Harmonic analysis

Characterization of Lipschitz spaces via commutators of the Hardy–Littlewood maximal function ⋆

Caractérisation des espaces de Lipschitz via les commutateurs de l’opérateur maximal de Hardy–Littlewood

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A B S T R A C T

Let M be the Hardy–Littlewood maximal function and b be a locally integrable function. Denote by $M_b$ and $[b, M]$ the maximal commutator and the (nonlinear) commutator of M with b. In this paper, the author considers the boundedness of $M_b$ and $[b, M]$ on Lebesgue spaces and Morrey spaces when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given.

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1. Introduction and results

Let T be the classical singular integral operator, the commutator $[b, T]$ generated by T and a suitable function b is given by

$$[b, T] f = b T(f) - T(b f).$$

(1.1)

A well-known result due to Coifman, Rochberg and Weiss [6] (see also [13]) states that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 1978, Janson [13] gave some characterizations of the Lipschitz spaces...
space \( \hat{\Lambda}_\beta(\mathbb{R}^n) \) (see Definition 1.1 below) via commutator \([b, T]\) and proved that \( b \in \hat{\Lambda}_\beta(\mathbb{R}^n)(0 < \beta < 1) \) if and only if \([b, T]\) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) where \( 1 < p < n/\beta \) and \( 1/p − 1/q = \beta/n \) (see also Paluszyński [18]).

For a locally integrable function \( f \), the Hardy–Littlewood maximal function \( M \) is given by

\[
M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy,
\]

the maximal commutator of \( M \) with a locally integrable function \( b \) is defined by

\[
M_b(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) − b(y)||f(y)|\,dy,
\]

where the supremum is taken over all cubes \( Q \subset \mathbb{R}^n \) containing \( x \).

The mapping properties of the maximal commutator \( M_b \) have been studied intensively by many authors. See [3,9,11,12, 20,21] and [25] for instance. The following result is proved by García-Cuerva et al. [9]. See also [20] and [21].

**Theorem A** ([9]). Let \( b \) be a locally integrable function and \( 1 < p < \infty \). Then the maximal commutator \( M_b \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) if and only if \( b \in \text{BMO}(\mathbb{R}^n) \).

The first part of this paper is to study the boundedness of \( M_b \) when the symbol \( b \) belongs to a Lipschitz space. Some characterizations of the Lipschitz space via such commutator are given.

**Definition 1.1.** Let \( 0 < \beta < 1 \), we say a function \( b \) belongs to the Lipschitz space \( \hat{\Lambda}_\beta(\mathbb{R}^n) \) if there exists a constant \( C \) such that for all \( x, y \in \mathbb{R}^n \),

\[
|b(x) − b(y)| \leq C|x − y|^\beta.
\]

The smallest such constant \( C \) is called the \( \hat{\Lambda}_\beta \) norm of \( b \) and is denoted by \( \|b\|_{\hat{\Lambda}_\beta} \).

Our first result can be stated as follows.

**Theorem 1.1.** Let \( b \) be a locally integrable function and \( 0 < \beta < 1 \), then the following statements are equivalent:

1. \( b \in \hat{\Lambda}_\beta(\mathbb{R}^n) \);
2. \( M_b \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) for all \( p, q \) with \( 1 < p < n/\beta \) and \( 1/q = 1/p − \beta/n \);
3. \( M_b \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) for some \( p, q \) with \( 1 < p < n/\beta \) and \( 1/q = 1/p − \beta/n \);
4. \( M_b \) satisfies the weak-type \( (1, n/(n − \beta)) \) estimates, namely, there exists a positive constant \( C \) such that for all \( \lambda > 0 \),

\[
|\{x \in \mathbb{R}^n : M_b(f)(x) > \lambda\}| \leq C(\lambda+1)^{−1}\|f\|_{L^1(\mathbb{R}^n)}^{n/(n−\beta)};
\]

5. \( M_b \) is bounded from \( L^{(n/\beta)}(\mathbb{R}^n) \) to \( L^\infty(\mathbb{R}^n) \).

Morrey spaces were originally introduced by Morrey in [17] to study the local behavior of solutions to second-order elliptic partial differential equations. Many classical operators of harmonic analysis were studied in Morrey-type spaces during the last decades. We refer the readers to Adams [2] and references therein.

