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Harmonic-counting measures and spectral theory of lens spaces

Mesures de comptage harmonique et théorie spectrale des espaces lenticulaires

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A B S T R A C T

In this article, associated with each lattice $T \subseteq \mathbb{Z}^n$, the concept of a harmonic-counting measure ν_T on a sphere S^{n-1} is introduced and is applied to determine the asymptotic behavior of the cardinality of the set of independent eigenfunctions of the Laplace–Beltrami operator on a lens space L corresponding to the elements of the associated lattice T of L lying in a cone.

R É S U M É

Dans cette Note, on associe à tout réseau $T \subseteq \mathbb{Z}^n$ une mesure de comptage harmonique ν_T sur la sphère S^{n-1}. On l'utilise pour déterminer le comportement asymptotique du cardinal d'un ensemble de fonctions propres indépendantes de l'opérateur de Laplace–Beltrami sur un espace lenticulaire L, correspondant aux éléments du réseau T de L appartenant à un cône.

1. Introduction

Counting the number of points of a lattice in a convex body has been well studied by many mathematicians including Minkowski, Ehrhart and Stanley. The asymptotic behavior of such counting functions leads to the definition of lattice-counting-measures on the sphere S^{n-1} [3,4,10]. In this paper we define the parallel notion of a harmonic-counting measure. We say that a polynomial $P \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ is harmonic if $\Delta(P) = 0$ where $\Delta = \sum_{i=1}^n (\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2})$. The restrictions of the harmonic homogeneous polynomials to S^{2n-1} are the eigenfunctions of the Laplace–Beltrami operator on (S^{2n-1}, g),

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where g is the metric induced by the Euclidean inner product of $\mathbb{R}^{2n}$. Let us identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, and let H denote the set of harmonic homogeneous polynomials which are invariant under the action

$$(z_1, z_2, ..., z_n) \to (e^{2i\pi \frac{p_1}{q}} z_1, ..., e^{2i\pi \frac{p_n}{q}} z_n)$$

of the homotopy group of the lens space $\mathfrak{L}(p_1, ..., p_n; q)$. It is proved that there is a correspondence between H and the set of eigenfunctions of the Laplace–Beltrami operator of this lens space [5,6,8]. In [8] the lattice associated with the lens space $\mathfrak{L}(p_1, ..., p_n; q)$ is defined as $T = \{(a_1, ..., a_n) \in \mathbb{Z}^n | \sum_{j=1}^n a_j p_j \equiv 0 \pmod q\}$. This lattice is used to provide a criterion for isospectrality of lens spaces (Theorem 3.6 of [8]). See also [7]. Let

$$z_{\sigma(j)}^{|a_j|} = \begin{cases} z_j^{a_j} & \text{if } a_j \geq 0 \\ \bar{z}_j^{-a_j} & \text{otherwise,} \end{cases}$$

and let $H_{s,(a_1,...,a_n)} =$

$$\{h \in H | \deg h = s, h = k(|z_1|^2, \ldots, |z_n|^2) \prod_{i=1}^n z_{\sigma(i)}^{|a_i|} \text{ for some } k \in \mathbb{C}[x_1, \cdots, x_n]\},$$

be the vector space of harmonic homogeneous polynomials of degree s associated with $(a_1, ..., a_n) \in T$. In [9] we used these subspaces to provide a proof of Theorem 3.6. of [8]. In this paper we use the dimension of the vector space $H_{s,(a_1,...,a_n)}$ to define a multiplicity for each point $(a_1, ..., a_n)$ of the lattice T. Counting points with such multiplicities we are led to the definition of harmonic-counting measures. In fact we consider the asymptotic behavior of the function,

$$F_{T\cap K}(t) = \sum_{s=0}^{t} \sum_{x \in T\cap K(s)} \dim H_{s,x},$$

where $T \cap K(s)$ denotes the set of elements in the intersection of T and the spherical cone K with l_1 norm equal to s, to provide a measure ν_T on S^{n-1}. The measure ν_T is a tool to compare the cardinality of harmonic homogeneous polynomials (or eigenfunctions of a lens space) associated with lattice points in two different cones. In Theorem 2.5 we calculate the values of the measure ν_T. This theorem provides more information than Weyl's law for Laplace–Beltrami operator in the case of lens spaces. (See Remark 1.) Using Theorem 2.5 we can see that the number of independent eigenfunctions of the Laplace–Beltrami operator associated with the integral points of an l_1-spherical sector of radius t is asymptotically $\frac{B(n-1,n+1)}{(n-2)!2^{n-1}} t^{n-1}$ times the number of lattice points in this region, where B is the beta function.

