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On the topology of the set of singularities of a solution to the Hamilton–Jacobi equation

Sur la topologie des singularités d'une solution de l'équation de Hamilton–Jacobi

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A R T I C L E   I N F O

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A B S T R A C T

We address the topology of the set of singularities of a solution to a Hamilton–Jacobi equation. For this, we will apply the idea of the first two authors (Cannarsa and Cheng, Generalized characteristics and Lax–Oleinik operators: global result, preprint, arXiv:1605.07581, 2016) to use the positive Lax–Oleinik semi-group to propagate singularities.

R É S U M É

Nous étudions l'ensemble des singularités d'une solution de l'équation de Hamilton–Jacobi. Pour cette étude, nous utilisons une idée due aux deux premiers auteurs (Cannarsa and Cheng, Generalized characteristics and Lax–Oleinik operators: global result, preprint, arXiv:1605.07581, 2016) pour propager les singularités en utilisant le semi-groupe positif de Lax–Oleinik.

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Version française abrégée

Nous étudions la topologie de l'ensemble des singularités d'une solution de l'équation de Hamilton–Jacobi et leur propagation.

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Nous supposons connue la notion de solution de viscosité ainsi que la théorie KAM faible, voir [6], qui est bien adapté à nos besoins.

Soit $H : T^*M \to \mathbb{R}$ un hamiltonien de Tonelli sur la variété compacte connexe M. En particulier, le hamiltonien H est au moins C^2. On considère une solution de viscosité (ou KAM faible) $u : M \to \mathbb{R}$ de l'équation stationnaire de Hamilton–Jacobi

$$H(x, d_x u) = c[0], \tag{1}$$

où $c[0]$ est la constante critique de Mañé. Nous désignons par $\Sigma(u)$ l'ensemble des points $x \in M$ où u n'est pas différentiable, et par $\mathcal{I}(u)$ l'ensemble d'Aubry de u. Nous introduisons aussi l'ensemble $\mathrm{Cut}(u)$ des points de coupure de u comme étant l'ensemble des points $x \in M$ où aucune courbe caractéristique en temps négatif de u aboutissant en x ne peut être étendue au-delà de x en une courbe u-calibrée ; de manière équivalente, si $\gamma : [a, b] \to M$ est une courbe u-calibrée avec $x \in \gamma([a, b])$, alors $x = \gamma(b)$. On a $\Sigma(u) \subset \mathrm{Cut}(u) \subset M \setminus \mathcal{I}(u)$, et $\Sigma(u) \subset \mathrm{Cut}(u) \subset \overline{\Sigma(u)}$.

Théorème 0.1. *Les inclusions $\Sigma(u) \subset \mathrm{Cut}(u) \subset \overline{\Sigma(u)} \cap (M \setminus \mathcal{I}(u)) \subset M \setminus \mathcal{I}(u)$ sont toutes des équivalences d'homotopies.*

Il en résulte le corollaire suivant.

Corollaire 0.2. *Pour toute composante connexe C de $M \setminus \mathcal{I}(u)$, les trois intersections $\Sigma(u) \cap C$, $\mathrm{Cut}(u) \cap C$ et $\overline{\Sigma(u)} \cap C$ sont connexes par arcs.*

Théorème 0.3. *Les espaces $\Sigma(u)$, and $\mathrm{Cut}(u)$ sont localement contractiles, c'est-à-dire que pour tout $x \in \Sigma(u)$ (resp. $x \in \mathrm{Cut}(u)$) et tout voisinage V de x dans $\Sigma(u)$ (resp. $\mathrm{Cut}(u)$), on peut trouver un voisinage W de x dans $\Sigma(u)$ (resp. $\mathrm{Cut}(u)$), tel que $W \subset V$ et que W soit homotope à une constante dans V.*

Par conséquent, $\Sigma(u)$ et $\mathrm{Cut}(u)$ sont localement connexes par arcs.

