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New nonlinear estimates for surfaces in terms of their fundamental forms

Nouvelles estimations pour des surfaces en fonction de leurs formes fondamentales

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ABSTRACT

We establish several estimates of the distance between two surfaces immersed in the three-dimensional Euclidean space in terms of the distance between their fundamental forms, measured in various Sobolev norms. These estimates, which can be seen as nonlinear versions of linear Korn inequalities on a surface appearing in the theory of linearly elastic shells, generalize in particular the nonlinear Korn inequality established in 2005 by P. G. Ciarlet, L. Gratie, and C. Mardare.

RÉSUMÉ

Nous établissons plusieurs majorations de la distance entre deux surfaces immergées dans l'espace euclidien tridimensionnel en fonction de la distance entre leurs formes fondamentales, mesurée à l'aide de diverses normes de Sobolev. Ces estimations, qui peuvent être vues comme des versions non linéaires des inégalités de Korn linéaires sur une surface apparaissant dans la théorie de coques linéairement élastiques, généralisent en particulier l'inégalité de Korn non linéaire sur une surface établie en 2005 par P. G. Ciarlet, L. Gratie et C. Mardare.

1. Introduction

The various notions and notations, notably from the differential geometry of surfaces, used in this introduction are defined in Sect. 2.

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A *nonlinear Korn inequality on a surface* is an inequality that estimates the deformation of a surface in terms of the variation of the fundamental forms of the surface induced by this deformation.

We establish here such inequalities for *deformations with as little regularity as possible*, that is, just enough to define the fundamental forms of the deformed surface in appropriate Lebesgue spaces L^q. This is motivated by the theory of nonlinearly elastic shells, in particular by the well-known nonlinear Koiter shell model, where the deformations with finite energy are those whose first two fundamental forms have covariant components in the Lebesgue space L^2.

More specifically, let $\omega \subset \mathbb{R}^2$ be a domain, let γ_0 be a non-empty relatively open subset of the boundary of ω, and let $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ be a sufficiently smooth immersion. Consider a shell with middle surface $\boldsymbol{\theta}(\overline{\omega}) \subset \mathbb{R}^3$ and half-thickness $h > 0$, made of a homogeneous and isotropic nonlinearly elastic material whose Lamé constants λ and μ satisfy $3\lambda + 2\mu > 0$ and $\mu > 0$.

In the *nonlinear* Koiter shell model, so named after Koiter [11], the *strain energy* associated with a sufficiently smooth deformation $\tilde{\boldsymbol{\theta}} : \omega \to \mathbb{R}^3$ of the middle surface $\boldsymbol{\theta}(\overline{\omega})$ of a shell is defined by (the notations for the fundamental forms associated with $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$ are defined in Sect. 3)

$$\mathcal{E}(\tilde{\boldsymbol{\theta}}) := \frac{1}{2} \int_\omega \Big\{ h W\big(\frac{1}{2}\boldsymbol{I}^{-1}(\tilde{\boldsymbol{I}} - \boldsymbol{I})\big) + \frac{h^3}{3} W\big(\boldsymbol{I}^{-1}(\tilde{\boldsymbol{II}} - \boldsymbol{II})\big) \Big\} \sqrt{\det \boldsymbol{I}}\, \mathrm{d}y$$

where the function $W : \mathbb{M}^2 \to \mathbb{R}$ is the *two-dimensional stored energy function* defined by

$$W(\boldsymbol{A}) := \frac{2\lambda\mu}{\lambda + 2\mu} (\operatorname{tr} \boldsymbol{A})^2 + 2\mu \operatorname{tr} \boldsymbol{A}^2 \text{ for each } \boldsymbol{A} \in \mathbb{M}^2.$$

It is easy to prove that there exist two constants $0 < C_1 \le C_2$, depending only on h, ω, and $\boldsymbol{\theta}$, such that

$$C_1 \Big\{ \|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|^2_{L^2(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|^2_{L^2(\omega)} \Big\} \le \mathcal{E}(\tilde{\boldsymbol{\theta}}) \le C_2 \Big\{ \|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|^2_{L^2(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|^2_{L^2(\omega)} \Big\}$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,4}(\omega; \mathbb{R}^3)$ that satisfy $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) \in W^{1,4}(\omega; \mathbb{R}^3)$.