2. Preliminaries on lens spaces

2.1. Lattices

In this paper a lattice T is a subgroup of the group $\mathbb{Z}^n$. T is of rank n if $T \bigotimes \mathbb{R} = \mathbb{R}^n$.

Definition 2.1. A preliminary lattice group T is defined as

$$T = \{(a_1, ..., a_n) \in \mathbb{Z}^n | \sum_{j=1}^n a_j p_j \equiv 0 \pmod q\}$$

where integers $\{p_1, ..., p_n\}$ are prime to the positive integer q.

The measures defined in this article can be used in general lattices, but we limit ourselves to preliminary lattices that are useful for the study of lens spaces. Let $\{v_1, ..., v_n\}$ be a basis for T. A matrix A whose columns are $v_1, ..., v_n$ is called a generating matrix of T. Another matrix B is a generating matrix of T if and only if there is a unimodular matrix U such that $A = UB$. An essential parallelepiped of a lattice $T \subset \mathbb{R}^n$ is a parallelepiped $P_T = \{\sum_{i=1}^n a_i v_i | 0 \leq a_i \leq 1, i = 1, \cdots, n\}$. Let K be a cone in $\mathbb{R}^n$ whose apex is the origin.

2.2. Harmonic-counting measure

Let $N_{T\cap K}(s)$ be the number of elements in $T \cap K$ with l_1-norm s. For a cone $K \subset \mathbb{R}^n$ set

$$F_{T\cap K}(t) = \sum_{s=0}^{t} \sum_{r=0}^{[\frac{s}{2}]} \binom{r+n-2}{n-2} N_{T\cap K}(s-2r).$$

Definition 2.2. The cone constructed from a set $U \subseteq \mathbb{R}^n$ is the set $\{tx | t \in \mathbb{R}^+, x \in U\}$. This set is denoted by $C(U)$.

Definition 2.3. The harmonic-counting measure associated with the lattice T, is a measure ν_T on the Borel σ-algebra of S^{n-1} which is defined as

$$\nu_T(U) := \lim_{t\to\infty} \frac{F_{T\cap C(U)}(t)}{t^{2n-1}}. \tag{1}$$

By Lemma 3.2 of [8],

$$\dim H_{s,(a_1,\ldots,a_n)} = \begin{cases} \binom{r+n-2}{n-2} & \| (a_1,\ldots,a_n) \|_{l_1} = s-2r \\ 0 & \textit{otherwise} \end{cases} \tag{2}$$

which is equal to the number of independent harmonic homogeneous polynomials of degree s associated with the element $(a_1,\ldots,a_n)$. So the resulting measure is named harmonic-counting measure.

In order to study the asymptotic behavior of the function $F_{T\cap K}(t)$, we need the asymptotic behavior of $N_{\mathbb{Z}^n\cap K}(t)$. It is a well-known fact that $\sum_{t=0}^{s} N_{\mathbb{Z}^n\cap K}(t) \sim \alpha_K s^n$ where α_K is the volume of the intersection of K and the l_1-sphere of radius 1 (Ehrhart–Stanley–Minkowski). This provides a combinatorial approach to a well-known measure μ_T on the sphere S^{n-1} [3]. Precisely

$$\mu_T(U) = \lim_{s\to\infty} \frac{\sum_{t=0}^{s} N_{\mathbb{Z}^n\cap A^{-1}C(U)}(t)}{s^n} \tag{3}$$

is a finite measure, where A is the generating matrix of T.