Par soucis de brièveté et de clarté, nous avons énoncé nos résultats pour l'équation de Hamilton–Jacobi stationnaire (1). Toutefois, ils sont aussi valides pour l'équation de Hamilton–Jacobi sous forme évolution

$$\partial_t U + H(x, \partial_x U) = 0, \tag{2}$$

où $U : M \times [0, +\infty[\to \mathbb{R}$ est continue et est solution de viscosité de (2) sur $M \times]0, +\infty[$.

De plus, sous des conditions appropriées sur l'hamiltonien, les résultats sont encore valables quand M n'est pas compacte. Nous pouvons aussi considérer le cas des problèmes de Dirichlet.

Il est aussi possible d'utiliser nos méthodes pour obtenir les mêmes résultats pour la fonction distance à un fermé dans une variété riemannienne complète.

La version complète de ce travail comportera les détails nécessaires.

Notons que pour propager globalement les singularités, nous utilisons l'idée due aux deux premiers auteurs (Generalized characteristics and Lax–Oleinik operators: global result, preprint, arXiv:1605.07581, 2016) de se servir du semi-groupe positif de Lax–Oleinik. Notons aussi que, bien que les résultats et les démonstrations des deux théorèmes énoncés ci-dessus soient originaux, un cas particulier du premier théorème pour les fonctions distances sur les variétés riemanniennes était déjà connu, voir [1].

1. Introduction

We address the problem of propagation of singularities and the topology of the set of singularities of a viscosity solution to a Hamilton–Jacobi equation.

We assume familiarity with the notion of viscosity solution and weak KAM theory, see [6], which is relevant to our manifold framework. Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian on the compact connected manifold M; in particular, the Hamiltonian H is at least C^2. We consider $u : M \to \mathbb{R}$ a viscosity (or weak KAM) solution to the Hamilton–Jacobi equation

$$H(x, d_x u) = c[0], \tag{3}$$

where $c[0]$ is Mañé's critical value. We denote by $\Sigma(u)$ the set of points $x \in M$, where u is not differentiable, and by $\mathcal{I}(u)$ the Aubry set of u. We also introduce the set $\mathrm{Cut}(u)$ of cut points of u, as the set of points $x \in M$ where no backward characteristic for u ending at x can be extended to a u-calibrating curve beyond x; equivalently, if $\gamma : [a, b] \to M$ is a u-calibrating curve with $x \in \gamma([a, b])$ then $x = \gamma(b)$. We have $\Sigma(u) \subset \mathrm{Cut}(u) \subset M \setminus \mathcal{I}(u)$, and $\Sigma(u) \subset \mathrm{Cut}(u) \subset \overline{\Sigma(u)}$.

Theorem 1.1. *The inclusion $\Sigma(u) \subset \mathrm{Cut}(u) \subset \overline{\Sigma(u)} \cap (M \setminus \mathcal{I}(u)) \subset M \setminus \mathcal{I}(u)$ are all homotopy equivalences.*

This theorem obviously implies the following corollary (see, for instance, [5]).

Corollary 1.2. *For every connected component C of $M \setminus \mathcal{I}(u)$ the three intersections $\Sigma(u) \cap C$, $\mathrm{Cut}(u) \cap C$, and $\overline{\Sigma(u)} \cap C$ are path-connected.*

Theorem 1.3. *The spaces $\Sigma(u)$, and $\mathrm{Cut}(u)$ are locally contractible, i.e. for every $x \in \Sigma(u)$ (resp. $x \in \mathrm{Cut}(u)$) and every neighborhood V of x in $\Sigma(u)$ (resp. $\mathrm{Cut}(u)$), we can find a neighborhood W of x in $\Sigma(u)$ (resp. $\mathrm{Cut}(u)$), such that $W \subset V$, and W in null-homotopic in V.*

Therefore $\Sigma(u)$ and $\mathrm{Cut}(u)$ are locally path connected.