**Definition 1.2.** Let \( 1 \leq p < \infty \) and \( 0 \leq \lambda \leq n \). The classical Morrey space is defined by

\[
L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}} < \infty \right\},
\]

where

\[
\|f\|_{L^{p,\lambda}} := \sup_Q \left( \frac{1}{|Q|^{\lambda/n}} \int_Q |f(x)|^p \,dx \right)^{1/p}.
\]

It is well known that if \( 1 \leq p < \infty \) then \( L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \) and \( L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) \).

**Theorem 1.2.** Let \( b \) be a locally integrable function and \( 0 < \beta < 1 \). Suppose that \( 1 < p < n/\beta, 0 < \lambda < n − \beta p \) and \( 1/q = 1/p − \beta/(n − \lambda) \). Then \( b \in \hat{\Lambda}_\beta(\mathbb{R}^n) \) if and only if \( M_b \) is bounded from \( L^{p,\lambda}(\mathbb{R}^n) \) to \( L^{q,\lambda}(\mathbb{R}^n) \).

**Theorem 1.3.** Let \( b \) be a locally integrable function and \( 0 < \beta < 1 \). Suppose that \( 1 < p < n/\beta, 0 < \lambda < n − \beta p, 1/q = 1/p − \beta/n \) and \( \lambda/p = \mu/q \). Then \( b \in \hat{\Lambda}_\beta(\mathbb{R}^n) \) if and only if \( M_b \) is bounded from \( L^{p,\lambda}(\mathbb{R}^n) \) to \( L^{q,\lambda}(\mathbb{R}^n) \).
On the other hand, similar to (1.1), we can define the (nonlinear) commutator of the Hardy–Littlewood maximal function $M$ with a locally integrable function $b$ by

$$[b, M](f)(x) = b(x)M(f)(x) - M(bf)(x).$$

Using real interpolation techniques, Milman and Schonbek [16] established a commutator result. As an application, they obtained the $L^p$-boundedness of $[b, M]$ when $b \in BMO(\mathbb{R}^n)$ and $b \geq 0$. This operator can be used in studying the product of a function in $H^1$ and a function in $BMO$ (see [5] for instance). In 2000, Bastero, Milman and Ruiz [4] studied the necessary and sufficient conditions for the boundedness of $[b, M]$ on $L^p$ spaces when $1 < p < \infty$. Zhang and Wu obtained similar results for the fractional maximal function in [24] and extended the mentioned results to variable exponent Lebesgue spaces in [25] and [26]. Recently, Agcayazi et al. [3] gave the end-point estimates for the commutator $[b, M]$. Zhang [23] extended these results to the multilinear setting.

We would like to remark that operators $M_b$ and $[b, M]$ essentially differ from each other. For example, $M_b$ is positive and sublinear, but $[b, M]$ is neither positive nor sublinear.

The second part of this paper aims to study the mapping properties of the (nonlinear) commutator $[b, M]$ when $b$ belongs to some Lipschitz space. To state our results, we recall the definition of the maximal operator with respect to a cube. For a fixed cube $Q_0$, the Hardy–Littlewood maximal function with respect to $Q_0$ of a function $f$ is given by

$$M_{Q_0}f(x) = \sup_{Q_0 \supseteq Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is taken over all the cubes $Q$ with $Q \subseteq Q_0$ and $Q \ni x$.

**Theorem 1.4.** Let $b$ be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. Then the following statements are equivalent:

1. $b \in \dot{A}_\beta(\mathbb{R}^n)$ and $b \geq 0$;
2. $[b, M]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$;
3. there exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int_Q |b(x) - M_{Q}(b)(x)|^q \, dx \right)^{1/q} \leq C. \quad (1.3)$$

**Theorem 1.5.** Let $b \geq 0$ be a locally integrable function, $0 < \beta < 1$ and $b \in \dot{A}_\beta(\mathbb{R}^n)$. Then there is a positive constant $C$ such that, for all $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : |[b, M](f)(x)| > \lambda\}| \leq C \lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)}^{n/(n-\beta)}.$$

**Theorem 1.6.** Let $b$ be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \lambda < n - \beta p$ and $1/q = 1/p - \beta/n - \beta/(n - \lambda)$. Then the following statements are equivalent:

1. $b \in \dot{A}_\beta(\mathbb{R}^n)$ and $b \geq 0$;
2. $[b, M]$ is bounded from $L^{p, \lambda}(\mathbb{R}^n)$ to $L^{q, \lambda}(\mathbb{R}^n)$.

