C. R. Acad. Sci. Paris, Ser. I 355 (2017) 187–192

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial differential equations/Mathematical physics

Cauchy problem on a characteristic cone for the Einstein–Vlasov system: (I) The initial data constraints

Problème de Cauchy sur un cône caractéristique pour le système Einstein–Vlasov : (I) contraintes initiales

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ARTICLE INFO

Article history:
Received 16 July 2016
Accepted after revision 3 November 2016
Available online 18 January 2017

Presented by the Editorial Board

ABSTRACT

In this paper, one considers a Cauchy problem with data on a characteristic cone for the Einstein–Vlasov system in temporal gauge. One highlights gauge-dependent constraints that, supplemented by the standard constraints i.e. the Hamiltonian and the momentum constraints, define the full set of constraints for the considered setting. One studies their global resolution from a suitable choice of some free data, the behavior of the deduced initial data at the vertex of the cone, and the preservation of the gauge.

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RÉSUMÉ

Dans cet article, on considère le problème de Cauchy caractéristique sur un cône pour le système des équations d'Einstein–Vlasov en jauge temporelle. On met en évidence les contraintes dépendant de la jauge, qui ensemble avec les contraintes hamiltoniennes et impulsionnelles constituent l'ensemble des équations des contraintes pour le cadre considéré. On étudie la résolution globale de ces équations à partir de certaines données indépendantes, le comportement des données initiales ainsi déduites au voisinage du sommet du cône et la préservation de la jauge.

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Version française abrégée

Dans le cadre du problème de Cauchy caractéristique en relativité générale, les équations des contraintes se scindent en deux types : les contraintes classiques (hamiltoniennes et impulsionnelles) et d'autres contraintes, qui dépendent du choix de la jauge, du système d'évolution considéré et de la matière considérée, et dont la hiérarchie dépend aussi du choix des données initiales indépendantes prescrites sur l'hypersurface intiale adoptée. Dans cet article, on fait une distinction

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http://dx.doi.org/10.1016/j.crma.2016.11.018
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claire entre ces deux types de contraintes lorsque la jauge temporelle est considérée, l'hypersurface initiale est un cône caractéristique $\mathcal{C}$ (5) de sommet O, et la matière considérée est cinétique. L'analyse du problème des contraintes initiales se fait dans un système de coordonnées (y^μ) (7), adapté à la géométrie du cône. On montre (théorème 2.1) que pour toute solution $(\bar{g}, \rho)$ du système d'évolution $(H_{\bar{g}}, H_\rho)$ (1)–(2) satisfaisant sur le cône caractéristique les équations des contraintes (14)–(15), (g, ρ) est solution des équations d'Einstein–Vlasov où g est de la forme (6), à condition qu'une partie des équations d'Einstein et leurs dérivées soit satisfaite au sommet du cône. On résout (théorème 4.1) globalement les contraintes et on étudie le comportement des solutions au voisinage du sommet du cône. Les difficultés mentionnées en [4,15,18] lorsqu'on considère la jauge harmonique en présence du champ de Vlasov dans le cadre du problème de Cauchy caractéristique sont levées. La question laissée en suspens ($\mathcal{C}$ ayant une singularité à son sommet O) est celle de la détermination de la classe des données initiales indépendantes qui permet au final de résoudre le problème de Cauchy caractéristique de l'évolution associée à $(H_{\bar{g}}, H_\rho)$.

1. Introduction and various issues

There are mainly two types of Cauchy problems in general relativity: the ordinary space-like Cauchy problem for the Einstein equations, and the characteristic initial value problem for these same equations. In the first case, the constraints are standard, i.e. the Hamiltonian and the momentum constraints [3,5]. In the case of the characteristic Cauchy problem, the set of constraint equations includes the standard constraints and other ones that are gauge-dependent. These latter are induced by the evolution system considered, the free data, and the form of the stress-energy momentum tensor of the matter involved. The usual gauge for the characteristic Cauchy problem in general relativity is the harmonic gauge [4,6,8,9,13,15,17,18], which fits well to some types of matter. However, another gauge, which is now used and principally in vacuum, is the "double null foliation gauge" [2,14,11], see also [16]. For all these gauges, the question of the existence of global solutions for the constraint equations is of great interest [8,9,14]. As we are dealing here with kinetic matters, we opt for the temporal gauge [3,5,10,16]. This has a clear advantage for the initial data constraints problem. Indeed, the presence of all the components of the metric in each component of the momentum tensor of matter due to the Vlasov's field makes difficult the use of the Rendall's scheme [17] of resolution of the initial data constraints problem. Such difficulties are revealed in [4,15,18]. In the temporal gauge setting, the shift is null and the time is in wave gauge [3,5]. In a global set of coordinates $(x^\alpha) = (x^0, x^1, x^a)$, $(\alpha = 0, 1, ..., n;\ a = 2, ..., n)$ of $\mathbb{R}^{n+1} = \mathbb{R}^2 \times \mathbb{R}^{n-1}$, $(n \geq 3)$, this is equivalent to: $g_{0i} = 0$, $\Gamma^0 \equiv g^{\lambda\delta}\Gamma^0_{\lambda\delta} = 0$, $(i = 1, ..., n;\ \lambda, \delta = 0, ..., n)$, where g is the searched metric. The evolution system $(H_{\bar{g}}, H_\rho)$ induced by this gauge [3,5] is