Combined with the above estimate, the *nonlinear Korn inequalities on a surface* established in this paper *show that the strain energy* $\mathcal{E}(\tilde{\boldsymbol{\theta}})$ *"controls" the "magnitude" of the deformation of the middle surface of the shell.* In particular, Theorem 3.2(b) implies that there exists a constant C_3, which depends only on h, ω, and $\boldsymbol{\theta}$, such that

$$\Big\{ \|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^4_{W^{1,4}(\omega)} + \|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|^4_{W^{1,4}(\omega)} \Big\} \le C_3 \Big\{ \|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|^2_{L^2(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|^2_{L^2(\omega)} \Big\} \le \frac{C_3}{C_1} \mathcal{E}(\tilde{\boldsymbol{\theta}})$$

and

$$\Big\{ \|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2_{H^1(\omega)} + \|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|^2_{H^1(\omega)} \Big\} \le C_3 \Big\{ \|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|^2_{L^2(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|^2_{L^2(\omega)} \Big\} \le \frac{C_3}{C_1} \mathcal{E}(\tilde{\boldsymbol{\theta}})$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,4}(\omega; \mathbb{R}^3)$ that satisfy $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) \in W^{1,4}(\omega; \mathbb{R}^3)$, $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$ on γ_0, $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) = \boldsymbol{a}_3(\boldsymbol{\theta})$ on γ_0, and

$$|\tilde{R}_\alpha| \ge \varepsilon \text{ a.e. in } \omega, \; |\tilde{\boldsymbol{I}}| \le \frac{1}{\varepsilon} \text{ a.e. in } \omega, \text{ and } \big|\tilde{\boldsymbol{I}}^{-1}\big| \le \frac{1}{\varepsilon} \text{ a.e. in } \omega.$$

Note that a *formal linearization* of the latter nonlinear Korn inequality with respect to the displacement field

$$\boldsymbol{\zeta} := \tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}$$

coincides with the following *linear Korn inequality on a surface*, due to Bernadou & Ciarlet [2] (see also [3,4]), which is the key to proving the existence and uniqueness of a solution to the *linear Koiter shell model* (in the strain energy of which the differences $(\tilde{\boldsymbol{I}} - \boldsymbol{I})$ and $(\tilde{\boldsymbol{II}} - \boldsymbol{II})$ are replaced by their linear parts with respect to $\boldsymbol{\zeta}$): There exists a constant C_4 depending only on h, ω, and $\boldsymbol{\theta}$, such that

$$\Big\{ \sum_{\alpha=1}^{2} \|\zeta_\alpha\|^2_{H^1(\omega)} + \|\zeta_3\|^2_{H^2(\omega)} \Big\} \le C_4 \Big\{ \|\boldsymbol{\gamma}(\boldsymbol{\zeta})\|^2_{L^2(\omega)} + \|\boldsymbol{\rho}(\boldsymbol{\zeta})\|^2_{L^2(\omega)} \Big\},$$

for all vector fields $\boldsymbol{\zeta} : \omega \to \mathbb{R}^3$ whose components $\zeta_i := \boldsymbol{\zeta} \cdot \boldsymbol{a}_i(\boldsymbol{\theta}) : \omega \to \mathbb{R}$, $1 \le i \le 3$, satisfy

$$\zeta_\alpha \in H^1(\omega) \text{ and } \zeta_\alpha = 0 \text{ on } \gamma_0, \text{ and } \zeta_3 \in H^2(\omega) \text{ and } \zeta_3 = \partial_\alpha \zeta_3 = 0 \text{ on } \gamma_0,$$

where $\boldsymbol{\gamma}(\boldsymbol{\zeta})$, resp. $\boldsymbol{\rho}(\boldsymbol{\zeta})$, denotes the linear part with respect to $\boldsymbol{\zeta}$ of the difference $(\boldsymbol{I}(\boldsymbol{\theta} + \boldsymbol{\zeta}) - \boldsymbol{I}(\boldsymbol{\theta}))$, resp. of the difference $(\boldsymbol{II}(\boldsymbol{\theta} + \boldsymbol{\zeta}) - \boldsymbol{II}(\boldsymbol{\theta}))$.

2. Notation

A *domain* in $\mathbb{R}^m$ is a connected, bounded, and open, subset of $\mathbb{R}^m$, whose boundary is Lipschitz-continuous (in the sense of Adams [1] or Nečas [12]), and which is locally on the same side of its boundary.

Let $\omega \subset \mathbb{R}^2$ be a domain. A generic point in ω is denoted $y = (y_\alpha)$, and partial derivatives with respect to y_α, both in the classical or distributional sense, are denoted ∂_α. Here and subsequently, Greek indices range in the set $\{1, 2\}$.

The notation $\mathbb{S}^2$ designates the space of 2×2 real symmetric matrices and the notations $\mathbb{M}^2$ and $\mathbb{M}^{3\times 2}$ respectively designate the spaces of 2×2 and 3×2 real matrices. The inner product and the vector product of vectors in $\mathbb{R}^3$ are respectively denoted $\cdot$ and $\wedge$. The Euclidean norm in $\mathbb{R}^3$, the Frobenius norm in $\mathbb{S}^2$, and the Frobenius norm in $\mathbb{M}^{3\times 2}$, are all denoted by $|\cdot|$.

