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Continuity properties of certain weighted log canonical thresholds

Propriétés de continuité de certains seuils log canoniques pondérés

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ABSTRACT

In this note, we prove a semicontinuity theorem for a class of weighted log canonical thresholds, and obtain some related results for restrictions of plurisubharmonic functions to k-dimensional subspaces and for multiplier ideal sheaves.

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RÉSUMÉ

Dans cette note, nous démontrons un théorème de semi-continuité pour une classe de seuils log-canoniques pondérés et obtenons des résultats connexes pour des restrictions de fonctions plurisubharmoniques à des sous-espaces k-dimensionnels et pour des faisceaux d'idéaux multiplicateurs.

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1. Introduction and main results

Let Ω be a domain in $\mathbb{C}^n$ and let φ be in the set PSH(Ω) of plurisubharmonic functions on Ω. Following Demailly and Kollár [7], we introduce the log canonical threshold of φ at point $0 \in \Omega$:

$$c(\varphi) = \sup\left\{c > 0 : \ \mathrm{e}^{-2c\varphi} \text{ is } L^1(\mathrm{d}V_{2n}) \text{ on a neighborhood of } 0\right\} \in (0, +\infty],$$

where $\mathrm{d}V_{2n}$ denotes the Lebesgue measure in $\mathbb{C}^n$. It is an invariant of the singularity of φ at 0. We refer to [1,3,4,6–8,11,12,15,16] for further information about this number.

For every non-negative Radon measure μ on a neighborhood of $0 \in \mathbb{C}^n$, we introduce the *weighted log canonical threshold* of φ with weight μ at 0 to be:

$$c_\mu(\varphi) = \sup\left\{c \geq 0 : \ \mathrm{e}^{-2c\varphi} \text{ is } L^1(\mu) \text{ on a neighborhood of } 0\right\} \in [0, +\infty].$$

In this note, we study the quantity

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$$(-n, +\infty) \times \mathrm{PSH}(\Omega) \ni (t, \varphi) \to c_{\|z\|^{2t} \mathrm{d}V_{2n}}(\varphi)$$

as a function of the two arguments (t, φ). The main results are contained in the following theorems.

Theorem 1.1. *Let $\{\varphi_j\}_{j\geq 1} \subset \mathrm{PSH}(\Omega)$ and $\varphi \in \mathrm{PSH}(\Omega)$ be such that $\varphi_j \to \varphi$ in $L^1_{\mathrm{loc}}(\Omega)$. Then*

$$\liminf_{j\to\infty} c_{\|z\|^{2t} \mathrm{d}V_{2n}}(\varphi_j) \geq c_{\|z\|^{2t} \mathrm{d}V_{2n}}(\varphi), \quad \forall t \in (-n, 1].$$

As in [13], we denote by $\mathcal{I}(\varphi)$ the sheaf of germs of holomorphic functions $f \in \mathcal{O}_{\mathbb{C}^n,z}$ such that

$$\int_U |f|^2 \mathrm{e}^{-2\varphi} < +\infty$$

on some neighborhood U of z. This is a coherent ideal sheaf over Ω (see [13]). Moreover, Theorem 1.1 and the main result of [10] imply as a consequence the following corollary.

Corollary 1.2. *Let $\{\varphi_j\}_{j\geq 1} \subset \mathrm{PSH}(\Omega)$ and $\varphi \in \mathrm{PSH}(\Omega)$ be such that $\varphi_j \to \varphi$ in $L^1_{\mathrm{loc}}(\Omega)$. Then the two following statements hold true:*

i) *if $\varphi_j \leq \varphi$ for all $j \geq 1$, then for $\Omega' \Subset \Omega$ there exists $j_0 \geq 1$ such that $\mathcal{I}(\varphi_j) = \mathcal{I}(\varphi)$ on Ω' for all $j \geq j_0$;*
ii) *if $\{z_1, ..., z_n\} \in \mathcal{I}(\varphi)_0$, then there exists $j_0 \geq 1$ such that $\{z_1, ..., z_n\} \in \mathcal{I}(\varphi_j)_0$ for all $j \geq j_0$.*

For $1 \leq k \leq n$, we denote

$$c_k(\varphi) = \sup\{c(\varphi_H) : \textit{ when } H \textit{ runs over all } k\textit{-dimensional linear subspaces through } 0\},$$

$$\tilde{c}_k(\varphi) = \sup\{c(\varphi_H) : \textit{ for all germs of smooth submanifolds } H \textit{ of dimension } k \textit{ through } 0\},$$

where φ_H is the restriction of φ to H.