$$H_{\bar{g}}:\ \partial_0 R_{ij} - \overline{\nabla}_i R_{j0} - \overline{\nabla}_j R_{i0} \ = \partial_0 \Lambda_{ij} - \overline{\nabla}_i \Lambda_{j0} - \overline{\nabla}_j \Lambda_{i0}; \tag{1}$$

$$H_\rho:\ p^\alpha \frac{\partial \rho}{\partial x^\alpha} - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial \rho}{\partial p^i} = 0; \tag{2}$$

with $\Lambda_{\mu\nu} = T_{\mu\nu} + \frac{g^{\lambda\delta}T_{\lambda\delta}}{1-n} g_{\mu\nu}$, and where the system $H_{\bar{g}}$ replaces the Einstein equations

$$H_g:\ G_{\mu\nu} \equiv R_{\mu\nu} - 2^{-1} g_{\mu\nu} R = T_{\mu\nu}, \tag{3}$$

and its principal part is $\Box \partial_0 \bar{g}_{ij}$, $\overline{\nabla}$ denotes the connection with respect to the induced metric $\bar{g}$ on $\Lambda_t : x^0 = t$. The Einstein equations H_g describe the gravitational potential g, while the Vlasov equation H_ρ gives a statistical description of a collection of particles of rest mass $\mathbf{m}$ and density $\rho \equiv \rho(x, p^0, p^i)$, which move towards the future $(p^0 > 0)$ in their mass shell $\mathbb{P} := \{(x, p^\mu) \in Y \times \mathbb{R}^{n+1} / g_{\mu\nu} p^\mu p^\nu = -\mathbf{m}^2,\ p^0 > 0\}$. The terms $R_{\mu\nu}$, R and $G_{\mu\nu}$ design respectively the components of the Ricci tensor, the scalar curvature, the components of the Einstein tensor G, relative to the searched metric g, while the $T_{\mu\nu}$ are the components of the stress-energy momentum tensor of matter. The $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols of g, the p^λ stand as the components of the momentum of the particles with respect to the basis $(\frac{\partial}{\partial x^\alpha})$ of the fiber $\mathbb{P}_x := \{(p^\alpha) \in \mathbb{R}^{n+1} / g_{\mu\nu}(x) p^\mu p^\nu = -\mathbf{m}^2,\ p^0 > 0\}$ of $\mathbb{P}$, and

$$T_{\alpha\beta} = -\int_{\{g(p,p) = -\mathbf{m}^2\}} \frac{\rho(x^\nu, p^\mu) p_\alpha p_\beta \sqrt{|g|}}{p^0} \, \mathrm{d}p^1 ... \mathrm{d}p^n. \tag{4}$$

In this note, we concentrate on the construction of the constraints to satisfy by a large class of initial data $(\bar{g}_0, k_0, \rho_0)$ on $\mathcal{C} \times \mathbb{R}^n$ with $\mathcal{C}$ of equation

$$\mathcal{C}:\ x^0 - r = 0,\ r := \sqrt{\sum_{i=1}^n (x^i)^2}, \tag{5}$$

such that, for any solution $(\bar{g}, \rho)$ of the evolution system $(H_{\bar{g}}, H_\rho)$ in a neighborhood $\widehat{Y}$ of $\mathcal{C} \times \mathbb{R}^n$ satisfying $\bar{g}_{|\mathcal{C}} = \bar{g}_0$, $(\partial_0 \bar{g})_{|\mathcal{C}} = k_0$, $\rho_{|\mathcal{C}} = \rho_0$, (g, ρ) is solution to the Einstein–Vlasov system (H_g, H_ρ) in $\widehat{Y}$, where g is of the form

$$g = -\tau^2 (\mathrm{d}x^0)^2 + \bar{g}_{ij} \mathrm{d}x^i \mathrm{d}x^j, \tag{6}$$

with $\tau^2 = (c(x^i))^2|\bar{g}|$, and c is determined by the prescribed data, such that $\Gamma^0 = 0$ in Y. One studies their global resolution from a suitable choice of some free data, the behavior of the deduced initial data, and the preservation of the gauge. The question left open is the study of the class of free data on a cone, which leads to a smooth solution to the Einstein–Vlasov system on a neighborhood of the vertex of the cone.