A *proper isometry* of $\mathbb{R}^3$ is a mapping $\boldsymbol{r} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\boldsymbol{r}(x) = \boldsymbol{a} + \boldsymbol{R}\,x \text{ for each } x \in \mathbb{R}^3,$$

where $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{R}$ is a 3×3 real orthogonal matrix with $\det \boldsymbol{R} = 1$. The set of all proper isometries of $\mathbb{R}^3$ is denoted $\mathbf{Isom}(\mathbb{R}^3)$.

Mappings from ω into a finite-dimensional vector space are denoted by boldface letters. Such a mapping is of class $\mathcal{C}^1(\omega)$, resp. of class $W^{1,p}(\omega)$, if all its components belong to $\mathcal{C}^1(\omega)$, resp. to $W^{1,p}(\omega)$. A function $f : \omega \to \mathbb{R}$ is of class $\mathcal{C}^1(\overline{\omega})$ if it can be extended into a function of class $\mathcal{C}^1$ over $\mathbb{R}^2$.

The $L^p(\omega)$-norm, $1 \le p < \infty$, of a measurable mapping $\boldsymbol{A} : \omega \to X$, where X is one of the spaces $\mathbb{R}^3$, $\mathbb{S}^2$, or $\mathbb{M}^{3\times 2}$, is denoted and defined by

$$\|\boldsymbol{A}\|_{L^p(\omega)} := \Big(\int_\omega |\boldsymbol{A}(y)|^p \, \mathrm{d}y \Big)^{1/p},$$

where $|\cdot|$ denotes the Euclidean or Frobenius norm in X.

The $W^{1,p}(\omega)$-norm of a mapping $\boldsymbol{\varphi} : \omega \to \mathbb{R}^3$ of class $W^{1,p}(\omega)$, $1 \le p < \infty$, is denoted and defined by

$$\|\boldsymbol{\varphi}\|_{W^{1,p}(\omega)} := \|\boldsymbol{\varphi}\|_{L^p(\omega)} + \|\boldsymbol{\nabla\varphi}\|_{L^p(\omega)},$$

where $\boldsymbol{\nabla\varphi} : \omega \to \mathbb{M}^{3\times 2}$ denotes the gradient of $\boldsymbol{\varphi}$ (that is, $\boldsymbol{\nabla\varphi}$ is the matrix field with $\partial_\alpha \boldsymbol{\varphi}$ as its column vector fields).

We now briefly review the notions from differential geometry of surfaces needed in this Note; for a detailed presentation of these notions, see, e.g., [5] or [10].

A mapping $\boldsymbol{\varphi} : \omega \to \mathbb{R}^3$ of class $\mathcal{C}^1(\overline{\omega})$, resp. of class $W^{1,p}(\omega)$, $p \ge 1$, is an *immersion* if the two vector fields $\partial_\alpha \boldsymbol{\varphi} : \overline{\omega} \to \mathbb{R}^3$ are linearly independent at each point of $\overline{\omega}$, resp. almost everywhere in ω.

Given any immersion $\boldsymbol{\varphi} : \omega \to \mathbb{R}^3$ of class $W^{1,2q}(\omega)$, $q \ge 1$, the vector field

$$\boldsymbol{a}_3(\boldsymbol{\varphi}) := \frac{\partial_1 \boldsymbol{\varphi} \wedge \partial_2 \boldsymbol{\varphi}}{|\partial_1 \boldsymbol{\varphi} \wedge \partial_2 \boldsymbol{\varphi}|} : \omega \to \mathbb{R}^3,$$

which is well-defined almost everywhere in ω, is a unit normal vector field to the surface $\boldsymbol{\varphi}(\omega)$, i.e.

$$\boldsymbol{a}_3(\boldsymbol{\varphi}) = 1 \text{ and } \boldsymbol{a}_3(\boldsymbol{\varphi}) \cdot \partial_\alpha \boldsymbol{\varphi} = 0 \text{ in } \omega.$$