For the sake of brevity and clarity, we stated our result for the stationary Hamilton–Jacobi equation (3), but they are also valid for the Hamilton–Jacobi equation in its evolution form

$$\partial_t U + H(x, \partial_x U) = 0, \tag{4}$$

where $U : M \times [0, +\infty[\to \mathbb{R}$ is continuous, and a viscosity solution to (4) on $M \times]0, +\infty[$.

Moreover, with the appropriate condition on the Hamiltonian, the results are also valid when M is not compact. We can also deal with Dirichlet-type problems.

It is also possible, using our method, to obtain the same results for the distance function to a closed set in a complete Riemannian manifold.

The details will appear in the complete version of this work.

We would like to note here that to obtain the global propagation of singularities, see Lemma 2.1 below, we apply the idea of the first two authors [4] to use the positive Lax–Oleinik group to propagate singularities. We also note that although the results and proofs of Theorems 1.1 and 1.3 are new, a particular version of Theorem 1.1 for distance functions in Riemannian manifolds is already known, see [1].

2. Proof of the theorems

The proof of both theorems uses the following lemma.

Lemma 2.1. *There exists some $t > 0$, and a (continuous) homotopy $F : M \times [0, t] \to M$, with the following properties:*

(a) *for all $x \in M$, we have $F(x, 0) = x$;*
(b) *if $F(x, s) \notin \Sigma(u)$, for some $s > 0$, and $x \in M$, then the curve $\sigma \mapsto F(x, \sigma)$ is u-calibrating on $[0, s]$;*
(c) *if there exists a u-calibrating curve $\gamma : [0, s] \to M$, with $\gamma(0) = x$, then $\sigma \mapsto F(x, \sigma) = \gamma(\sigma)$, for every $\sigma \in [0, \min(s, t)]$.*

The proof of the lemma will be sketched in the next section.

We extend F to a homotopy $F : M \times [0, +\infty[\to M$, using induction on $n \geqslant 1$, by

$$F(x, s) = F(F(x, nt), s - nt), \ \text{for} \ s \in [nt, (n+1)t].$$

It is not difficult to check that this extended F has the same properties (a), (b), and (c) stated above.

These properties imply:

(1) $F(\mathrm{Cut}(u) \times]0, +\infty[) \subset \Sigma(u)$;
(2) if $F(x, s)$ never enters $\Sigma(u)$, then $x \in \mathcal{I}(u)$, and $s \mapsto F(x, s), s \in [0, \infty[$ is the forward calibrating curve through x;
(3) if $x \notin \mathcal{I}(u)$, then $F(x, s) \notin \mathcal{I}(u)$, for every $s \in [0, \infty[$.

Before proceeding further, we note that this extended homotopy F shows that we have propagation of singularities in infinite time. It is convenient to introduce the cut time function $\tau : M \to [0, +\infty]$ for u, where $\tau(x)$ is the supremum of the $t \geq 0$ such that there exists a u-calibrating curve $\gamma : [0, t] \to M$, with $\gamma(0) = x$. The properties of τ are:

(i) $\tau(x) = 0$ if and only if $x \in \mathrm{Cut}(u)$;
(ii) $\tau(x) = +\infty$ if and only if $x \in \mathcal{I}(u)$;
(iii) the function τ is upper semi-continuous.

By (b) of Lemma 2.1, we have $F(s, x) \in \Sigma(u)$, for all $s > \tau(x)$.