Theorem 1.3. *Let $\varphi \in \mathrm{PSH}(\Omega)$. Then*

$$\tilde{c}_k(\varphi) = c_k(\varphi) = c_{\|z\|^{2(k-n)} \mathrm{d}V_{2n}}(\varphi).$$

Remark 1.4.

i) Consider $\varphi_j, \varphi \in \mathrm{PSH}(\Omega)$, $t \in \mathbb{R}$, $c \geq 0$ and a holomorphic function f on Ω such that $\varphi_j \leq \varphi$, $\varphi_j \to \varphi$ in $L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \mathrm{e}^{-2c\varphi} |f|^{2t} \mathrm{d}V_{2n} < +\infty.$$

Then $\mathrm{e}^{-2c\varphi_j} |f|^{2t} \to \mathrm{e}^{-2c\varphi} |f|^{2t}$ in $L^1_{\mathrm{loc}}(\Omega)$. Indeed, let $m \in \mathbb{N}$ be such that $m \geq t$. We have:

$$\int_\Omega \mathrm{e}^{-2c\varphi - 2(m-t)\log|f|} |f|^{2m} \mathrm{d}V_{2n} = \int_\Omega \mathrm{e}^{-2c\varphi} |f|^{2t} \mathrm{d}V_{2n} < +\infty.$$

By the main theorem in [10], we get that

$$\mathrm{e}^{-2c\varphi_j - 2(m-t)\log|f|} |f|^{2m} \to \mathrm{e}^{-2c\varphi - 2(m-t)\log|f|} |f|^{2m},$$

in $L^1_{\mathrm{loc}}(\Omega)$. This implies that $\mathrm{e}^{-2c\varphi_j} |f|^{2t} \to \mathrm{e}^{-2c\varphi} |f|^{2t}$ in $L^1_{\mathrm{loc}}(\Omega)$.
ii) The semicontinuity theorem for the weighted log canonical thresholds is not true in the case of the measure $\mu = |z_1|^2 \mathrm{d}V_{2n}$ without the condition $\varphi_j \leq \varphi$. Indeed, as in Remark 1.3 of [10], we can choose $\varphi(z) = \log|z_1|$ and $\varphi_j(z) = \log|z_1 + \frac{z_2}{j}|$ for $j \geq 1$. One has $\varphi_j \to \varphi$ in $L^1_{\mathrm{loc}}(\mathbb{C}^n)$, however $\forall j \geq 1$, we find $c_\mu(\varphi_j) = 1 < c_\mu(\varphi) = 2$.

Remark 1.5. Hölder's inequality implies that the function

$$(-n, +\infty) \ni t \to c_{\|z\|^{2t} \mathrm{d}V_{2n}}(\varphi)$$

is concave and increasing for all $\varphi \in \mathrm{PSH}(\Omega)$. In particular, this function is continuous and increasing in t for all $\varphi \in \mathrm{PSH}(\Omega)$. Moreover, by Theorem 1.3, we obtain inequalities similar to the ones proved in [9]:

$$c_k(\varphi) - c_{k-1}(\varphi) \leq c_{k-1}(\varphi) - c_{k-2}(\varphi), \quad \forall k = 2, ..., n.$$