2. The characteristic initial data constraints on $\mathcal{C}$

To carry out the analysis, we introduce null adapted coordinates with respect to the trace of the metric on the cone $\mathcal{C}$ (y^0, y^1, y^A) [4,6,8,9] defined by

$$y^0 = x^0 - r,\ y^1 = r = \sqrt{\sum_{i=1}^{n}(x^i)^2},\ y^A,\ A = 2, ..., n; \tag{7}$$

where (y^A) design local coordinates in the sphere $\mathcal{S}^{n-1} : \sum_{i=1}^{n}(x^i)^2 = 1$, then

$$x^0 = y^0 + y^1,\ x^i = y^1\theta^i(y^A),\ \sum_i(\theta^i)^2 = 1; \tag{8}$$

and the $\theta^i(y^A)$ are $\mathcal{C}^\infty$ functions of y^A, $A = 2, ..., n$; and require the assumption:

$$(\mathbf{A}) : \text{The vector fields } \frac{\partial}{\partial y^1} \text{ is tangent to the null geodesics generating } \mathcal{C}, \tag{9}$$

which is an aspect of the affine parameterization condition of [4,6,8–10], and is equivalent to the requirement that on the cone the lapse τ^{-2} is an eigenvalue of the Riemannian metric $\bar{g} = (g^{ij})$, the corresponding eigenvector being $-q = (-q_i)$, $q_i = -\frac{x^i}{r}$. The components of tensors in coordinates (y^μ) are equipped with a tilde $(\widetilde{\ })$. The assumption $(\mathbf{A})$ induces that the trace on the null cone $\mathcal{C} : y^0 = 0$, of the searched metric in temporal gauge is of the form

$$g_{|C} = \widetilde{g}_{01}\mathrm{d}y^0\mathrm{d}y^0 + \widetilde{g}_{01}(\mathrm{d}y^0\mathrm{d}y^1 + \mathrm{d}y^1\mathrm{d}y^0) + \widetilde{g}_{AB}\mathrm{d}y^A\mathrm{d}y^B. \tag{10}$$

The characteristic initial data constraints in this setting is obtained in coordinates (y^μ) as follows. First, consider the Hamiltonian and momentum constraints $\widetilde{X}_{1\mu} = 0$, $\mu = 0, ..., n$, where $\widetilde{X}_{\mu\nu} \equiv \widetilde{G}_{\mu\nu} - \widetilde{T}_{\mu\nu}$, with respect to the restrictions to $\mathcal{C}$ of the components of the Ricci tensor, these constraints comprise naturally only the Cauchy data for the evolution system $(H_{\bar{g}}, H_\rho)$. The others equations

$$\widetilde{X}_{00} = 0,\ \widetilde{X}_{0A} = 0,\ \widetilde{X}_{AB} = 0,\ A, B = 2, ..., n; \tag{11}$$

do not play the role of constraints as their expressions on $\mathcal{C}$ contain second-order outgoing derivatives of the metric that are not part of the initial data of the third-order characteristic problem for the evolution system $(H_{\bar{g}}, H_\rho)$. In particular, the following expressions reveal harmful terms that we have to deal with, and the terms not written explicitly are harmless:

$$[\widetilde{X}_{00}] = \frac{1}{2}\widetilde{g}^{01}\widetilde{g}^{AB}[\partial^2_{00}\widetilde{g}_{AB}] + H_1,\ [\widetilde{X}_{AB}] = \frac{1}{2}(\widetilde{g}^{01})^2[\partial^2_{00}\widetilde{g}_{11}]\widetilde{g}_{AB} + H_2, \tag{12}$$

$$\left[\frac{\partial}{\partial y^0}(\widetilde{\Gamma}^0 + \widetilde{\Gamma}^1)\right] = \frac{1}{2}(\widetilde{g}^{01})^2[\partial^2_{00}\widetilde{g}_{11}] - \frac{1}{2}\widetilde{g}^{01}\widetilde{g}^{AB}\partial^2_{00}\widetilde{g}_{AB} + H_3. \tag{13}$$

Appropriate modifications or combinations of these relations result in the following theorem.

