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An online intrinsic stabilization strategy for the reduced basis approximation of parametrized advection-dominated problems

Une stratégie intrinsèque de stabilisation en ligne pour l'approximation bases réduites de problèmes paramètrés avec transport dominant

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ABSTRACT

We propose a new, black-box online stabilization strategy for reduced basis (RB) approximations of parameter-dependent advection–diffusion problems in the advection-dominated case. Our goal is to stabilize the RB problem irrespectively of the stabilization (if any) operated on the high-fidelity (e.g., finite element) approximation, provided a set of stable RB functions have been computed. Inspired by the *spectral vanishing viscosity* method, our approach relies on the transformation of the basis functions into modal basis, then on the addition of a vanishing viscosity term over the high RB modes, and on a *rectification* stage – prompted by the *spectral filtering technique* – to further enhance the accuracy of the RB approximation. Numerical results dealing with an advection-dominated problem parametrized with respect to the diffusion coefficient show the accuracy of the RB solution on the whole parametric range.

RÉSUMÉ

Nous proposons une nouvelle strategie pour stabiliser l'approximation d'un problème de diffusion–transport avec transport dominant par une méthode de bases réduites. Cette strategie, opérée en ligne, est indépendante de la technique « haute fidélité » utilisée « hors ligne » ; elle trouve son inspiration dans la méthode de la *viscosité spectrale évanescente*. Par une diagonalisation sur l'espace de base réduite, on introduit une nouvelle base modale, qui permet d'ajouter au problème réduit un terme de viscosité évanescent sur les modes suffisant pour stabiliser l'approximation. Une méthode de rectification de la solution (semblable aux techniques de filtrage spectral) de ce problème est enfin opérée afin d'améliorer la précision de cette approximation. Les résultats numériques obtenus pour un problème

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avec transport dominant dont l'intensité est parametrisée montrent que l'approximation réduite résultante est stable et précise sur tout l'intervalle des paramètres.

Version française abrégée

Nous proposons dans cette note une nouvelle approche pour traiter le problème des instabilités apparaissant lors de l'approximation de problèmes de diffusion–transport avec transport dominant par des méthodes de bases réduites (RB). Ces instabilités sont un problème classique, quelle que soit la discrétisation utilisée, qui peut être traité en raffinant le maillage de façon extrême ou en ajoutant des termes de stabilisation appropriés. Dans le cas RB, nous pouvons nous appuyer sur un ensemble de N fonctions de base, dont chacune est obtenue grâce à une technique classique d'éléments finis *stable*, comme la méthode SUPG, basée sur $N_h \gg N$ degrés de liberté. Néanmoins, la combinaison de ces fonctions à travers une méthode de Galerkin pure ne suffit pas à assurer la stabilité du problème RB quand N augmente, d'où la nécessité d'introduire des termes de stabilisation appropriés au niveau RB (Section 1).

Naturellement, l'espace réduit pour le problème de diffusion–transport doit être indépendant de la méthode classique utilisée pour générer les fonctions de base réduite. Il en est de même pour le schéma, d'autant plus qu'il n'est pas souhaitable – difficile, voire impossible – de disposer de sa traduction en terme d'algèbre matricielle. Nous proposons ici une technique de stabilisation « en ligne », indépendante de la stabilisation utilisée lors de la création de l'espace réduit. Ceci nous permet d'introduire une stabilisation plus faible que celle utilisée dans la technique SUPG et plus simple à mettre en œuvre. Elle s'inspire de la méthode de la *viscosité spectrale évanescente* [8] et repose sur trois étapes (voir la Section 2). On construit tout d'abord « hors ligne » la matrice $\mathbb{Z}$ dont les colonnes sont les vecteurs de la base réduite (préalablement orthonormalisées pour assurer un bon conditionnement du système réduit) :

1) nous obtenons une nouvelle base modale $\tilde{\mathbb{Z}}$ par rotation de $\mathbb{Z}$, selon les vecteurs propres du Laplacien réduit ;
2) dans l'étape « en ligne », nous ajoutons un terme de *viscosité* diagonal (dans cette base), dont l'amplitude est nulle sur les modes les plus bas et augmente sur les modes plus hauts ; on calcule au point 2 la solution du problème réduit, notée $\tilde{u}_N^{vv}(\mu)$;
3) nous déterminons enfin l'approximation RB $u_N^{vv}(\mu)$ en opérant la méthode de *rectification* proposée dans [1] à la solution $\tilde{u}_N^{vv}(\mu)$ afin d'en augmenter la précision (voir la Section 3).