**Theorem 1.7.** Let $b$ be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \lambda < n - \beta p$, $1/q = 1/p - \beta/n$ and $\lambda/p = \mu/q$. Then the following statements are equivalent:

1. $b \in \dot{A}_\beta(\mathbb{R}^n)$ and $b \geq 0$;
2. $[b, M]$ is bounded from $L^{p, \lambda}(\mathbb{R}^n)$ to $L^{q, \mu}(\mathbb{R}^n)$.

This paper is organized as follows. In the next section, we recall some basic definitions and known results. In Section 3, we will prove Theorems 1.1–1.3. Section 4 is devoted to proving Theorems 1.4–1.7.

### 2. Preliminaries and lemmas

For a measurable set $E$, we denote by $|E|$ the Lebesgue measure and by $\chi_E$ the characteristic function of $E$. For $p \in [1, \infty]$, we denote by $p'$ the conjugate index of $p$, namely, $p' = p/(p - 1)$. For a locally integrable function $f$ and a cube $Q$, we denote by $f_0 = (f)_{Q} = \frac{1}{|Q|} \int_Q f(x) \, dx$.

To prove the theorems, we need some known results. It is known that the Lipschitz space $\dot{A}_\beta(\mathbb{R}^n)$ coincides with some Morrey–Companato space (see [14] for example) and can be characterized by mean oscillation as the following lemma, which is due to DeVore and Sharpley [7] and Janson, Taibleson and Weiss [14] (see also Paluszyński [18]).
Lemma 2.1. Let $0 < \beta < 1$ and $1 \leq q < \infty$. Define
\[
\dot{\Lambda}_{\beta,q} (\mathbb{R}^n) := \left\{ f \in L^1_{\text{loc}} (\mathbb{R}^n) : \| f \| \dot{\Lambda}_{\beta,q} = \sup_Q \frac{1}{|Q|^\beta/n} \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^q \, dx \right)^{1/q} < \infty \right\}.
\]
Then, for all $0 < \beta < 1$ and $1 \leq q < \infty$, $\dot{\Lambda}_{\beta} (\mathbb{R}^n) = \dot{\Lambda}_{\beta,q} (\mathbb{R}^n)$ with equivalent norms.

Let $0 < \alpha < n$ and $f$ be a locally integrable function, the fractional maximal function of $f$ is given by
\[
M_\alpha (f)(x) = \sup_{Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| \, dy
\]
where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$.

The following strong and weak-type boundednesses of $M_\alpha$ are well known, see [10] and [8].

Lemma 2.2. Let $0 < \alpha < n$, $1 \leq p \leq n/\alpha$ and $1/p = 1/p - \alpha/n$.

1. If $1 < p < n/\alpha$ then there exists a positive constant $C(n, \alpha, p)$ such that
\[
\| M_\alpha (f) \|_{L^p (\mathbb{R}^n)} \leq C(n, \alpha, p) \| f \|_{L^p (\mathbb{R}^n)}.
\]
2. If $p = n/\alpha$ then there exists a positive constant $C(n, \alpha)$ such that
\[
\| M_\alpha (f) \|_{L^{n/\alpha} (\mathbb{R}^n)} \leq C(n, \alpha) \| f \|_{L^{n/\alpha} (\mathbb{R}^n)}.
\]
3. If $p = 1$ then there exists a positive constant $C(n, \alpha)$ such that for all $\lambda > 0$
\[
\left| \left\{ x \in \mathbb{R}^n : M_\alpha (f)(x) > \lambda \right\} \right| \leq C(n, \alpha) (\lambda^{-1} \| f \|_{L^1 (\mathbb{R}^n)})^{n/(n-\alpha)}.
\]

Spanne (see [19]) and Adams [1] studied the boundedness of the fractional integral $I_\alpha$ in classical Morrey spaces. We note that the fractional maximal function enjoys the same boundedness as that of the fractional integral since the pointwise inequality $M_\alpha (f)(x) \leq I_\alpha (|f|)(x)$. These results can be summarized as follows (see also [22]).

Lemma 2.3. Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $0 < \lambda < n - \alpha p$.