Theorem 2.1. *Let $(\bar{g}, \rho)$ be any $\mathcal{C}^\infty$ solution to the evolution system $(H_{\bar{g}}, H_\rho)$ in a neighborhood $\mathcal{V}$ of $\mathcal{C} \times \mathbb{R}^n$, and let g associated with $\bar{g}$ of the form (6) such that the temporal gauge condition is satisfied in $Y_O = \{y^0 \geq 0\}$. One sets $\widetilde{X}_{\mu\nu} \equiv \widetilde{G}_{\mu\nu} - \widetilde{T}_{\mu\nu}$, and one assumes that, with respect to the metric g, the hypothesis $(\mathbf{A})$ (9) and the relations*

$$\widetilde{X}_{1\lambda} = 0, \lambda = 0, ..., n, \tag{14}$$

$$\widetilde{X}_{AB} - \frac{\widetilde{g}^{CD}\widetilde{X}_{CD}}{n-1}\widetilde{g}_{AB} = 0,\ \widetilde{X}_{00} - \widetilde{g}_{01}\frac{\widetilde{g}^{CD}\widetilde{X}_{CD}}{n-1} + \widetilde{g}_{01}\frac{\partial(\widetilde{\Gamma}^0 + \widetilde{\Gamma}^1)}{\partial y^0} = 0,\ A, B, C, D = 2, ..., n; \tag{15}$$

are satisfied on $\mathcal{C}$; if furthermore one has $X_{0k}(O) = 0$, $k = 1, ..., n$; $(\partial_0 X_{0s})(O) = 0$; $s = 1, ..., n$, $s \neq s_0$, $q^{s_0}(O) \neq 0$; then (g, ρ) is solution to the Einstein–Vlasov system (H_g, H_ρ) in Y_O.

Proof. Since $(\bar{g}, \rho)$ is a $\mathcal{C}^\infty$ solution to the evolution system $(H_{\bar{g}}, H_\rho)$, and according to the divergence free properties of the Einstein tensor $(G_{\mu\nu})$ and the stress energy momentum tensor of matter $(T_{\mu\nu})$ of g (6), one has:

$$\nabla^\alpha X_{\alpha\beta|C} = 0,\ \left(\partial_0(R_{ij} - \Lambda_{ij}) - \overline{\nabla}_i X_{j0} - \overline{\nabla}_j X_{i0}\right)_{|C} = 0. \tag{16}$$

These relations induce on $\mathcal{C}$ partial differential relations in terms of $[X_{\mu\nu}] = X_{\mu\nu}{}_{|\mathcal{C}}$, i.e.:

$$q^j \partial_j [X_{k0}] + q^j \partial_k [X_{j0}] - q_k g^{lm} \partial_l [X_{m0}] + g^{ij} \partial_i [X_{jk}] + A_k^{\mu\nu} [X_{\mu\nu}] = 0. \tag{17}$$

Now, if the relations (9), (14), (15) are satisfied for g of the form (6), then on $\mathcal{C}$ one has:

$$X_{00} = \widetilde{g}_{01} \frac{\widetilde{g}^{CD} \widetilde{X}_{CD}}{n-1} = -\widetilde{g}_{01} q^s X_{0s}, \ X_{0k} = \widetilde{g}_{01} \frac{\widetilde{g}^{CD} \widetilde{X}_{CD}}{n-1} q_k + \frac{\partial y^A}{\partial x^k} \widetilde{X}_{0A}; \tag{18}$$

$$X_{jk} = \left(q_j \frac{\partial y^A}{\partial x^k} + q_k \frac{\partial y^A}{\partial x^j} \right) \widetilde{X}_{0A} + (2\widetilde{g}_{01} q_j q_k + g_{jk}) \frac{\widetilde{g}^{CD} \widetilde{X}_{CD}}{n-1}; \ A, B, C, D = 2, ..., n; \tag{19}$$

$$X_{jk} = q_j X_{0k} + q_k X_{0j} - g_{jk} q^s X_{0s}. \tag{20}$$

By combining the relations (18)–(20) and (17), one obtains on $\mathcal{C}$ the homogeneous linear differential system

$$\frac{\partial [X_{0k}]}{\partial y^1} + L_k^s([X_{0s}]) = 0, \ k = 1, ..., n. \tag{21}$$