Given any immersion $\boldsymbol{\varphi} : \omega \to \mathbb{R}^3$ of class $W^{1,2q}(\omega)$, $q \ge 1$, such that $\boldsymbol{a}_3(\boldsymbol{\varphi})$ is also of class $W^{1,2q}(\omega)$, the matrix fields

$$\begin{aligned}
\boldsymbol{I}(\boldsymbol{\varphi}) &:= \boldsymbol{\nabla\varphi}^{\mathrm{T}} \boldsymbol{\nabla\varphi} : \omega \to \mathbb{S}^2, \\
\boldsymbol{II}(\boldsymbol{\varphi}) &:= -\boldsymbol{\nabla a}_3(\boldsymbol{\varphi})^{\mathrm{T}} \boldsymbol{\nabla\varphi} = -\boldsymbol{\nabla\varphi}^{\mathrm{T}} \boldsymbol{\nabla a}_3(\boldsymbol{\varphi}) : \omega \to \mathbb{S}^2, \\
\boldsymbol{III}(\boldsymbol{\varphi}) &:= \boldsymbol{\nabla a}_3(\boldsymbol{\varphi})^{\mathrm{T}} \boldsymbol{\nabla a}_3(\boldsymbol{\varphi}) : \omega \to \mathbb{S}^2,
\end{aligned}$$

are well-defined and belong to $L^q(\omega; \mathbb{S}^2)$. Note that the equality appearing in the definition of $\boldsymbol{II}(\boldsymbol{\varphi})$ holds thanks to the relations $\boldsymbol{a}_3(\boldsymbol{\varphi}) \cdot \partial_\alpha \boldsymbol{\varphi} = 0$ in ω; cf. [6, Lemma 3.3]. The components of the matrix fields $\boldsymbol{I}(\boldsymbol{\varphi})$, $\boldsymbol{II}(\boldsymbol{\varphi})$, and $\boldsymbol{III}(\boldsymbol{\varphi})$ are respectively the covariant components of the *first, second, and third fundamental forms* of the surface $\boldsymbol{\varphi}(\omega)$.

If $\boldsymbol{\varphi} \in W^{1,2q}(\omega; \mathbb{R}^3)$, $q \ge 1$, is an immersion, the matrix field $\boldsymbol{I}(\boldsymbol{\varphi})$ is positive definite almost everywhere in ω; hence the matrix field $\boldsymbol{I}(\boldsymbol{\varphi})^{-1} : \omega \to \mathbb{S}^2$, where

$$\boldsymbol{I}(\boldsymbol{\varphi})^{-1}(y) := \big(\boldsymbol{I}(\boldsymbol{\varphi})(y)\big)^{-1} \text{ for almost all } y \in \omega,$$

is well defined almost everywhere in ω. The components of the matrix fields $\boldsymbol{I}(\boldsymbol{\varphi})^{-1}$ are the contravariant components of the first fundamental form of the immersion $\boldsymbol{\varphi}$.

The *principal radii of curvature* $R_1(\boldsymbol{\varphi})$ and $R_2(\boldsymbol{\varphi})$ of the surface $\boldsymbol{\varphi}(\omega)$ are the inverses of the eigenvalues of the matrix field $(\boldsymbol{I}(\boldsymbol{\varphi})^{-1} \boldsymbol{II}(\boldsymbol{\varphi}))$; they are well defined almost everywhere in ω.

3. Main results

This section gathers the main results (Theorems 3.1 and 3.2 below) of this Note. The details of the proofs, which are only briefly sketched below, will appear in a forthcoming paper [9], where the case of *higher Sobolev norms* will also be considered.

The first theorem estimates the *distance in* $W^{1,p}(\omega;\mathbb{R}^3)$ *between two immersions* $\boldsymbol{\theta}:\omega\to\mathbb{R}^3$ *and* $\tilde{\boldsymbol{\theta}}:\omega\to\mathbb{R}^3$ *in terms of the distance in* $L^q(\omega;\mathbb{S}^2)$ *between their three fundamental forms*.

The second theorem estimates the *distance in* $W^{1,p}(\omega;\mathbb{R}^3)$ *between two immersions* $\boldsymbol{\theta}:\omega\to\mathbb{R}^3$ *and* $\tilde{\boldsymbol{\theta}}:\omega\to\mathbb{R}^3$ *in terms of the distance in* $L^q(\omega;\mathbb{S}^2)$ *between their first two fundamental forms*, under an additional assumption on the first fundamental form of $\tilde{\boldsymbol{\theta}}$ that intuitively "prevents infinite stretching or compression" of the deformed surfaces $\tilde{\boldsymbol{\theta}}(\omega)$.

For notational brevity, we let in what follows

$$\boldsymbol{I}:=\boldsymbol{I}(\boldsymbol{\theta}),\ \boldsymbol{II}:=\boldsymbol{II}(\boldsymbol{\theta}),\ \boldsymbol{III}:=\boldsymbol{III}(\boldsymbol{\theta}),$$
$$\tilde{\boldsymbol{I}}:=\boldsymbol{I}(\tilde{\boldsymbol{\theta}}),\ \tilde{\boldsymbol{II}}:=\boldsymbol{II}(\tilde{\boldsymbol{\theta}}),\ \tilde{\boldsymbol{III}}:=\boldsymbol{III}(\tilde{\boldsymbol{\theta}}).$$

Note also that the various constants $c_1(\varepsilon,\boldsymbol{\theta})$, $c_2(\varepsilon,\boldsymbol{\theta})$, $C_1(\boldsymbol{\Theta})$, etc., appearing in the statements of Theorems 3.1 and 3.2 below and in their proofs also depend on ω, p, and q, but these dependences are not mentioned, again for notational brevity.