2.3. Lens spaces

Let q be a positive integer, and let $p_1,\ldots,p_n$ be integers that are prime to q. Let

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \sim e^{i\theta} \tag{4}$$

and

$$g = R(2\pi p_1/q) \oplus \cdots \oplus R(2\pi p_n/q). \tag{5}$$

Suppose that $G \subset O(2n)$ is the finite cyclic group generated by g. If G (as a group of isometries) acts freely on S^{2n-1}, then the manifold S^{2n-1}/G, denoted by $\mathfrak{L}(p_1,\ldots,p_n;q)$, is called a lens space. Let $spec(M)$ denote the set of eigenvalues of the Laplace–Beltrami operator. $G_1 \subseteq G$ implies $spec(S^{2n-1}/G) \subseteq spec(S^{2n-1}/G_1)$. In particular $spec(S^{2n-1}/G) \subseteq spec(S^{2n-1})$. The Laplace–Beltrami eigenvalues of the manifold S^{2n-1} are of the form $k(k+2n-2)$, $k \in \mathbb{N}\cup\{0\}$ [5,6].

Definition 2.4. Let $p_1,\ldots,p_n$ be integers that are prime to q. The lens space associated with a lattice $T = \{(a_1,\ldots,a_n) \in \mathbb{Z}^n | \sum_{j=1}^{n} a_j p_j \equiv 0 \pmod q\}$ is the space S^{2n-1}/G ([8] Definition 3.2).

A nice relation between lattices and isospectrality is:

Theorem 2.1 *(Lauret, Miatello and Rossetti ([8], Theorem 3.6)). Two lens spaces $\mathfrak{L}_1 = S^{2n-1}/G_1$ and $\mathfrak{L}_2 = S^{2n-1}/G_2$ are isospectral if and only if for the associated lattices T_1 and T_2, $\mathrm{card}(\overline{B_{l_1}(0,k)} \cap T_1) = \mathrm{card}(\overline{B_{l_1}(0,k)} \cap T_2)$ for each $k \in \mathbb{N}$ where $B_{l_1}(0,k)$, is the l_1-ball of radius k centered at 0.*

As a result we have the next corollary (also see [5]).

Corollary 2.2. *Two isospectral lens spaces have the same dimension and the same homotopy group.*

Theorem 2.3. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be homotopy equivalent n-dimensional lens spaces with associated lattices T_1 and T_2. Then $\mu_{T_1} = \mu_{T_2}$*

Proof. It is well-known that for an arbitrary convex polytope $\Omega \subset \mathbb{R}^n$ we have $\lim_{s\to\infty} \frac{\mathrm{card}(\mathbb{Z}^n \cap s\Omega)}{s^n} = \mathrm{Vol}(\Omega)$ [2]. If A is a generating matrix of the lattice T and K is the part of $\overline{B_{l_1}(0,1)}$ opposite to $U \subseteq S^{n-1}$, then $\mathrm{card}(T \cap sK) = \mathrm{card}(\mathbb{Z}^n \cap sA^{-1}K)$. Therefore

$$\mu_T(U) = \lim_{s\to\infty} \frac{\mathrm{card}(\mathbb{Z}^n \cap sA^{-1}K)}{s^n} = \mathrm{Vol}(A^{-1}K) = \det A^{-1}\mathrm{Vol}(K). \tag{6}$$

On the other hand it is well known that by Theorem 2.1, the values of $\det A_1^{-1}$ and $\det A_2^{-1}$ are equal to q^{-1}. Therefore, these measures are equivalent. □

Theorem 2.4. *ν_T is a finite measure and its total value, $\nu_T(S^{n-1})$, is equal to*

$$\frac{1}{q}(2\pi)^{1-2n}\omega_{2n-1}\mathrm{Vol}(S^{2n-1}),$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

This is a corollary of Theorem 2.5. Here we provide another proof for preliminary lattices using the properties of lens spaces.