1. If $1/q = 1/(p - \alpha/(n - \lambda))$, then there is a constant $C > 0$ such that
\[
\| M_\alpha (f) \|_{L^{q,\lambda} (\mathbb{R}^n)} \leq C \| f \|_{L^{p,\lambda} (\mathbb{R}^n)} \text{ for every } f \in L^{p,\lambda} (\mathbb{R}^n).
\]
2. If $1/q = 1/(p - \alpha/n)$ and $\lambda/p = \mu/q$. Then there is a constant $C > 0$ such that
\[
\| M_\alpha (f) \|_{L^{q,\lambda} (\mathbb{R}^n)} \leq C \| f \|_{L^{p,\lambda} (\mathbb{R}^n)} \text{ for every } f \in L^{p,\lambda} (\mathbb{R}^n).
\]

Lemma 2.4 ([15]). Let $1 < p < \infty$ and $0 < \lambda < n$, then there is a constant $C > 0$ that depends only on $n$ such that
\[
\| \chi_Q \|_{L^{p,\lambda} (\mathbb{R}^n)} \leq C \| Q \|^{n/\lambda}.
\]

3. Proof of Theorems 1.1–1.3

Proof of Theorem 1.1. If $b \in \dot{\Lambda}_\beta (\mathbb{R}^n)$, then
\[
M_b (f)(x) = \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)| \, dy 
\leq C \| b \| \dot{\Lambda}_\beta \sup_{Q} \frac{1}{|Q|^{1-\beta/n}} \int_Q |f(y)| \, dy 
= C \| b \| \dot{\Lambda}_\beta M_{\beta} (f)(x).
\]

Theorem 1.1 and 2.2 follow from Lemma 2.2, Lemma 2.3 and (3.1).

(3) $\Rightarrow$ (1): Assume $M_b$ is bounded from $L^p (\mathbb{R}^n)$ to $L^q (\mathbb{R}^n)$ for some $p$, $q$ with $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. For any cube $Q \subset \mathbb{R}^n$, by Hölder’s inequality and noting that $1/p + 1/q = 1 + \beta/n$, one gets
\[
\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| \, dx \leq \frac{1}{|Q|^{1+\beta/n}} \int_Q \left( \frac{1}{|Q|} \int_Q |b(x) - b(y)| \, dy \right) \, dx
\]

\[
= \frac{1}{|Q|^{1+\beta/n}} \int_Q \left( \frac{1}{|Q|} \int_Q |b(x) - b(y)| \chi_Q(y) \, dy \right) \, dx
\]

\[
\leq \frac{1}{|Q|^{1+\beta/n}} \int Q M_b(\chi_Q)(x) \, dx
\]

\[
\leq \frac{1}{|Q|^{1+\beta/n}} \left( \int Q M_b(\chi_Q)(x)^{p'} \, dx \right)^{1/p'} \left( \int_Q \chi_Q(x) \, dx \right)^{1/q'}
\]

\[
\leq \frac{C}{|Q|^{1+\beta/n}} \left| M_b \right|_{L^p \rightarrow L^q} \left| \chi_Q \right|_{L^p(\mathbb{R}^n)} \left| \chi_Q \right|_{L^{p'}(\mathbb{R}^n)}
\]

\[
\leq C \left| M_b \right|_{L^p \rightarrow L^q}.
\]

This together with Lemma 2.1 gives \( b \in \dot{A}_\beta(\mathbb{R}^n) \).

(4) \( \implies \) (1): We assume (1.2) is true and will verify \( b \in \dot{A}_\beta(\mathbb{R}^n) \). For any fixed cube \( Q_0 \subset \mathbb{R}^n \), since for any \( x \in Q_0 \),

\[
|b(x) - b_Q| \leq \frac{1}{|Q_0|} \int_{Q_0} |b(x) - b(y)| \, dy,
\]

then, for all \( x \in Q_0 \),

\[
M_b(\chi_{Q_0})(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)| \chi_{Q_0}(y) \, dy
\]

\[
\geq \frac{1}{|Q_0|} \int_{Q_0} |b(x) - b(y)| \chi_{Q_0}(y) \, dy
\]

\[
= \frac{1}{|Q_0|} \int_{Q_0} |b(x) - b(y)| \, dy
\]

\[
\geq |b(x) - b_{Q_0}|.
\]

This together with (1.2) gives

\[
\left| \left\{ x \in Q_0 : |b(x) - b_{Q_0}| > \lambda \right\} \right| \leq \left| \left\{ x \in Q_0 : M_b(\chi_{Q_0})(x) > \lambda \right\} \right|
\]

\[
\leq C (\lambda^{-1} \left| \chi_{Q_0} \right|_{L^1(\mathbb{R}^n)})^{n/(n-\beta)}
\]

\[
= C (\lambda^{-1} |Q_0|)^{n/(n-\beta)}.
\]