2. Proof of Theorem 1.1

As we argued in Remark 1.4, we only need to prove the theorem for the case $t = 1$. Take $c < c_{\|z\|^2 dV_{2n}}(\varphi)$. Without loss of generality, we can assume that $\varphi_j, \varphi \in \mathrm{PSH}^-(\Delta^n)$ and

$$\int_{\Delta^n} e^{-2c\varphi}\|z\|^2 dV_{2n} < +\infty,$$

where Δ is the unit polydisc in $\mathbb{C}$. By Fubini's theorem we have

$$\int_{\Delta}\Big[\int_{\Delta^{n-1}} e^{-2c\,\varphi(z',z_n)} dV_{2n-2}(z')\Big]|z_n|^2 dV_2(z_n) < +\infty.$$

By well-known properties of pluripotential theory, the L^1 convergence of φ_j to φ implies that $\varphi_j \to \varphi$ almost everywhere with respect to the Lebesgue measure. Then $\varphi_j(\bullet, z_n) \to \varphi(\bullet, z_n)$ in the topology of $L^1_{\mathrm{loc}}(\Delta^{n-1})$ for almost every $z_n \in \Delta$. Therefore, we can find $w_n \in \Delta \setminus \{0\}$ such that

$$\int_{\Delta^{n-1}} e^{-2c\,\varphi(z',w_n)}|w_n|^2 dV_{2n-2}(z') \le \frac{\epsilon^2}{|w_n|^2},$$

and $\varphi_j(\bullet, w_n) \to \varphi(\bullet, w_n)$ in the topology of $L^1_{\mathrm{loc}}(\Delta^{n-1})$. By the effective version of the semicontinuity theorem for weighted log canonical thresholds (see [7] and see also [10]), we can find $j_0 \ge 1$ and $\rho > 0$ such that

$$\int_{\Delta_\rho^{n-1}} e^{-2c\,\varphi_j(z',w_n)}|w_n|^2 dV_{2n-2}(z') \le \frac{\epsilon^2}{|w_n|^2}, \quad \forall j \ge j_0.$$

Thanks to the L^2-extension theorem of Ohsawa and Takegoshi (see [14] and see also [2,5]), there exists a holomorphic function f_{jn} on $\Delta_\rho^{n-1} \times \Delta$ such that $f_{jn}(z', w_n) = w_n$ for all $z' \in \Delta_\rho^{n-1}$, and

$$\begin{aligned}
&\int_{\Delta_\rho^{n-1}\times\Delta} |f_{jn}(z)|^2 e^{-2c\,\varphi_j(z)} dV_{2n}(z) \\
&\le A \int_{\Delta_\rho^{n-1}} e^{-2c\,\varphi_j(z',w_n)}|w_n|^2 dV_{2n-2}(z') \\
&\le \frac{A\epsilon^2}{|w_n|^2},
\end{aligned}$$

where A is a constant. By the mean value inequality for the plurisubharmonic function $|f_{jn}|^2$, we get

$$\begin{aligned}
|f_{jn}(z)|^2 &\le \frac{1}{\pi^n(\rho-|z_1|)^2\ldots(\rho-|z_n|)^2}\int_{\Delta_{\rho-|z_1|}(z_1)\times\ldots\times\Delta_{\rho-|z_n|}(z_n)} |f_{jn}|^2 dV_{2n} \\
&\le \frac{A\epsilon^2}{\pi^n(\rho-|z_1|)^2\ldots(\rho-|z_n|)^2|w_n|^2},
\end{aligned}$$

where $\Delta_\rho(z)$ is the disc of center z and radius ρ. Hence, for any $r < \rho$, we infer

$$\|f_{jn}\|_{L^\infty(\Delta_r^n)} \le \frac{2^n A^{\frac{1}{2}}\epsilon}{\pi^{\frac{n}{2}}(\rho-r)^n|w_n|}.$$