Proof. Let T be a preliminary lattice and let $\mathfrak{L}$ be its associated lens space. According to [8] (or [9]) the number of independent eigenfunctions of the Laplace–Beltrami operator on a lens space with eigenvalue $s(s+(2n-1)-1)$ is equal to

$$\sum_{r=0}^{[\frac{s}{2}]}\binom{r+n-2}{n-2}N_{T\cap\mathbb{R}^n}(s-2r).$$

By Weyl's law [1] we have

$$\lim_{x\to\infty}\frac{N(x)}{x^{\frac{2n-1}{2}}}=(2\pi)^{-(2n-1)}\omega_{2n-1}\mathrm{Vol}(\mathfrak{L}),$$

where $N(x)$ denotes the number of eigenvalues less than x and $2n-1$ is the dimension of the lens space $\mathfrak{L}$. So

$$\begin{aligned}\lim_{t\to\infty}\frac{F_{T\cap\mathbb{R}^n}(t)}{t^{2n-1}}&=\lim_{t\to\infty}\frac{N(t(t+2n-2))}{t^{2n-1}}=\lim_{t\to\infty}\frac{N(t(t+2n-2))}{(t(t+2n-2))^{\frac{2n-1}{2}}}\\&=(2\pi)^{-(2n-1)}\omega_{2n-1}\mathrm{Vol}(S^{2n-1}/G).\end{aligned}\tag{7}$$

S^{2n-1} is a q-sheeted covering space of S^{2n-1}/G and therefore $\mathrm{Vol}(S^{2n-1}/G)=\frac{1}{q}\mathrm{Vol}(S^{2n-1})$. $\square$

Now we compute the value of $\nu_T(U)$ where U is a Borel subset of the sphere S^{n-1}. Let A be the generating matrix of T. Also let

$$\alpha(U)=\mathrm{Vol}(A^{-1}(C(U))\cap B_{l_1}(0,1)).$$

Theorem 2.5. *The value of $\nu_T(U)$ is equal to*

$$\lim_{t\to\infty}\frac{F_{T\cap C(U)}(t)}{t^{2n-1}}=\frac{B(n-1,n+1)}{(n-2)!2^{n-1}}\alpha(U),\tag{8}$$

where the beta function is defined as $B(z,t)=\int_0^1 x^{z-1}(1-x)^{t-1}\,\mathrm{d}x$.

Proof. We have

$$F_{T\cap C(U)}(t)=\sum_{s=0}^{t}\sum_{r=0}^{[\frac{s}{2}]}\binom{r+n-2}{n-2}N_{T\cap C(U)}(s-2r),\tag{9}$$

where

$$\sum_{s=0}^{t}N_{T\cap C(U)}(s)=\alpha(U)t^n+O(t^{n-1}).\tag{10}$$

By changing the order of summation in (9), we have

$$F_{T\cap C(U)}(t)=\sum_{r=0}^{[\frac{t}{2}]}\left(\binom{r+n-2}{n-2}\sum_{i=0}^{t-2r}N_{T\cap C(U)}(i)\right).$$

So by (10),

$$\frac{1}{t^{2n-1}}\sum_{r=0}^{[\frac{t}{2}]}\left(\binom{r+n-2}{n-2}\alpha(U)(t-2r)^n - M(t-2r)^{n-1}\right) \leq$$

$$\frac{1}{t^{2n-1}}\sum_{r=0}^{[\frac{t}{2}]}\left(\binom{r+n-2}{n-2}\sum_{i=0}^{t-2r} N_{T\cap C(U)}(i)\right) \leq \tag{$**$}$$

$$\frac{1}{t^{2n-1}}\sum_{r=0}^{[\frac{t}{2}]}\left(\binom{r+n-2}{n-2}(\alpha(U)(t-2r)^n + M(t-2r)^{n-1})\right).$$