Cette stratégie permet de stabiliser de façon intrinsèque l'approximation en base réduite lors de l'étape en ligne, reposant des structures algébriques qui peuvent être construites indépendamment de l'approximation classique (ici, SUPG) utilisée. Les résultats numériques présentés dans la Section 4 montrent la faisabilité de la technique proposée et la notable amélioration des résultats ainsi stabilisés fournie par la méthode de rectification.

1. Existing stabilization approaches for Galerkin–RB approximations

Let us consider, for the sake of exposition, the following parametrized advection–diffusion problem:

$$\begin{cases} L(\mu)u := -\mu\Delta u + \mathbf{b}\cdot\nabla u = f & \text{in } \Omega = (0,1)^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mu \in \mathcal{P} = [10^{-6}, 1]$ denotes the diffusion coefficient and $\mathbf{b} = (1,1)^{\mathrm{T}}$ the (constant) transport field. The weak form of problem (1) is: find $u(\mu) \in V$ such that

$$a(u(\mu), v; \mu) = F(v) \qquad \forall v \in V, \tag{2}$$

being $V = H_0^1(\Omega)$,

$$a(u, v; \mu) = \mu \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} + \int_\Omega \mathbf{b}\cdot\nabla u\, v \, \mathrm{d}\mathbf{x}, \qquad F(v) = \int_\Omega f\, v \, \mathrm{d}\mathbf{x}$$

a continuous and coercive bilinear form, for any $\mu \in \mathcal{P}$, and a linear and continuous functional, respectively; note that only a is μ-dependent. The so-called *high-fidelity* solution to problem (2) is obtained by a Galerkin–finite element (FE) method introducing a high-fidelity space $V_h \subset V$ of dimension $\dim(V_h) = N_h < \infty$ as: find $u_h(\mu) \in V_h$ such that

$$a(u_h(\mu), v_h; \mu) = F(v_h) \qquad \forall v_h \in V_h. \tag{3}$$

Although problem (2) is well-posed for any choice of $\mu > 0$, when dealing with advection-dominated problems, i.e. when $|\mathbf{b}|/\mu \ll 1$, it is well recognized that (3) yields spurious numerical oscillations unless *(i)* a sufficiently fine mesh is considered, or *(ii)* suitable stabilization techniques are introduced. In this paper, we consider the latter option; hence, we seek $u_h^S(\mu) \in V_h$ such that

$$a_S(u_h^S(\mu), v_h; \mu) = F_S(v_h; \mu) \qquad \forall v_h \in V_h \tag{4}$$

being $a_S(\cdot,\cdot;\mu)$ and $F_S(\cdot;\mu)$ suitable bilinear and linear forms including the stabilization terms. Note that, because of the latter, also the right-hand side might now be μ-dependent. Usual choices are given by the Streamline Upwind Petrov–Galerkin (SUPG) method; the Galerkin–Least Squares method, but also Variational Multi-Scale or Subgrid stabilization methods can be recast under the form (4), see, e.g., [2,7,9]. For the sake of simplicity, hereon we rely on the SUPG method, even if also other stabilizations can fit the proposed framework. In this case, our high-fidelity FE problem is given by (4), where

$$a_S(u_h^S(\mu), v_h; \mu) = a(u_h^S(\mu), v_h; \mu) + b_h(u_h^S(\mu), v_h; \mu), \qquad F_S(v_h; \mu) = F(v_h) + G_h(v_h; \mu),$$

being

$$b_h(u_h, v_h; \mu) = \sum_{K\in\mathcal{T}_h} (L(\mu)u_h, \delta_K \mathcal{S}_K(v_h))_{L^2(K)}, \qquad G_h(v_h; \mu) = \sum_{K\in\mathcal{T}_h} (f, \delta_K \mathcal{S}_K(v_h))_{L^2(K)}$$

where $\mathcal{S}_K(v_h) = L_{SS}v_h$; here we denote by $L_S(\mu)u = -\mu\Delta u$ the symmetric part of L, by $L_{SS}u = \mathbf{b}\cdot\nabla u$ its skew-symmetric part, and by $\delta_K > 0$ a suitable scaling function. Note that when dealing with linear ($\mathbb{P}_1$) finite elements, i.e. when $V_h = X_h^1 \cap V$, the stabilization term is μ-independent, since, $\forall u_h \in V_h = X_h^1 \cap V$, over each element K, $L(\mu)u_h = L_{SS}u_h$. It is straightforward to show that both problems (3) and (4) are coercive, with coercivity factors given by $\alpha_h(\mu)$ and $\alpha_h^S(\mu)$, respectively; nevertheless, the stabilized problem (4) features a larger coercivity factor, since