Let \( t > 0 \) be a constant to be determined later, then

\[
\int_{Q_0} |b(x) - b_{Q_0}| \, dx = \int_0^\infty \left| \left\{ x \in Q_0 : |b(x) - b_{Q_0}| > \lambda \right\} \right| \, d\lambda
\]

\[
= \int_0^t \left| \left\{ x \in Q_0 : |b(x) - b_{Q_0}| > \lambda \right\} \right| \, d\lambda
\]

\[
+ \int_t^\infty \left| \left\{ x \in Q_0 : |b(x) - b_{Q_0}| > \lambda \right\} \right| \, d\lambda
\]

\[
\leq t |Q_0| + C \int_t^\infty (\lambda^{-1} |Q_0|)^{n/(n-\beta)} \, d\lambda
\]
\[ t |Q_0| + C |Q_0|^{\frac{n}{(n-\beta)}} \int_{t}^{\infty} \lambda^{-\frac{n}{(n-\beta)}} d\lambda \]
\[ \leq C(n, \beta) \left( t |Q_0| + |Q_0|^{\frac{n}{(n-\beta)}} t^{1-n/(n-\beta)} \right). \]

Set \( t = |Q_0|^{\beta/n} \) in the above estimate, we have
\[ \int_{Q_0} |b(x) - b_{Q_0}| dx \leq C |Q_0|^{1+\beta/n}. \]

It follows from \textbf{Lemma 2.1} that \( b \in \dot{A}_p(\mathbb{R}^n) \) since \( Q_0 \) is an arbitrary cube in \( \mathbb{R}^n \).

(5) \( \implies \) (1): If \( M_b \) is bounded from \( L^{p, \lambda}(\mathbb{R}^n) \) to \( L^{\infty}(\mathbb{R}^n) \), then for any cube \( Q \subset \mathbb{R}^n \),
\[ \frac{1}{|Q|^{1+\beta/n}} \int_{Q} |b(x) - b_Q| dx \leq \frac{1}{|Q|^{1+\beta/n}} \int_{Q} \left( \frac{1}{|Q|} \int_{Q} |b(x) - b(y)| \chi_Q(y) dy \right) dx \]
\[ \leq \frac{1}{|Q|^{1+\beta/n}} \int_{Q} M_b(\chi_Q)(x) dx \]
\[ \leq \frac{1}{|Q|^{1+\beta/n}} \| M_b(\chi_Q) \|_{L^{\infty}(\mathbb{R}^n)} \]
\[ \leq \frac{C}{|Q|^{1+\beta/n}} \| M_b \|_{L^{p, \lambda} \rightarrow L^{\infty}} \| \chi_Q \|_{L^{p, \lambda}(\mathbb{R}^n)} \]
\[ \leq C \| M_b \|_{L^{p, \lambda} \rightarrow L^{\infty}}. \]

This together with \textbf{Lemma 2.1} gives \( b \in \dot{A}_p(\mathbb{R}^n) \).

The proof of \textbf{Theorem 1.1} is completed since (2) \( \implies \) (1) follows from (3) \( \implies \) (1). \( \square \)

\textbf{Proof of Theorem 1.2.} Assume \( b \in \dot{A}_p(\mathbb{R}^n) \). By (3.1) and \textbf{Lemma 2.3} (1), we have
\[ \| M_b(f) \|_{L^{p, \lambda}} \leq \| b \|_{\dot{A}_p} \| \| \nabla M_b(f) \|_{L^{p, \lambda}} \leq C \| b \|_{\dot{A}_p} \| f \|_{L^{p, \lambda}}. \]