Proof of Theorem 1.1. The function τ is upper semi-continuous and finite on $M \setminus \mathcal{I}(u)$; therefore (by Proposition 7.20 in [3]) we can find a continuous function $\alpha : M \setminus \mathcal{I}(u) \to]0, +\infty[$, with $\alpha > \tau$ on $M \setminus \mathcal{I}(u)$. We now define $G : (M \setminus \mathcal{I}(u)) \times [0, 1] \to M \setminus \mathcal{I}(u)$ by

$$G(x, s) = F(x, s\alpha(x)).$$

The map G is a homotopy of $M\setminus\mathcal{I}(u)$ into itself starting with the identity, such that $G(M\setminus\mathcal{I}(u),1)\subset\Sigma(u)$, and $G(\mathrm{Cut}(u)\times]0,1])\subset\Sigma(u)$. It is not difficult to check that the time one map of G gives a homotopy inverse for each one of the inclusions $\Sigma(u)\subset\mathrm{Cut}(u)\subset\overline{\Sigma(u)}\cap(M\setminus\mathcal{I}(u))\subset M\setminus\mathcal{I}(u)$. □

Proof of Theorem 1.3. We first construct for every open subset $O\subset M\setminus\mathcal{I}(u)$, an open subset $\tilde{O}\subset O$, such that $\Sigma(u)\cap\tilde{O}=\Sigma(u)\cap O$ and $\mathrm{Cut}(u)\cap\tilde{O}=\mathrm{Cut}(u)\cap O$, together with a homotopy $G_O:\tilde{O}\times[0,1]\to O$, which satisfies:

(i) $G_O(x,0)=x$ for every $x\in\tilde{O}$;
(ii) $G_O\big((\mathrm{Cut}(u)\cap O)\times]0,1]\big)\subset\Sigma(u)\cap O=\Sigma(u)\cap\tilde{O}$;
(iii) $G_O(\tilde{O}\times\{1\})\subset\Sigma(u)\cap O=\Sigma(u)\cap\tilde{O}$.

To define $\tilde{O}$, we introduce the function $\eta_O:O\to[0,\infty]$ defined by

$$\eta_O(x)=\sup\{t\in[0,+\infty[:F(x,s)\in O,\ \text{for all } s\in[0,t]\}.$$

Since O is open and F is continuous, the function η_O is lower semi-continuous and everywhere >0 on O. Using that η_O is lower semi-continuous and the cut time function τ is upper semi-continuous, we conclude that the subset

$$\tilde{O}=\{x\in O:\tau(x)<\eta_O(x)\}\subset O$$

is indeed open. Furthermore, since τ is 0 on $\mathrm{Cut}(u)\supset\Sigma(u)$, we get $\Sigma(u)\cap\tilde{O}=\Sigma(u)\cap O$ and $\mathrm{Cut}(u)\cap\tilde{O}=\mathrm{Cut}(u)\cap O$.

It remains to construct the homotopy G_O. For this, we observe that $\tau<\eta_O$ on $\tilde{O}\subset O$, with τ upper semi-continuous, and η_O lower semi-continuous. Hence, Baire's interpolation theorem (Proposition 7.21 in [3] or Section VIII.4.3 in [5]) guarantees the existence of a continuous function $\alpha_O:\tilde{O}\to]0,+\infty[$ such that $\tau<\alpha_O<\eta_O$ everywhere on $\tilde{O}$. It is not difficult to check that the map $G_O:\tilde{O}\times[0,1]\to O$, defined by

$$G_O(x,s)=F(x,s\alpha_O(x)),$$

satisfies the required conditions (i), (ii), and (iii).