$$\alpha_h^S(\mu) = \inf_{v_h\in V_h} \frac{a_S(v_h, v_h; \mu)}{\|v_h\|_V^2} \geq \inf_{v_h\in V_h} \frac{a(v_h, v_h; \mu)}{\|v_h\|_V^2} = \alpha_h(\mu). \tag{5}$$

From an algebraic standpoint, in the SUPG case problem (4) yields the following linear system,

$$(\mathbb{A}_h^G(\mu) + \mathbb{A}_h^{SUPG}(\mu))\mathbf{u}_h(\mu) = \mathbf{f}_h^G + \mathbf{f}_h^S \tag{6}$$

being $\mathbf{u}_h^S(\mu) \in \mathbb{R}^{N_h}$ the vector whose components are the degrees of freedom of $u_h^S(\mu)$ and, for $i, j = 1, \ldots, N_h$, $(\mathbb{A}_h^G(\mu))_{ij} = a(\varphi_j, \varphi_i; \mu)$, $(\mathbb{A}_h^{SUPG}(\mu))_{ij} = b_h(\varphi_j, \varphi_i; \mu)$, whereas $(\mathbf{f}_h^G)_i = F(\varphi_i)$, $(\mathbf{f}_h^S)_i = G_h(\varphi_i; \mu)$; $\{\varphi_i\}_{i=1}^{N_h}$ denote the set of (Lagrangian) basis functions on V_h.

The reduced basis (RB) method allows one to speedup the solution to a parametrized PDE under the form (2) by seeking for its approximation in a low-dimensional subspace V_N, of dimension $N = \dim(V_N) \ll \dim(V_h) = N_h$ built from a set of snapshots, that is, high-fidelity solutions computed for properly selected parameter values [5,10]. The RB problem is then obtained by employing a pure Galerkin method over V_N and reads as follows: find $u_N(\mu) \in V_N$ such that

$$a(u_N(\mu), v_N; \mu) = F(v_N) \qquad \forall v_N \in V_N. \tag{7}$$

The RB space V_N can be obtained *offline,* e.g. , through the greedy algorithm (see, e.g., [10]), thus yielding, for the case at hand, $V_N = \text{span}\{u(\mu^n), n = 1, \ldots, N\} = \text{span}\{\zeta_1, \ldots, \zeta_N\}$; note that, in practice, the *high-fidelity* code is used to get a very accurate approximation of each $u(\mu^n) \approx u_h^S(\mu^n)$. The Galerkin–RB (G–RB) approximation is thus a linear combination of *stable* FE approximations, obtained for $\mu \in S_N = \{\mu^1, \ldots, \mu^N\}$; a Gram–Schmidt orthonormalization is then performed to get the orthonormal basis $\{\zeta_n\}_{n=1}^N$.

Unfortunately, this approximation computed by solving *online* the reduced problem (7) is not stable: similarly to the standard FE approximation, this pure G–RB approximation shows spurious oscillations already for $\mu \leq 10^{-2}$, even if the space V_N is built starting from a set of stabilized snapshots. This shortcoming can be explained by using, e.g., the Strang lemma; see, e.g., [11] for numerical results.

On the other hand, performing a Galerkin projection of the stabilized problem (4) onto V_N yields stable RB approximations on the whole parameter range. In this case, the RB approximation of problem (4) reads as: find $u_N^S(\mu) \in V_N$ such that

$$a_S(u_N^S(\mu), v_N; \mu) = F_S(v_N; \mu) \qquad \forall v_N \in V_N. \tag{8}$$

Note that problem (8) – as the Galerkin–RB approximation of any strongly coercive problem – is automatically stable, thanks to the Cea's lemma and the Galerkin orthogonality.