One deduces that if the subset $X_{0k} = 0$, $k = 1, ..., n$ of the Einstein equations is satisfied at the vertex O of the cone, then $X_{0k} = 0$; $k = 1, ..., n$, on $\mathcal{C}$, and after that $X_{\mu\nu} = 0$, $\forall \mu, \nu$, on $\mathcal{C}$, thanks to the relations (18)–(20). Now, to prove that $\partial_0 X_{\mu\nu} = 0$ on $\mathcal{C}$, one restricts to $\mathcal{C}$ the following homogeneous linear system satisfied by the $X^{\mu\nu}$ in Y_O (see [5], pp. 407–414):

$$\partial_0 X^{00} + L^{00}(X^{\gamma\alpha}, \partial_i X^{i0}) = 0 \tag{22}$$

$$\partial_0 X^{ij} + L^{ij}(X^{\gamma\alpha}, \partial_s X^{k0}) = 0 \tag{23}$$

$$\Box_g X^{0j} + L^{0j}(X^{\gamma\alpha}, \partial_s X^{\delta\beta}) = 0. \tag{24}$$

By combining these restrictions, the system (24) restricted to $\mathcal{C}$ appears as a homogeneous linear differential system of propagation equations on $\mathcal{C}$ of unknowns $[\partial_0 X^{0i}]$, $i = 1, ..., n$. As a consequence, in virtue of the Bianchi relations $\nabla_\alpha X^{\alpha i} = 0$, $i = 1, ..., n$ and the evolution system $H_{\bar{g}}$ written at O, it suffices that $\partial_0 X_{0k}(O) = 0$, $k = 1, ..., n$, $k \neq k_0$, $q^{k_0}(O) \neq 0$ so that at O $\partial_0 X^{\mu\nu} = 0$. Finally $X^{\mu\nu} = 0$ in Y_O thanks to another linear homogeneous hyperbolic system in Y_O, derived from (22)–(24), which is of principal part $\Box \partial_0 X^{\mu\nu}$ (see [5], pp. 407–414), provided that $X_{0k}(O) = 0$, $k = 1, ..., n$; $\partial_0 X_{0k}(O) = 0$, $k \neq k_0$, where $q^{k_0}(O) \neq 0$. ■

3. Constraints and Cauchy data for $(H_{\bar{g}}, H_\rho)$

An exhaustive description of the constraints in terms of the Cauchy data for the evolution system $(H_{\bar{g}}, H_\rho)$ is as follow, where the terms not written explicitly will be known quantities at the corresponding order in the scheme of resolution of the constraints. Indeed, setting $\widetilde{g}_{01|\mathcal{C}} = \theta$, $\widetilde{g}_{AB|\mathcal{C}} = \Theta_{AB}$, $\Theta = (\Theta_{AB})$, $\rho_{|\mathcal{C}} = \mathbf{f}$, $\psi_{\mu\nu} = \frac{\partial \widetilde{g}_{\mu\nu}}{\partial y^0}{}_{|\mathcal{C}}$, $\partial_\mu = \frac{\partial}{\partial y^\mu}$, $\pi^\alpha = \frac{\partial y^\alpha}{\partial x^\delta} p^\delta$, $\mathrm{d}\pi = \mathrm{d}\pi^1 ... \mathrm{d}\pi^n$, the Hamiltonian constraint $\widetilde{X}_{11} = 0$ and the momentum constraints $\widetilde{X}_{1A} = 0$, $A = 2, ..., n$ reduce on $\mathcal{C}$ to the following partial differential relations in terms of the Cauchy data for $(H_{\bar{g}}, H_\rho)$:

$$\theta^{-1}(\Theta^{AB} \partial_1 \Theta_{AB}) \psi_{11} + 2\partial_1(\Theta^{AB} \partial_1 \Theta_{AB}) + (\Theta^{CB} \partial_1 \Theta_{BD})(\Theta^{DE} \partial_1 \Theta_{CE}) - 2\theta^{-1}(\Theta^{AB} \partial_1 \Theta_{AB}) \partial_1 \theta$$
$$= 4\widetilde{T}_{11} = -4 \int_{\mathbb{R}^n} \frac{\mathbf{f} \, |\theta|^3 \left(\sqrt{(\pi^1)^2 - \theta^{-1}(\mathbf{m}^2 + \Theta_{AB} \pi^A \pi^B)} - \pi^1 \right)^2 \sqrt{|\Theta|}}{\sqrt{(\pi^1)^2 - \theta^{-1}(\mathbf{m}^2 + \Theta_{AB} \pi^A \pi^B)}} \mathrm{d}\pi; \tag{25}$$

$$\partial_1 \psi_{1A} + (2^{-1} \Theta^{AB} \partial_1 \Theta_{AB} - \theta^{-1} \partial_1 \theta) \psi_{1A} + H_{1A} = 2\theta \widetilde{T}_{1A}, \ A = 2, ..., n. \tag{26}$$