Since $f_{jn}(z', w_n) - w_n = 0$, $\forall z' \in \Delta_\rho^{n-1}$, we can write $f_{jn}(z) = z_n + (z_n - w_n)g_{jn}(z)$ for some function $g_{jn}(z) = \sum_{\alpha\in\mathbb{N}^n} a_{jn,\alpha} z^\alpha$ on $\Delta_\rho^{n-1} \times \Delta$. We have

$$\begin{aligned}
\|g_{jn}\|_{\Delta_r^n} = \|g_{jn}\|_{\Delta_r^{n-1}\times\partial\Delta_r} &\le \frac{1}{r-|w_n|}\Big(\|f_{jn}\|_{L^\infty(\Delta_r^n)} + 1\Big) \\
&\le \frac{1}{r-|w_n|}\Big(\frac{2^n A^{\frac{1}{2}}\epsilon}{\pi^{\frac{n}{2}}(\rho-r)^n|w_n|} + 1\Big).
\end{aligned}$$

Thanks to the Cauchy integral formula, we find

$$|a_{jn,\alpha}| \le \frac{\|g_j\|_{\Delta_r^n}}{r^{|\alpha|}} \le \frac{1}{(r-|w_n|)r^{|\alpha|}}\Big(\frac{2^n A^{\frac{1}{2}}\epsilon}{\pi^{\frac{n}{2}}(\rho-r)^n|w_n|}+1\Big).$$

We take in any case $\eta \le \epsilon \le \frac{1}{2}r$. As $|w_n| < \eta \le \frac{1}{2}r$, this implies

$$|w_n||a_{jn,\alpha}|r^{|\alpha|} \le \frac{2}{r}\Big(\frac{2^n A^{\frac{1}{2}}\epsilon}{\pi^{\frac{n}{2}}(\rho-r)^n}+|w_n|\Big) \le A'\epsilon.$$

Similarly, for $\epsilon_1, ..., \epsilon_n > 0$, we can find $w_1, ..., w_n \in \Delta_{\frac{1}{4}}\backslash\{0\}$, holomorphic functions f_{jk} and $g_{jk} = \sum_{\alpha\in\mathbb{N}^n} a_{jk,\alpha}z^\alpha$ on Δ_ρ^n with $|w_k\|a_{jk,\alpha}| \le 2^{|\alpha|}\epsilon_k$ such that

$$\int_{\Delta_\rho^n} |f_{jk}(z)|^2 e^{-2c\,\varphi_j(z)} dV_{2n}(z) \le \frac{\epsilon_k^2}{|w_k|^2},$$

$$f_{jk}(z) = z_k + (z_k - w_k)g_{jk}(z),$$

for all $1 \le k \le n$, $j \ge j_0$. Now, we only need to prove that there exist $\delta_j, \theta_j > 0$ such that

$$\sum_{1\le k\le n} |f_{jk}(z)|^2 \ge \theta_j\|z\|^2,$$

for all $z \in \Delta_{\delta_j}^n$, $j \ge j_0$. First, if $f_{jk}(0) = w_k g_{jk}(0) \ne 0$ for some $k \in \{1, ..., n\}$ then there exist $\delta_j, \theta_j > 0$ such that

$$\sum_{1\le k\le n} |f_{jk}(z)|^2 \ge \theta_j, \quad \forall z \in \Delta_{\delta_j}^n.$$

Now, we only consider the case of $f_{jk}(0) = w_k g_{jk}(0) = 0$ for all $k \in \{1, ..., n\}$. Since $|w_k\|a_{jk,\alpha}| \le 2^{|\alpha|}\epsilon_k$, we get

$$|g_{jk}(z)| \le \frac{4n\epsilon_k}{|w_k|}\|z\|, \quad \forall z \in \Delta_{\frac{1}{4}}^n.$$

Hence

$$|f_{jk}(z)| \ge |z_k| - 8n\epsilon_k\|z\|, \quad \forall z \in \Delta^n_{\min(|w_1|,...,|w_n|)}.$$

By choosing $\epsilon_1, ..., \epsilon_n > 0$ small enough, we get

$$\sum_{1\le k\le n} |f_{jk}(z)|^2 \ge \theta_j\|z\|^2, \quad \forall z \in \Delta_{\delta_j}^n, \quad j \ge j_0.$$

