta| \leq l' := [\frac{n}{p_1}]+1$.

The operator A is bounded from $B^r_{p_1,q}(G)$ *into* $B^r_{p_2,q}(G)$ *for all* $r \in \mathbb{R}$ *and* $0 < q < \infty$. *Also, if we consider a symbol* σ_A *satisfying*

6′. $\max_{|\alpha| \leq l}[\sup_{z \in G, [\xi] \in \widehat{G}} \|(\partial_x^\alpha a)(x,\xi)|_{x=z}\|_{op} + \|(\partial_x^\alpha a)(x,\cdot)|_{x=z}\eta(r^{-2}\mathcal{L}_G)\|_{\dot{H}^s(\widehat{G})}] < \infty$,

the corresponding operator A is bounded on $B^r_{p,q}(G)$ *for all* $r \in \mathbb{R}$, $1 < p < \infty$ *and* $0 < q \leq \infty$.

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