Now setting $\chi = \Theta^{AB} \psi_{AB}$, the momentum constraint $\widetilde{X}_{01} = 0$, which is equivalent to $\widetilde{g}^{AB} \widetilde{R}_{AB} = \widetilde{g}^{01}(\widetilde{R}_{11} - 2\widetilde{T}_{01})$ is in turn a partial differential relation on $\mathcal{C}$ of the form

$$\partial_1 \chi + 2^{-1}(\Theta^{AB} \partial_1 \Theta_{AB} - \theta^{-1} \psi_{11}) \chi + H_\chi = \widetilde{R}_{11} - 2\widetilde{T}_{01}. \tag{27}$$

For the constraints $\widetilde{X}_{AB} - \frac{\widetilde{g}^{CD} \widetilde{X}_{CD}}{n-1} \widetilde{g}_{AB} = 0$, they are equivalent to $\widetilde{R}_{AB} - \widetilde{T}_{AB} - \frac{\widetilde{g}^{CD}(\widetilde{R}_{CD} - \widetilde{T}_{CB})}{n-1} \widetilde{g}_{AB} = 0$, and agree on $\mathcal{C}$ with the form

$$\partial_1 \psi_{AB} - 2^{-1}(2^{-1} \theta^{-1} \psi_{11} \delta_A^C \delta_B^D + \Theta^{ED} \partial_1 \Theta_{EB} \delta_A^C + \Theta^{ED} \partial_1 \Theta_{DA} \delta_B^C) \psi_{CD} + 2^{-1}(\partial_1 \Theta_{AB}) \chi + H_{AB}$$
$$= \widetilde{T}_{AB} + \frac{\Theta^{CD}(\widetilde{R}_{CD} - \widetilde{T}_{CB})}{n-1} \Theta_{AB}, \ A, B, C, D, E = 2, ..., n. \tag{28}$$

We remark that similar constraints appear in the case of the double null foliation gauge [2,14,16]. The last constraint in our scheme corresponds to $\widetilde{X}_{00} - \widetilde{g}_{01}\frac{\widetilde{g}^{CD}\widetilde{X}_{CD}}{n-1} + \widetilde{g}_{01}\frac{\partial(\widetilde{\Gamma}^0+\widetilde{\Gamma}^1)}{\partial y^0} = 0$, $A, B, C, D = 2, ..., n$; it is equivalent to $\widetilde{R}_{00} - \widetilde{T}_{00} - \widetilde{g}_{01}\frac{\widetilde{g}^{AB}(\widetilde{R}_{AB}-\widetilde{T}_{AB})}{n-1} + \widetilde{g}_{01}\frac{\partial(\widetilde{\Gamma}^0+\widetilde{\Gamma}^1)}{\partial y^0} = 0$, and reads finally:

$$\partial_1\psi_{01} + 2^{-1}(2^{-1}\Theta^{AB}\partial_1\Theta_{AB} - \theta^{-1}\psi_{11})\psi_{01} + H_{01} = \widetilde{T}_{00} + \widetilde{g}_{01}\frac{\widetilde{g}^{AB}(\widetilde{R}_{AB} - \widetilde{T}_{AB})}{n-1}. \tag{29}$$

4. On the resolution of the constraints

The cone $\mathcal{C}$ admits the equation $x^0 - \sqrt{\sum_i (x^i)^2} = 0$ in coordinates (x^μ), and this induces that a regular metric in a neighborhood of the cone and Minkowskian at O must satisfy the expansion

$$g = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = (\eta_{\mu\nu} + O(r^2))\mathrm{d}x^\mu\mathrm{d}x^\nu. \tag{30}$$

Its behavior at O (see also [6] for more details) in coordinates (y^μ) is thus:

$$\widetilde{g}_{01} = -1 + O(r^2),\ \widetilde{g}_{AB} = r^2\sum_i \frac{\partial\theta^i}{\partial y^A}\frac{\partial\theta^i}{\partial y^B} + O(r^4),\ \widetilde{g}^{AB} = \sum_s \frac{\partial y^A}{\partial x^s}\frac{\partial y^B}{\partial x^s} + O(r). \tag{31}$$