3. Proof of Theorem 1.3

First, we will prove that

$$c_k(\varphi) \ge c_{\|z\|^{2(k-n)}dV_{2n}}(\varphi).$$

Indeed, take $c < c_{\|z\|^{2(k-n)}dV_{2n}}(\varphi)$. We choose $\delta > 0$ such that

$$\int_{\mathbb{B}(0,\delta)} e^{-2c\varphi}\|z\|^{2(k-n)}dV_{2n} < +\infty,$$

where $\mathbb{B}(0, \delta)$ is the ball with center at 0 and radius δ. By Fubini's theorem we have

$$\int_{H\in \mathrm{Gr}(k,n)} d\mu(H) \int_{H\cap\mathbb{B}(0,\delta)} e^{-2c\varphi}dV_{2k} = 0(1) \int_{\mathbb{B}(0,\delta)} e^{-2c\varphi}\|z\|^{2(k-n)}dV_{2n} < +\infty,$$

where $\mathrm{Gr}(k, n)$ is the Grassmannian manifold of k-dimensional subspaces in $\mathbb{C}^n$ and $d\mu$ is the Haar measure on $\mathrm{Gr}(k, n)$. This implies that there exists $H \in \mathrm{Gr}(k, n)$ such that

$$\int_{H\cap\mathbb{B}(0,\delta)} \mathrm{e}^{-2c\varphi}\mathrm{d}V_{2k} < +\infty.$$

Hence $c_k(\varphi) \geq c$. Second, we will prove that

$$c_k(\varphi) \leq c_{\|z\|^{2(k-n)}\mathrm{d}V_{2n}}(\varphi).$$

Indeed, take $c < c_k(\varphi)$. We choose $\delta > 0$ and $H \in \mathrm{Gr}(k,n)$ such that

$$\int_{H\cap\mathbb{B}(0,\delta)} \mathrm{e}^{-2c\varphi}\mathrm{d}V_{2k} < +\infty.$$

Without loss of generality, we can assume that $H = \{z \in \mathbb{C}^n : z_{k+1} = ...z_n = 0\}$. As in the proof of Theorem 2.5 in [7], thanks to the L^2-extension theorem of Ohsawa and Takegoshi (see [14]), we can find a holomorphic function f on $\mathbb{B}(0,\delta)$ such that $f = 1$ on H and

$$\int_{\mathbb{B}(0,\delta)} |f|^2\mathrm{e}^{-2c\varphi}\left(\sum_{j=k+1}^{n}|z_j|^2\right)^{(k-n)+\epsilon}\mathrm{d}V_{2k} \leq 0(1)\int_{H\cap\mathbb{B}(0,\delta)} \mathrm{e}^{-2c\varphi}\mathrm{d}V_{2k} < +\infty,$$

for all $\epsilon > 0$. This implies that there exists $0 < \delta_1 < \delta$ such that

$$\int_{\mathbb{B}(0,\delta_1)} \mathrm{e}^{-2c\varphi}\|z\|^{2(k-n)+2\epsilon}\mathrm{d}V_{2k} < +\infty,$$

for all $\epsilon > 0$. Hence

$$c_{\|z\|^{2(k-n)+2\epsilon}}(\varphi) \geq c, \quad \forall \epsilon > 0.$$

Letting $\epsilon \to 0$, we get

$$c_{\|z\|^{2(k-n)}}(\varphi) \geq c.$$

Now, we will only need to show that

$$\tilde{c}_k(\varphi) \leq c_k(\varphi).$$

We choose a smooth k-dimensional submanifold H through 0 such that $\tilde{c}_k(\varphi) = c(\varphi|_H)$. We can find a biholomorphic $\Phi : U \to V$ such that $\Phi(0) = 0$ and $\Psi(H)$ is a k-dimensional subspace in $\mathbb{C}^n$, where U, V are neighborhoods of $0 \in \mathbb{C}^n$. Since $c_k(\varphi) = c_{\|z\|^{2(k-n)}\mathrm{d}V_{2n}}(\varphi)$, we have

$$\tilde{c}_k(\varphi) = c(\varphi|_H) = c(\varphi_o\Phi^{-1}|_{\Phi(H)}) \leq c_k(\varphi_o\Phi^{-1}) = c_k(\varphi).$$

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