Theorem 3.1. *Let* $\omega\subset\mathbb{R}^2$ *be a domain, let* $\boldsymbol{\theta}:\overline{\omega}\to\mathbb{R}^3$ *be an immersion of class* $\mathcal{C}^1$ *such that* $\boldsymbol{a}_3(\boldsymbol{\theta}):\overline{\omega}\to\mathbb{R}^3$ *is also of class* $\mathcal{C}^1$, *and let* $p>1$ *and* $q\geq 1$ *be two parameters such that* $p/2\leq q\leq p$. *Then the following nonlinear Korn inequalities hold.*

(a) *For each* $\varepsilon>0$, *there exist two constants* $c_1(\varepsilon,\boldsymbol{\theta})$ *and* $c_2(\varepsilon,\boldsymbol{\theta})$ *such that*

$$\inf_{\boldsymbol{r}\in\mathbf{Isom}(\mathbb{R}^3)}\left\{\|\boldsymbol{r}\circ\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\|_{W^{1,p}(\omega)}+\|\boldsymbol{a}_3(\boldsymbol{r}\circ\tilde{\boldsymbol{\theta}})-\boldsymbol{a}_3(\boldsymbol{\theta})\|_{W^{1,p}(\omega)}\right\}$$
$$\leq c_1(\varepsilon,\boldsymbol{\theta})\left\{\|\tilde{\boldsymbol{I}}-\boldsymbol{I}\|_{L^q(\omega)}+\|\tilde{\boldsymbol{II}}-\boldsymbol{II}\|_{L^q(\omega)}+\|\tilde{\boldsymbol{III}}-\boldsymbol{III}\|_{L^q(\omega)}\right\}^{q/p},$$

and

$$\|\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\|_{W^{1,p}(\omega)}+\|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}})-\boldsymbol{a}_3(\boldsymbol{\theta})\|_{W^{1,p}(\omega)}\leq c_2(\varepsilon,\boldsymbol{\theta})\Big(\left\{\|\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\|_{L^p(\omega)}+\|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}})-\boldsymbol{a}_3(\boldsymbol{\theta})\|_{L^p(\omega)}\right\}$$
$$+\left\{\|\tilde{\boldsymbol{I}}-\boldsymbol{I}\|_{L^q(\omega)}+\|\tilde{\boldsymbol{II}}-\boldsymbol{II}\|_{L^q(\omega)}+\|\tilde{\boldsymbol{III}}-\boldsymbol{III}\|_{L^q(\omega)}\right\}^{q/p}\Big),$$

for all immersions $\tilde{\boldsymbol{\theta}}\in W^{1,2q}(\omega;\mathbb{R}^3)$ *satisfying* $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}})\in W^{1,2q}(\omega;\mathbb{R}^3)$ *and* $|\tilde{R}_\alpha|\geq\varepsilon$ *a.e. in* ω.

(b) *For each* $\varepsilon>0$ *and each non-empty relatively open subset* $\gamma_0\subset\partial\omega$, *there exists a constant* $c_3(\varepsilon,\boldsymbol{\theta},\gamma_0)$ *such that*

$$\|\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\|_{W^{1,p}(\omega)}+\|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}})-\boldsymbol{a}_3(\boldsymbol{\theta})\|_{W^{1,p}(\omega)}$$
$$\leq c_3(\varepsilon,\boldsymbol{\theta},\gamma_0)\left\{\|\tilde{\boldsymbol{I}}-\boldsymbol{I}\|_{L^q(\omega)}+\|\tilde{\boldsymbol{II}}-\boldsymbol{II}\|_{L^q(\omega)}+\|\tilde{\boldsymbol{III}}-\boldsymbol{III}\|_{L^q(\omega)}\right\}^{q/p}$$

for all immersions $\tilde{\boldsymbol{\theta}}\in W^{1,2q}(\omega;\mathbb{R}^3)$ *satisfying* $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}})\in W^{1,2q}(\omega;\mathbb{R}^3)$, $|\tilde{R}_\alpha|\geq\varepsilon$ *a.e. in* ω, $\tilde{\boldsymbol{\theta}}=\boldsymbol{\theta}$ *on* γ_0, *and* $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}})=\boldsymbol{a}_3(\boldsymbol{\theta})$ *on* γ_0.