The free data

In order to solve the constraints (25)–(29), the prescribed free data comprise:

(**a**) $\mathcal{C}^\infty$ functions $\widetilde{\gamma}_{AB} = \widetilde{\gamma}_{AB}(y^i)$ that make up (for $y^1 \neq 0$) a symmetric positive definite matrix, and satisfy

$$\left|\widetilde{\gamma}^{AB}\frac{\partial\widetilde{\gamma}_{AB}}{\partial y^1}\right| > 0,\ y^1 \neq 0; \tag{32}$$

(**b**) a smooth function $\theta \equiv \theta(y^i)$ on $\mathcal{C}$, and $\mathbf{f} \equiv \mathbf{f}(y^i, \pi^j)$ on $\mathcal{C} \times \mathbb{R}^n$ such that θ is negative, $\mathbf{f}$ is non-negative of compact support contained in $\{\pi^1 > c_1 > 0\}$ for a mass $\mathbf{m} \neq 0$; and for a zero mass the support of $\mathbf{f}$ is contained in $\{\pi^1 > c_1 > 0, \sum_{A=2}^n (\pi^A)^2 > c_2 > 0\}$, besides that, $\mathrm{Supp}(\mathbf{f}) \cap (\{O\} \times \mathbb{R}^n) = \varnothing$. These free data satisfy

$$\theta = -1 + O(r^2),\ \widetilde{\gamma}_{AB} = r^2\sum_i \frac{\partial\theta^i}{\partial y^A}\frac{\partial\theta^i}{\partial y^B} + O(r^4),\ \widetilde{\gamma}^{AB} = \sum_s \frac{\partial y^A}{\partial x^s}\frac{\partial y^B}{\partial x^s} + O(r). \tag{33}$$

Remark 1. Given a metric $\widetilde{\gamma} = (\widetilde{\gamma}_{AB})$, a class of metric $(e^\omega\gamma_{AB})$ conformal to $\widetilde{\gamma}$ and satisfying the condition (32) can be determined; for instance, it suffices to define ω as

$$\omega(y^i) = -\frac{1}{n-1}\int_0^{y_1}(\widetilde{\gamma}^{AB}\partial_1\widetilde{\gamma}_{AB} - \omega_0)(\lambda, y^A)\mathrm{d}\lambda \tag{34}$$

for any function $\omega_0(y^1, y^A)$ such that $|\omega_0| > 0$, $y^1 \neq 0$. We note, however, that the condition (32) is not a necessary condition in the scheme of resolution of the constraints provided some function ω_0 is prescribed freely on $\mathcal{C}$.

Theorem 4.1. *Given the free data as described above by (**a**)–(**b**), there exists a unique global solution* $(\theta,\ \Theta_{AB},\ \psi_{1\nu},\ \psi_{AB},\ \mathbf{f})$ *on* $\mathcal{C} \times \mathbb{R}^n$ *of the initial data constraints (25)–(29) for the Einstein–Vlasov system.*

Proof. Given the free data (**a**)–(**b**), one solves the constraints in a hierarchical scheme. Indeed, setting $\Theta_{AB}(y^1, y^A) = \widetilde{\gamma}_{AB}(y^1, y^A)$, then $|\Theta^{AB}\partial_1\Theta_{AB}| > 0$, and ψ_{11} solves algebraically the Hamiltonian constraint $\widetilde{X}_{11} = 0$ as described by (25), with $\psi_{11} = O(y^1)$. If condition (32) is not satisfied and if a function ω_0 is given on $\mathcal{C}$, one sets $\psi_{11} = \omega_0$ and solves the constraint $\widetilde{X}_{11} = 0$ (25) in term of the conformal factor ω (as in [17]) s.t. $\Theta_{AB} = e^\omega\widetilde{\gamma}_{AB}$; in this case, the solution is not necessarily global. For the constraints $\widetilde{X}_{1A} = 0$ (26) and $\widetilde{X}_{01} = 0$ (27), they can be written in the forms:

$$\frac{\mathrm{d}\psi_{1A}}{\mathrm{d}y^1} + \frac{(n-1)}{y^1}\psi_{1A} + \psi_A(y^i, \psi_{1C}) = 0,\ \frac{\mathrm{d}\chi}{\mathrm{d}y^1} + \frac{(n-1)}{y^1}\chi + \psi(y^i, \chi) = 0. \tag{35}$$