From the properties of G_O, we obtain that the identity on $\Sigma(u)\cap O=\Sigma(u)\cap\tilde{O}$ (resp. $\mathrm{Cut}(u)\cap O=\mathrm{Cut}(u)\cap\tilde{O}$) is homotopic to the restriction $G_{O,1}:\Sigma(u)\cap O\to\Sigma(u)\cap O$ (resp. $G_{O,1}:\mathrm{Cut}(u)\cap O\to\mathrm{Cut}(u)\cap O$) of the time one map of G_O as maps with values in $\Sigma(u)\cap O$ (resp. $\mathrm{Cut}(u)\cap O$). Let B be an open subset included in $\tilde{O}$, which is homeomorphic to an Euclidean ball. Since B is contractible, the restriction of $G_{O,1}$ to B is homotopic to a constant as maps from B to $\Sigma(u)\cap O$. Therefore the restriction of $G_{O,1}$ to $\Sigma(u)\cap B$ (resp. to $\mathrm{Cut}(u)\cap B$) is homotopic to a constant as maps with values in $\Sigma(u)\cap O$ (resp. in $\mathrm{Cut}(u)\cap O$). Since $\Sigma(u)\cap B\subset\Sigma(u)\cap O$ (resp. $\mathrm{Cut}(u)\cap B\subset\mathrm{Cut}(u)\cap O$), it follows that the inclusion $\Sigma(u)\cap B\hookrightarrow\Sigma(u)\cap O$ (resp. $\mathrm{Cut}(u)\cap B\hookrightarrow\mathrm{Cut}(u)\cap O$) is homotopic to a constant as maps with values in $\Sigma(u)\cap O$ (resp. in $\mathrm{Cut}(u)\cap O$). □

3. Proof of Lemma 2.1

By [2] we know that there exists $t>0$ such that T_s^+u is $C^{1,1}$ for all $s\in]0,t]$, where $T_s^+u:M\to\mathbb{R}$ is given by

$$T_s^+u(x):=\sup_{y\in M}\{u(y)-h_s(x,y)\}\tag{5}$$

and $h_s(x,y)$ is the minimal action of a curve joining x to y in time s.

The supremum defining $T_s^+u(x)$ in (5) is attained at some point y_s. We first show that y_s is unique. Indeed, if $\gamma:[0,s]\to M$ is a minimizer with $\gamma(0)=x$ and $\gamma(s)=y_s$, we know that

$$\frac{\partial L}{\partial v}(x,\dot\gamma(0))=d_xT_s^+u(x)\quad\text{and}\quad\frac{\partial L}{\partial v}(y_s,\dot\gamma(s))\in D^+u(y_s).\tag{6}$$

The first equality implies that y_s is unique. Therefore, we can define the map $F:M\times[0,t]\to M$ by $F(x,s)=y_s$. Continuity of F follows from compactness and the fact that the set

$$\{(x,y,s)\in M\times M\times[0,+\infty[\,:T_s^+u(x)=u(y)-h_s(x,y)\}$$

is closed.

Moreover, if $y_s\notin\Sigma(u)$, the second part of (6) implies

$$d_yu(y_s)=\frac{\partial L}{\partial v}(y_s,\dot\gamma(s)).$$

If we denote by $\tilde\gamma:]-\infty,0]\to M$ a u-calibrating curve such that $\tilde\gamma(0)=y_s$, we also know that

$$d_y u(y_s) = \frac{\partial L}{\partial v}(y_s, \dot{\tilde{\gamma}}(0)).$$

Since both γ and $\tilde{\gamma}$ are solutions to the Euler–Lagrange equation, we get $\gamma(\sigma) = \tilde{\gamma}(\sigma - s)$, for all $\sigma \in [0, s]$. Therefore, the minimizer $\gamma : [0, s] \to M$ is a u-calibrating curve with $\gamma(0) = x$.

To finish the proof, it remains to show that if $\gamma : [0, s] \to M$ is a u-calibrating curve, then

$$T^+_\sigma(\gamma(0)) = u(\gamma(\sigma)) - h_\sigma(\gamma(0), \gamma(\sigma)), \text{ for all } \sigma \in [0, s]. \tag{7}$$

Indeed, since u is a weak KAM solution, and γ is u-calibrating, we get $u(y) - u(\gamma(0)) \le h_\sigma(\gamma(0), y) + c[0]\sigma$, with equality at $y = \gamma(\sigma)$. Therefore, we conclude that $u(y) - h_\sigma(\gamma(0), y) \le u(\gamma(0)) + c[0]\sigma$, with equality at $y = \gamma(\sigma)$. This clearly proves (7).

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