Sketch of proof. Let $\tilde{\boldsymbol{\theta}}:\omega\to\mathbb{R}^3$ be an immersion that satisfies the assumptions of part (a) of the theorem.

First, we show that there exists a constant $\delta=\delta(\boldsymbol{\theta})$ such that the mappings $\boldsymbol{\Theta}:\Omega\to\mathbb{R}^3$ and $\tilde{\boldsymbol{\Theta}}:\Omega\to\mathbb{R}^3$ defined over the three-dimensional open set $\Omega:=\omega\times(-\varepsilon\delta,\varepsilon\delta)$ by

$$\boldsymbol{\Theta}(y,x_3):=\boldsymbol{\theta}(y)+x_3\boldsymbol{a}_3(\boldsymbol{\theta})(y)\text{ for all }(y,x_3)\in\Omega,$$
$$\tilde{\boldsymbol{\Theta}}(y,x_3):=\tilde{\boldsymbol{\theta}}(y)+x_3\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}})(y)\text{ for almost all }(y,x_3)\in\Omega,$$

satisfy

$$\boldsymbol{\Theta}\in\mathcal{C}^1(\overline{\Omega})\text{ and }\det\nabla\boldsymbol{\Theta}>0\text{ in }\overline{\Omega},$$
$$\tilde{\boldsymbol{\Theta}}\in W^{1,2q}(\Omega)\text{ and }\det\nabla\tilde{\boldsymbol{\Theta}}>0\text{ a.e. in }\Omega.$$

Note that the assumption that ω is a domain in $\mathbb{R}^2$ implies that Ω is a domain in $\mathbb{R}^3$.

The above relations then allow us to use the special case $m=3$ of the *nonlinear Korn inequalities on open domains in* $\mathbb{R}^m$ established in [8, Thm. 1(b) and Thm. 2(b)], which assert that there exist constants $C_1(\boldsymbol{\Theta})$ and $C_2(\boldsymbol{\Theta})$ such that

$$\inf_{\boldsymbol{R}\in\mathbb{O}^3_+}\|\boldsymbol{R}\nabla\tilde{\boldsymbol{\Theta}}-\nabla\boldsymbol{\Theta}\|_{L^p(\Omega)}\leq C_1(\boldsymbol{\Theta})\|\nabla\tilde{\boldsymbol{\Theta}}^{\mathrm{T}}\nabla\tilde{\boldsymbol{\Theta}}-\nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\|^{q/p}_{L^q(\Omega)},$$
$$\|\tilde{\boldsymbol{\Theta}}-\boldsymbol{\Theta}\|_{W^{1,p}(\Omega)}\leq C_2(\boldsymbol{\Theta})\Big(\|\tilde{\boldsymbol{\Theta}}-\boldsymbol{\Theta}\|_{L^p(\Omega)}+\|\nabla\tilde{\boldsymbol{\Theta}}^{\mathrm{T}}\nabla\tilde{\boldsymbol{\Theta}}-\nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\|^{q/p}_{L^q(\Omega)}\Big).$$

Second, using in particular Clarkson's inequality in $L^p(\Omega)$ (see, e.g., [1]) and Fubini's theorem, we show that there exist constants $C_3(\varepsilon, \boldsymbol{\theta}) > 0$ and $C_4(\varepsilon, \boldsymbol{\theta}) > 0$ such that

$$\|\boldsymbol{R}\nabla\tilde{\boldsymbol{\Theta}} - \nabla\boldsymbol{\Theta}\|_{L^p(\Omega)} \geq C_3(\varepsilon, \boldsymbol{\theta})\Big\{\|\boldsymbol{R}\nabla\tilde{\boldsymbol{\theta}} - \nabla\boldsymbol{\theta}\|_{L^p(\omega)} + \|\boldsymbol{R}\tilde{\boldsymbol{a}}_3(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|_{W^{1,p}(\omega)}\Big\},$$
$$\|\tilde{\boldsymbol{\Theta}} - \boldsymbol{\Theta}\|_{W^{1,p}(\Omega)} \geq C_4(\varepsilon, \boldsymbol{\theta})\Big\{\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{W^{1,p}(\omega)} + \|\tilde{\boldsymbol{a}}_3(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|_{W^{1,p}(\omega)}\Big\}.$$

Besides, the Poincaré–Wirtinger inequality implies that there exists a vector $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, \boldsymbol{R}) \in \mathbb{R}^3$ such that

$$\|\boldsymbol{R}\nabla\tilde{\boldsymbol{\theta}} - \nabla\boldsymbol{\theta}\|_{L^p(\omega)} \geq C_5\|(\boldsymbol{v} + \boldsymbol{R}\tilde{\boldsymbol{\theta}}) - \boldsymbol{\theta}\|_{W^{1,p}(\omega)},$$

where $C_5 > 0$ is a constant depending only on ω and p.