Conversely, if \( M_b \) is bounded from \( L^{p, \lambda}(\mathbb{R}^n) \) to \( L^{q, \lambda}(\mathbb{R}^n) \), then for any cube \( Q \subset \mathbb{R}^n \),
\[ \frac{1}{|Q|^{1+\beta/n}} \left( \frac{1}{|Q|} \int_{Q} |b(x) - b_Q|^{ql} dx \right)^{\frac{1}{q}} \leq \frac{1}{|Q|^{1+\beta/n}} \left( \frac{1}{|Q|} \int_{Q} \left( \frac{1}{|Q|} \int_{Q} |b(x) - b(y)| \chi_Q(y) dy \right)^{ql} dx \right)^{\frac{1}{q}} \]
\[ \leq \frac{1}{|Q|^{1+\beta/n}} \left( \frac{1}{|Q|} \int_{Q} \left( M_b(\chi_Q)(x) \right)^{ql} dx \right)^{\frac{1}{q}} \]
\[ = \frac{1}{|Q|^{1+\beta/n}} \left( \frac{|Q|^{\lambda/n}}{|Q|} \right)^{\frac{1}{q}} \left( \frac{1}{|Q|^{\lambda/n}} \int_{Q} \left( M_b(\chi_Q)(x) \right)^{ql} dx \right)^{\frac{1}{q}} \]
\[ \leq |Q|^{-\beta/n-1/q+\lambda/(nq)} \| M_b(\chi_Q) \|_{L^{q, \lambda}(\mathbb{R}^n)} \]
\[ \leq C |Q|^{-\beta/n-1/q+\lambda/(nq)} \| M_b \|_{L^{p, \lambda} \rightarrow L^{q, \lambda}} \| \chi_Q \|_{L^{p, \lambda}(\mathbb{R}^n)} \]
\[ \leq C \| M_b \|_{L^{p, \lambda} \rightarrow L^{q, \lambda}}, \]

where in the last step we have used \( 1/q = 1/p - \beta/(n - \lambda) \) and \textbf{Lemma 2.4}.

It follows from \textbf{Lemma 2.1} that \( b \in \dot{A}_p(\mathbb{R}^n) \). This completes the proof. \( \square \)

\textbf{Proof of Theorem 1.3.} By a similar proof to the one of \textbf{Theorem 1.2}, we can obtain \textbf{Theorem 1.3}. \( \square \)
4. Proof of Theorems 1.4–1.7

Proof of Theorem 1.4. (1) $\implies$ (2): For any fixed $x \in \mathbb{R}^n$ such that $M(f)(x) < \infty$, since $b \geq 0$ then

$$
[b, M](f)(x) = |b(x)M(f)(x) - M(bf)(x)| \\
= \left| \sup_{Q \ni x} \frac{1}{|Q|} \int b(y)|f(y)|dy - \sup_{Q \ni x} \frac{1}{|Q|} \int b(y)|f(y)|dy \right| \\
\leq \sup_{Q \ni x} \frac{1}{|Q|} \int |b(x) - b(y)||f(y)|dy \\
= \sup_{Q \ni x} \frac{1}{|Q|} \int |b(x) - b(y)||f(y)|dy
$$

(4.1)

It follows from Theorem 1.1 that $[b, M]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ since $b \in \dot{A}_\beta(\mathbb{R}^n)$.

(2) $\implies$ (3): For any fixed cube $Q \subset \mathbb{R}^n$ and all $x \in Q$, we have (see the proof of Proposition 4.1 in [4], see also (2.4) in [24])

$$
M(\chi_Q)(x) = \chi_Q(x) \quad \text{and} \quad M(b\chi_Q)(x) = M_Q(b)(x).
$$

Then,

$$
\frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int b(x) - M_Q(b)(x) \right)^{q/2} dx = \frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int b(x)M(\chi_Q)(x) - M_Q(b\chi_Q)(x) \right)^{q/2} dx
$$

(4.2)

$$
\leq \frac{C}{|Q|^{\beta/n+1/q}} \left\| b, M(\chi_Q) \right\|_{L^q(\mathbb{R}^n)} \\
\leq \frac{C}{|Q|^{\beta/n+1/q}} \left\| \chi_Q \right\|_{L^p(\mathbb{R}^n)} \\
\leq C.
$$

which implies (3) since the cube $Q \subset \mathbb{R}^n$ is arbitrary.

(3) $\implies$ (1): To prove $b \in \dot{A}_\beta(\mathbb{R}^n)$, by Lemma 2.1, it suffices to verify that there is a constant $C > 0$ such that for all cubes $Q$,

$$
\frac{1}{|Q|^{1+\beta/n}} \int b(x) - b_Q |dx \leq C.
$$

(4.3)