Here we have

$$\|u_h^S(\mu) - u_N^S(\mu)\|_V \leq \left(\frac{M_h^S(\mu)}{\alpha_h^S(\mu)}\right)^{1/2} \inf_{w_N\in V_N} \|u_h(\mu) - w_N\|_V \tag{9}$$

being $M_h^S(\mu)$ and $\alpha_h^S(\mu)$ the discrete continuity and the coercivity factors, respectively, of $a_S(\cdot,\cdot;\mu)$. Thanks to the improved stability expressed by (5), (9) yields a good control of the error $\|u_h^S(\mu) - u_N^S(\mu)\|_V$ in terms of the best approximation error; the same estimate holds for the solution to problem (7) as well, but including a much larger factor $(M_h(\mu)/\alpha_h(\mu))^{1/2}$. We

also point out that, as soon as a pure Galerkin method is used for the high-fidelity approximation (on a sufficiently fine mesh) like in (3), a pure Galerkin–RB approximation then yields automatically stable solutions (see, e.g., Lemma 3.1 in [10]), but entailing a very expensive generation of the RB space.

Algebraically, in the SUPG case the RB approximation (8) (which we can refer to as the SUPG–RB method) turns into the solution to the following RB system:

$$(\mathbb{A}_N^{\mathrm{G}}(\mu) + \mathbb{A}_{N,h}^{\mathrm{SUPG}}(\mu))\mathbf{u}_N^{\mathrm{S}}(\mu) = \mathbf{f}_N^{\mathrm{G}} + \mathbf{f}_{N,h}^{\mathrm{SUPG}} \tag{10}$$

where $(\mathbb{A}_N^{\mathrm{G}}(\mu))_{mn} = a(\zeta_n, \zeta_m; \mu)$, $(\mathbf{f}_N^{\mathrm{G}})_m = F(\zeta_m)$, and $(\mathbb{A}_{N,h}^{\mathrm{SUPG}}(\mu))_{mn} = b_h(\zeta_n, \zeta_m)$, $(\mathbf{f}_{N,h}^{\mathrm{SUPG}})_m = G_h(\zeta_m)$, $m, n = 1, \ldots, N$. Hence, also the stabilization term appearing in the SUPG-RB problem is μ-independent; however, its assembling requires to access the matrices of the stabilization terms in the SUPG–FE problem, as

$$\mathbb{A}_N^{\mathrm{G}}(\mu) = \mathbb{Z}^{\mathrm{T}}\mathbb{A}_h^{\mathrm{G}}(\mu)\mathbb{Z}, \quad \mathbb{A}_{N,h}^{\mathrm{SUPG}}(\mu) = \mathbb{Z}^{\mathrm{T}}\mathbb{A}_h^{\mathrm{SUPG}}(\mu)\mathbb{Z}, \qquad \mathbf{f}_N^{\mathrm{G}} = \mathbb{Z}^{\mathrm{T}}\mathbf{f}_h^{\mathrm{G}}, \quad \mathbf{f}_{N,h}^{\mathrm{SUPG}} = \mathbb{Z}^{\mathrm{T}}\mathbf{f}_h^{\mathrm{SUPG}}. \tag{11}$$

Here, $\mathbb{Z} \in \mathbb{R}^{N_h \times N}$ is the basis matrix, such that $\mathbb{Z}_{im} = \zeta_m^{(i)}$; the columns of $\mathbb{Z}$ are nothing but the vectors of degrees of freedom corresponding to the basis.

In order to be fast, i.e. with a complexity that depends on N only and not on $\mathrm{card}(V_h)$, the solution method requires that each new algebraic system associated with (10)–(11) be built rapidly for any new value of μ. This makes the use of matrix multiplications as indicated in (11) unsuitable, since their cost scales like $\mathrm{card}(V_h)$. In this respect, the "golden standard" would be to pre-compute many "reduced" matrices, multiply them by some μ-dependent functions and finally sum them online (see, e.g., [10,5] for further details). This avenue is typically followed when the high-fidelity code is accessible, an assumption that we would like to avoid here. After generating the N snapshots $u_h^{\mathrm{S}}(\mu^i)$, $i = 1, \ldots, N$, the reduced model algorithm that we are about to propose will not involve the stabilization terms reported in (11). This is even more crucial as such terms are mesh-dependent.

2. An *online vanishing viscosity* stabilization method

Here we propose a new intrinsic way to stabilize the RB problem independently of the stabilization procedure operated on the FE approximation, provided a set of stable RB functions has been computed offline, no matter how. As a matter of fact, the stabilization of the RB problem is built directly at the online stage, without relying on the stabilization terms possibly employed offline. Moreover, we aim at introducing a *weaker* stabilization than the one appearing in the RB approximation of a stabilized FE problem like (4), by adding a suitable *diffusion term* on the RB problem, depending on N and vanishing on the lower modes – that is, the higher the mode, the stronger is the added stabilization.