Third, the definition of the mapping $\tilde{\boldsymbol{\Theta}} : \Omega \to \mathbb{R}^3$, resp. $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3$, in terms of the immersion $\tilde{\boldsymbol{\theta}} : \omega \to \mathbb{R}^3$, resp. $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$, allows to express the matrix field $\nabla\tilde{\boldsymbol{\Theta}}^{\mathrm{T}}\nabla\tilde{\boldsymbol{\Theta}} : \Omega \to \mathbb{S}^3$, resp. $\nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta} : \Omega \to \mathbb{S}^3$, in terms of the fundamental forms of the surface $\tilde{\boldsymbol{\theta}}(\omega)$, resp. of the surface $\boldsymbol{\theta}(\omega)$. Combined with Fubini's theorem, this implies that there exists a constant $C_6(\varepsilon, \boldsymbol{\theta})$ such that

$$\|\nabla\tilde{\boldsymbol{\Theta}}^{\mathrm{T}}\nabla\tilde{\boldsymbol{\Theta}} - \nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\|_{L^q(\Omega)} \leq C_6(\varepsilon, \boldsymbol{\theta})\Big\{\|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|_{L^q(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|_{L^q(\omega)} + \|\tilde{\boldsymbol{III}} - \boldsymbol{III}\|_{L^q(\omega)}\Big\}.$$

Besides, Minkowski's inequality in $L^p(\Omega)$ and Fubini's theorem together imply that there exists a constant $C_7(\varepsilon, \boldsymbol{\theta})$ such that

$$\|\tilde{\boldsymbol{\Theta}} - \boldsymbol{\Theta}\|_{L^p(\Omega)} \leq C_7(\varepsilon, \boldsymbol{\theta})\Big\{\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{L^p(\omega)} + \|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|_{L^p(\omega)}\Big\}.$$

Combining the above inequalities then yields the nonlinear Korn inequalities of part (a) of the theorem.

Next, let an immersion $\tilde{\boldsymbol{\theta}} : \omega \to \mathbb{R}^3$ satisfy the assumptions of part (b) of the theorem, and let the immersions $\tilde{\boldsymbol{\Theta}}$ and $\boldsymbol{\Theta}$ be defined as above in terms of the immersions $\tilde{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$. We already saw that

$$\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}) \text{ and } \det \nabla\boldsymbol{\Theta} > 0 \text{ in } \overline{\Omega},$$
$$\tilde{\boldsymbol{\Theta}} \in W^{1,2q}(\Omega) \text{ and } \det \nabla\tilde{\boldsymbol{\Theta}} > 0 \text{ a.e. in } \Omega.$$

Besides,

$$\tilde{\boldsymbol{\Theta}} = \boldsymbol{\Theta} \text{ on } \Gamma_0,$$

where $\Gamma_0 := \gamma_0 \times (-\varepsilon\delta, \varepsilon\delta)$ is a relatively open subset of the boundary of Ω. This allows us to apply the special case $m = 3$ of another *nonlinear Korn inequality on open domains in* $\mathbb{R}^m$, established in [8, Thm. 3(b)], and to deduce that there exists a constant $C_8(\boldsymbol{\Theta}, \Gamma_0)$ such that

$$\|\tilde{\boldsymbol{\Theta}} - \boldsymbol{\Theta}\|_{W^{1,p}(\Omega)} \leq C_8(\boldsymbol{\Theta}, \Gamma_0)\|\nabla\tilde{\boldsymbol{\Theta}}^{\mathrm{T}}\nabla\tilde{\boldsymbol{\Theta}} - \nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\|_{L^q(\Omega)}^{q/p}.$$

Combining this inequality with the estimates of its left- and right-hand sides established (under weaker assumptions on $\tilde{\boldsymbol{\theta}}$, so that they are still valid here) in the proof of part (a) of the theorem yields the nonlinear Korn inequality of part (b). □

Remark 1. The first inequality of Theorem 3.1 has been established by Ciarlet, Gratie & Mardare [6, Theorem 4.1] in the particular case where $p = 2$ and $q = 1$. □

Remark 2. The first inequality of Theorem 3.2 with $p = 2$ and $q = 1$ implies the rigidity theorem for surfaces of Ciarlet & Mardare [7, Theorem 3]. □

Theorem 3.2. *Let $\omega \subset \mathbb{R}^2$ be a domain, let $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ be an immersion of class $\mathcal{C}^1$ such that $\boldsymbol{a}_3(\boldsymbol{\theta}) : \overline{\omega} \to \mathbb{R}^3$ is also of class $\mathcal{C}^1$, and let $p > 1$ and $q \geq 1$ be two parameters such that $p/2 \leq q \leq p$. Then the following nonlinear Korn inequalities hold:*