For any fixed cube $Q$, let $E = \{x \in Q : b(x) \leq b_Q \}$ and $F = \{x \in Q : b(x) > b_Q \}$. The following equality is trivially true (see [4] page 3331):

$$
\int \limits_E |b(x) - b_Q| dx = \int \limits_F |b(x) - b_Q| dx.
$$

Since for any $x \in E$ we have $b(x) \leq b_Q \leq M_Q(b)(x)$, then for any $x \in E$,

$$
|b(x) - b_Q| \leq |b(x) - M_Q(b)(x)|.
$$
Thus,
\[
\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| \, dx = \frac{1}{|Q|^{1+\beta/n}} \int_{E \cup F} |b(x) - b_Q| \, dx = \frac{2}{|Q|^{1+\beta/n}} \int_E |b(x) - b_Q| \, dx \\
\leq \frac{2}{|Q|^{1+\beta/n}} \int_E |b(x) - M_Q(b(x))| \, dx \\
\leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |b(x) - M_Q(b(x))| \, dx.
\] (4.4)

On the other hand, it follows from Hölder’s inequality and (1.3) that
\[
\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - M_Q(b(x))| \, dx \leq \frac{1}{|Q|^{1+\beta/n}} \left( \int_Q |b(x) - M_Q(b(x))|^q \, dx \right)^{1/q} |Q|^{1/q'} \\
\leq \frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int_Q |b(x) - M_Q(b(x))|^q \, dx \right)^{1/q} \\
\leq C.
\]

This together with (4.4) gives (4.3), and so we achieve \( b \in \hat{A}_\beta(\mathbb{R}^n) \).

In order to prove \( b \geq 0 \), it suffices to show \( b^- = 0 \), where \( b^- = -\min\{b, 0\} \). Let \( b^+ = |b| - b^- \), then \( b = b^+ - b^- \). For any fixed cube \( Q \), observe that

\[
0 \leq b^+(x) \leq |b(x)| \leq M_Q(b)(x)
\]

for \( x \in Q \) and therefore we have that, for \( x \in Q \),

\[
0 \leq b^-(x) \leq M_Q(b)(x) - b^+(x) + b^-(x) = M_Q(b)(x) - b(x).
\]

Then, it follows from (1.3) that, for any cube \( Q \),

\[
\frac{1}{|Q|} \int_Q b^+(x) \, dx \leq \frac{1}{|Q|} \int_Q |M_Q(b)(x) - b(x)| \\
\leq \left( \frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^q \, dx \right)^{1/q} \\
= |Q|^{\beta/n} \left\{ \frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^q \, dx \right)^{1/q} \right\} \\
\leq C|Q|^{\beta/n}.
\]

Thus, \( b^- = 0 \) follows from Lebesgue’s differentiation theorem.

The proof of Theorem 1.4 is completed. \( \square \)

**Proof of Theorem 1.5.** Obviously, Theorem 1.5 follows from (4.1) and Theorem 1.1. \( \square \)

**Proof of Theorem 1.6.** (1) \( \implies \) (2): Assume \( b \geq 0 \) and \( b \in \hat{A}_\beta(\mathbb{R}^n) \), then by (4.1) and Theorem 1.2 we see that \( |b, M| \) is bounded from \( L^{p,\lambda}(\mathbb{R}^n) \) to \( L^{q,\lambda}(\mathbb{R}^n) \).

(2) \( \implies \) (1): Assume that \( |b, M| \) is bounded from \( L^{p,\lambda}(\mathbb{R}^n) \) to \( L^{q,\lambda}(\mathbb{R}^n) \). Similarly to (4.2), we have, for any cube \( Q \subset \mathbb{R}^n \),

\[
\frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^q \, dx \right)^{1/q} \\
= \frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int_Q |b(x)|^q \, dx \right)^{1/q}.
\]
\[
\begin{align*}
&\leq \frac{\|Q^{\lambda/(nq)}\|_q}{\|Q^{\beta/(p+1/q)}\|_p} \|b, M\|_{L^1}(\mathbb{X}_1)\|_{L^q}(\mathbb{R}^n) \\
&\leq \frac{\|Q^{\lambda/(nq)}\|_q}{\|Q^{\beta/(p+1/q)}\|_p} \|\mathbb{X}_1\|_{L^1}(\mathbb{R}^n) \\
&\leq C,
\end{align*}
\]

where in the last step we have used \(1/q = 1/p - \beta/(n - \lambda)\) and Lemma 2.4. This shows by Theorem 1.4 that \(b \in \Lambda^\beta_{\mathbb{R}^n}\) and \(b \geq 0\).

**Proof of Theorem 1.7.** By the same way of the proof of Theorem 1.6, Theorem 1.7 can be proven. We omit the details.

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**References**