To do this, we rely on a revisitation of the *spectral vanishing viscosity* technique [8]. First, we rewrite the diffusion operator over the orthonormal reduced basis $\mathbb{Z}$ built offline, that is, we *rotate* the reduced basis $\mathbb{Z}$ by the matrix $\mathbb{W}$ of the eigenvectors of the (reduced) diffusion operator, in order to deal with a *diagonalized* (reduced) diffusion operator. For ease of notation, we consider an algebraic formulation of this new scheme. Let us denote by $\mathbb{K}_N \in \mathbb{R}^{N\times N}$ and $\mathbb{M}_N \in \mathbb{R}^{N\times N}$ the reduced stiffness and mass operators, respectively, i.e.,

$$(\mathbb{K}_N)_{mn} = \int_\Omega \nabla\zeta_n \cdot \nabla\zeta_m \, \mathrm{d}\mathbf{x}, \qquad (\mathbb{M}_N)_{mn} = \int_\Omega \zeta_n\zeta_m \, \mathrm{d}\mathbf{x}, \qquad m, n = 1, \ldots, N,$$

obtained from the full-order matrices as $\mathbb{K}_N = \mathbb{Z}^{\mathrm{T}}\mathbb{K}_h\mathbb{Z}$, $\mathbb{M}_N = \mathbb{Z}^{\mathrm{T}}\mathbb{M}_h\mathbb{Z}$, being

$$(\mathbb{K}_h)_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, \mathrm{d}\mathbf{x}, \qquad (\mathbb{M}_h)_{mn} = \int_\Omega \varphi_i\varphi_j \, \mathrm{d}\mathbf{x}, \qquad i, j = 1, \ldots, N_h.$$

Then, we solve the following *generalized eigenvalue problem*:

$$\mathbb{K}_N\mathbf{w}_j = \lambda_j\mathbb{M}_N\mathbf{w}_j, \qquad j = 1, \ldots, N$$

and denote by $\mathbb{W} = [\mathbf{w}_1 \mid \ldots \mid \mathbf{w}_N] \in \mathbb{R}^{N\times N}$ the matrix of the eigenvectors; note that $\mathbb{W}^{\mathrm{T}}\mathbb{K}_N\mathbb{W} = \mathbb{D}_N$, that is, $\mathbf{w}_i\mathbb{K}_N\mathbf{w}_j = \lambda_j\mathbf{w}_i\mathbb{M}_N\mathbf{w}_j = \lambda_j\delta_{ij}$, being $\mathbb{D}_N = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$.

We then rotate the columns of the basis matrix $\mathbb{Z} = [\boldsymbol{\zeta}_1 \mid \ldots \mid \boldsymbol{\zeta}_N]$ by the matrix $\mathbb{W}$ in order to get the *transformed* basis matrix $\tilde{\mathbb{Z}} = \mathbb{Z}\mathbb{W}$. With respect to this new basis, the stiffness term appearing in the RB problem is diagonal, that is, $\tilde{\mathbb{Z}}^{\mathrm{T}}\mathbb{K}_h\tilde{\mathbb{Z}} = \mathbb{W}^{\mathrm{T}}\mathbb{Z}^{\mathrm{T}}\mathbb{K}_h\mathbb{Z}\mathbb{W} = \mathbb{W}^{\mathrm{T}}\mathbb{K}_N\mathbb{W} = \mathbb{D}_N$. We denote the RB space obtained as the span of the new basis functions by $\tilde{V}_N = \mathrm{span}\{\tilde{\zeta}_1, \ldots, \tilde{\zeta}_N\}$, being $\tilde{\boldsymbol{\zeta}}_n$ the algebraic representation (as vector of degrees of freedom) of $\tilde{\zeta}_n$, $n = 1, \ldots, N$.