(a) *For each $\varepsilon > 0$, there exist two constants $c_4(\varepsilon, \boldsymbol{\theta})$ and $c_5(\varepsilon, \boldsymbol{\theta})$ such that*

$$\inf_{\boldsymbol{r} \in \mathbf{Isom}(\mathbb{R}^3)} \Big\{\|\boldsymbol{r} \circ \tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{W^{1,p}(\omega)} + \|\boldsymbol{a}_3(\boldsymbol{r} \circ \tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|_{W^{1,p}(\omega)}\Big\}$$
$$\leq c_4(\varepsilon, \boldsymbol{\theta})\Big\{\|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|_{L^q(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|_{L^q(\omega)}\Big\}^{q/p},$$

and

$$\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{W^{1,p}(\omega)} + \|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|_{W^{1,p}(\omega)} \leq c_5(\varepsilon, \boldsymbol{\theta})\Big(\Big\{\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{L^p(\omega)} + \|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|_{L^p(\omega)}\Big\}$$
$$+ \Big\{\|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|_{L^q(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|_{L^q(\omega)}\Big\}^{q/p}\Big),$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,2q}(\omega; \mathbb{R}^3)$ satisfying $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) \in W^{1,2q}(\omega; \mathbb{R}^3)$, and

$$|\tilde{R}_\alpha| \geq \varepsilon \text{ a.e. in } \omega, \ |\tilde{\boldsymbol{I}}| \leq \frac{1}{\varepsilon} \text{ a.e. in } \omega, \text{ and } \left|\tilde{\boldsymbol{I}}^{-1}\right| \leq \frac{1}{\varepsilon} \text{ a.e. in } \omega.$$

(b) *For each $\varepsilon > 0$ and each non-empty relatively open subset $\gamma_0 \subset \partial\omega$, there exists a constant $c_6(\varepsilon, \boldsymbol{\theta}, \gamma_0)$ such that*

$$\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{W^{1,p}(\omega)} + \|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|_{W^{1,p}(\omega)} \leq c_6(\varepsilon, \boldsymbol{\theta}, \gamma_0)\left\{\|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|_{L^q(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|_{L^q(\omega)}\right\}^{q/p}$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,2q}(\omega; \mathbb{R}^3)$ satisfying $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) \in W^{1,2q}(\omega; \mathbb{R}^3)$, $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$ on γ_0, $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) = \boldsymbol{a}_3(\boldsymbol{\theta})$ on γ_0, and

$$|\tilde{R}_\alpha| \geq \varepsilon \text{ a.e. in } \omega, \ |\tilde{\boldsymbol{I}}| \leq \frac{1}{\varepsilon} \text{ a.e. in } \omega, \text{ and } \left|\tilde{\boldsymbol{I}}^{-1}\right| \leq \frac{1}{\varepsilon} \text{ a.e. in } \omega.$$

Sketch of proof. This theorem is a consequence of Theorem 3.1 combined with the following identity:

$$\tilde{\boldsymbol{III}} - \boldsymbol{III} = \tilde{\boldsymbol{II}}(\tilde{\boldsymbol{I}}^{-1})\Big((\tilde{\boldsymbol{II}} - \boldsymbol{II}) - (\tilde{\boldsymbol{I}} - \boldsymbol{I})\boldsymbol{I}^{-1}\boldsymbol{II}\Big) + (\tilde{\boldsymbol{II}} - \boldsymbol{II})\boldsymbol{I}^{-1}\boldsymbol{II}.$$

It then suffices to observe that the above identity implies the existence of a constant $C(\boldsymbol{\theta})$ such that

$$\|\tilde{\boldsymbol{III}} - \boldsymbol{III}\|_{L^p(\omega)} \leq C(\boldsymbol{\theta})\Big(1 + \|\tilde{\boldsymbol{II}}(\tilde{\boldsymbol{I}}^{-1})\|_{L^\infty(\omega)}\Big)\Big(\|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|_{L^p(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|_{L^p(\omega)}\Big),$$

and then to estimate the $L^\infty(\omega)$-norm of the matrix field $\tilde{\boldsymbol{II}}(\tilde{\boldsymbol{I}}^{-1})$ by noting that

$$|\tilde{\boldsymbol{II}}(\tilde{\boldsymbol{I}}^{-1})| \leq |\tilde{\boldsymbol{I}}|^{1/2}\left|\tilde{\boldsymbol{I}}^{-1}\right|^{1/2} \max\left\{\frac{1}{|\tilde{R}_1|}, \frac{1}{|\tilde{R}_2|}\right\} \leq \frac{1}{\varepsilon^2} \text{ a.e. in } \omega. \qquad \square$$

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