Their solutions satisfy the integral systems

$$\psi_{1A} = \frac{1}{y^1}\int_0^{y^1}[-\lambda\psi_A + (2-n)\psi_{1A}](\lambda, y^A)\mathrm{d}\lambda,\ \chi = \frac{1}{y^1}\int_0^{y^1}[-\lambda\psi + (2-n)\chi](\lambda, y^A)\mathrm{d}\lambda. \tag{36}$$

Since the functions under the integral's sign are continuous w.r.t. to y^1 and Lipschitzian w.r.t. the corresponding unknowns, the solutions to the systems (35) exist, are unique and global thanks to linearity, furthermore $\psi_{1A} = O((y^1)^2)$, $\chi = O(y^1)$. One first solves the system in ψ_{1A} and, after that, the equation regarding χ. Now, we consider the constraints $Z_{AB} \equiv \widetilde{X}_{AB} - \frac{\widetilde{g}^{CD}\widetilde{X}_{CD}}{n-1}\widetilde{g}_{AB} = 0$ described in (28) for $(A, B) \neq (2, 2)$ since (Z_{AB}), $(A, B = 2, ..., n)$ is a traceless tensor, of unknowns ψ_{AB} for $(A, B) \neq (2, 2)$ provided ψ_{22} takes the value $\psi_{22} = \frac{1}{\Theta^{22}}(\chi - \sum_{(A,B)\neq(2,2)} \Theta^{AB}\psi_{AB})$, and where $\Theta^{AB}\widetilde{R}_{AB} \equiv \widetilde{R}^{(n-1)}$ equals $\frac{(\widetilde{T}_{11}-2\widetilde{T}_{01})}{\theta}$ since $\widetilde{X}_{01} = 0$ is satisfied. This system is of the form

$$\frac{d\psi_{AB}}{dy^1} - \frac{2}{y^1}\psi_{AB} + \psi'_{AB}(y^i, \psi_{BC}) = 0; \tag{37}$$

its solution is unique, global, satisfies the integral system $\psi_{AB} = \frac{1}{y^1}\int_0^{y^1}[-\lambda\psi'_{AB} + 3\psi_{AB}](\lambda, y^A)d\lambda$, and $\psi_{AB} = O((y^1)^3)$. That $Z_{22} = 0$ is also satisfied with $\psi_{22} = \frac{1}{\Theta^{22}}(\chi - \sum_{(a,b)\neq(2,2)} \Theta^{ab}\psi_{ab})$ follows from the traceless property of (Z_{ab}). One also has $\psi_{22} = O((y^1)^3)$. The last constraint (29) has the form

$$\frac{d\psi_{01}}{dy^1} + \frac{n-1}{2\,y^1}\psi_{01} + \psi'_{01}(y^i, \psi_{01}) = 0, \tag{38}$$

its solution is unique, global, agrees with the integral equation $\psi_{01} = \frac{1}{y^1}\int_0^{y^1}[-\lambda\psi'_{01} + \frac{(3-n)}{2}\psi_{01}](\lambda, y^A)d\lambda$, and $\psi_{01} = O(y^1)$. ■

5. On the evolution system $(H_{\bar{g}}, H_\rho)$

The treatment of the evolution system requires more analysis of the free data, namely their behavior near O, and obviously the behavior near the vertex O of the constraints's solutions. We remark that for this purpose, the lapse is related to the metric $\bar{g}$ by the relation $\tau = \frac{|\frac{D(x)}{D(y)}|}{\sqrt{|\widetilde{\gamma}|}}\sqrt{|\bar{g}|}$. It would be interesting to study the class of free data that induces the constraints's solutions which are trace on $\mathcal{C}$ of $\mathcal{C}^\infty$ functions in a neighborhood of $\mathcal{C}$. This will be done in a subsequent work. Another more technical issue would be to derive the above results under lower regularity assumptions. The construction of a large class of initial data sets offers also here the possibility to study without any symmetry assumptions the global future evolution of small data with appropriate fall-off behavior at infinity. The strong nonlinear features of the Einstein equations require one to rely on a quite rigid analytic approach based on energy estimates and other many tools; for this, a general review of methods for global existence of D. Christodoulou and S. Klainerman [7], S. Klainerman and F. Nicolo [11], H. Linblad and I. Rodniansky [13], Y. Choquet-Bruhat [3], L. Bieri and N. Zipser [1], and now P.G. LeFloch and Y. Ma [12] appears as a start point.

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