Now we have

$$\begin{aligned}
&\frac{1}{t^{2n-1}}\sum_{r=0}^{[\frac{t}{2}]}\left(\binom{r+n-2}{n-2}\alpha(U)(t-2r)^n - M(t-2r)^{n-1}\right) \\
&= \alpha(U)\frac{1}{(n-2)!}\frac{1}{t^{2n-1}}\sum_{r=0}^{[\frac{t}{2}]}(r^{n-2}(t-2r)^n + O(t^{2n-3})) \\
&= (\alpha(U)\frac{1}{(n-2)!}\frac{1}{t^{2n-1}}\sum_{r=0}^{[\frac{t}{2}]}r^{n-2}(t-2r)^n) + \alpha(U)\frac{1}{(n-2)!}\frac{1}{t^{2n-1}}O(t^{2n-2}) \\
&= (\alpha(U)\frac{1}{(n-2)!}\frac{1}{2^{n-1}}\frac{2}{t}\sum_{r=0}^{[\frac{t}{2}]}(\frac{2r}{t})^{n-2}(1-\frac{2r}{t})^n) + \alpha(U)\frac{1}{(n-2)!}\frac{1}{t^{2n-1}}O(t^{2n-2}).
\end{aligned}$$

Also we have

$$\lim_{t\to\infty}\frac{2}{t}\sum_{i=0}^{[\frac{t}{2}]}g(\frac{2i}{t}) = \int_0^1 g(x)\,\mathrm{d}x. \tag{11}$$

Applying (11), we see that the limits of the left and the right parts of $(**)$ are equal to

$$\frac{\alpha(U)}{(n-2)!2^{n-1}}\int_0^1 x^{n-2}(1-x)^n\,\mathrm{d}x.$$

So,

$$\lim_{t\to\infty}\frac{\sum_{r=0}^{[\frac{t}{2}]}\left(\binom{r+n-2}{n-2}\sum_{i=0}^{t-2r}N_{T\cap C(U)}(i)\right)}{t^{2n-1}} = \frac{\alpha(U)}{(n-2)!2^{n-1}}B(n-1,n+1). \quad \square$$

This shows that the normalization of ν_T is a uniform measure with respect to surface area on each face of $B_{l_1}(0,1)$.

Remark 1. When $\mathfrak{L}$ is the lens space associated with the preliminary lattice T, the set of independent eigenfunctions of the Laplace–Beltrami operator on $\mathfrak{L}$ associated with the elements of $T\cap C(U)\cap B_{l_1}(0, m(m+2n-2))$ is the same as $F_{C(U)\cap T}(m)$. Thus the number of independent eigenfunctions with eigenvalues less than $s(s+(2n-1)-1)$ is equal to $F_{T\cap\mathbb{R}^n}(s)$. So, Theorem 2.5 provides more information than Weyl's law which asymptotically computes the number of independent eigenfunctions with eigenvalues less than $t = s(s+(2n-1)-1)$. Also, Theorem 2.5 shows that the number of independent eigenfunctions of the Laplace–Beltrami operator associated with the integral points of $C(U)\cap B_{l_1}(0,t)$ is asymptotically $\frac{B(n-1,n+1)}{(n-2)!2^{n-1}}t^{n-1}$ times the number of lattice points in $C(U)\cap B_{l_1}(0,t)$.

Remark 2. Harmonic-counting measures are constant multiples of lattice counting measures where the constant is an explicit function of the dimension of the lattice.

Remark 3. Let T be the lattice associated with the Lens space $\mathfrak{L}$. Let $P\in\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ be a harmonic polynomial which is invariant under the action of the homotopy group of the lens space and let $z_j = x_j + iy_j$, $j=1,\ldots,n$. Then P can be written uniquely as the sum of harmonic polynomials of the form $k(|z_1|^2,\cdots,|z_n|^2)\prod_{i=1}^n z_{\sigma(i)}^{|a_i|}$ where $(a_1,\cdots,a_n)\in T$ and

$$z_{\sigma(j)}^{|a_j|} = \begin{cases} z_j^{a_j} & \text{if } a_j \geq 0 \\ \bar{z}_j^{-a_j} & \text{otherwise.} \end{cases}$$

(See [9].) Theorem 2.5 asymptotically determines the number of independent homogeneous polynomials k such that $k(|z_1|^2, \cdots, |z_n|^2) \prod_{i=1}^{n} z_{\sigma(i)}^{|a_i|}$ is harmonic for some $(a_1, \cdots, a_n) \in C(U) \cap T$.

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