We now consider the following *vanishing viscosity* RB-VV approximation: find $\tilde{u}_N^{\nu v}(\mu) \in \tilde{V}_N$ such that

$$a(\tilde{u}_N^{\nu v}(\mu), v_N; \mu) + d_N(\tilde{u}_N^{\nu v}(\mu), v_N) = F(v_N) \qquad \forall v_N \in \tilde{V}_N, \tag{12}$$

being $d_N(\cdot,\cdot)$ an additional viscosity term, whose action on each couple of basis functions $(\tilde{\zeta}_m, \tilde{\zeta}_n)$, $m, n = 1, \ldots, N$ is such that

$$d_N(\tilde{\zeta}_m, \tilde{\zeta}_n) = f(\lambda_n) \int_\Omega \nabla\tilde{\zeta}_n \cdot \nabla\tilde{\zeta}_m \, \mathrm{d}\mathbf{x},$$

with $f(\lambda_n)$ to be properly defined. Algebraically, we turn to solve the following reduced system:

$$(\tilde{\mathbb{Z}}^{\mathrm{T}}\mathbb{A}_h(\mu)\tilde{\mathbb{Z}} + \mathbb{S}_N)\mathbf{u}_N(\mu) = \tilde{\mathbb{Z}}^{\mathrm{T}}\mathbf{f}_h$$

where $\mathbb{S}_N \in \mathbb{R}^{N\times N}$ is a diagonal matrix, whose generic element is given by

$$(\mathbb{S}_N)_{mn} = d_N(\tilde{\zeta}_m, \tilde{\zeta}_n) = f(\lambda_n) \int_\Omega \nabla\tilde{\zeta}_n \cdot \nabla\tilde{\zeta}_m \, \mathrm{d}\mathbf{x} = \begin{cases} 0, & n \neq m \\ f(\lambda_n)(\mathbb{D}_N)_{nn} = f(\lambda_n)\lambda_n, & n = m. \end{cases}$$

In the simpler case, $f(\lambda_n) = c$, $c \in \mathbb{R}$, $c > 0$, we would add a viscosity contribution on each mode proportional to λ_n for any $n = 1, \ldots, N$. This is, in fact, what the RB–SUPG method (8) does somehow in practice (upon rotating the basis functions of V_N), and what we want to avoid; we rather seek for a *less intrusive* technique yielding a non-negligible effect only on those (energetic) modes that effectively need to be stabilized.

By choosing a nonconstant function $f(\lambda_n)$, we add an artificial viscosity only to those (energetic) modes which effectively need to be stabilized. To determine $f(\lambda_n)$, we exploit a strategy similar to the *spectral vanishing viscosity* method by Maday and Tadmor [8], i.e., we consider a *non-uniform artificial viscosity* term under the form $(\mathbb{S}_N)_{nn} = f(\lambda_n)\lambda_n$, where

$$f(\lambda_n) = \begin{cases} 0 & n < \bar{N}_1 \\ c\dfrac{(\lambda_n - \lambda_{\bar{N}_1})^2}{(\lambda_{\bar{N}_2} - \lambda_{\bar{N}_1})^3}\left(2\lambda_{\bar{N}_2}^2 - (\lambda_{\bar{N}_1} + \lambda_{\bar{N}_2})\lambda_n\right) & \bar{N}_1 \le n \le \bar{N}_2 \\ c\lambda_n & n > \bar{N}_2 \end{cases} \tag{13}$$

and $\bar{N}_1 \geq 0$, $\bar{N}_2 \leq N$ are prescribed indices. The expression of f in the range $N_1 \leq n \leq N_2$ is determined by ensuring a $\mathcal{C}^1$ regularity on the weighting coefficient. In other words, we consider no stabilization over the first N_1 modes; on the *intermediate* modes ($\bar{N}_1 \leq n \leq \bar{N}_2$) a stabilization term of the form

$$(\mathbb{S}_N)_{nn} = f(\lambda_n)\lambda_n = (\lambda_n - \lambda_{\bar{N}_1})^2 \left(\frac{2c\lambda_{\bar{N}_2}^2}{(\lambda_{\bar{N}_2} - \lambda_{\bar{N}_1})^3} - \frac{\lambda_{\bar{N}_1} + \lambda_{\bar{N}_2}}{(\lambda_{\bar{N}_2} - \lambda_{\bar{N}_1})^3} c\lambda_n\right)\lambda_n,$$

and, finally, a stabilization term proportional to λ_n^2 on the higher modes, that is, $(\mathbb{S}_N)_{nn} = f(\lambda_n)\lambda_n = c\lambda_n^2$ for those modes $n \geq \bar{N}_2$ that require a stronger stabilization. The constant $c = \nu/\lambda_N > 0$ is the ratio between the desired added viscosity ν on the highest mode (to be selected, depending on the problem at hand) and the largest eigenvalue λ_N, that is, $f(\lambda_N) = \nu$ on the highest mode.

3. A further post-processing based on a rectification method

Once the problem (12) has been solved, we perform a further *rectification* (as introduced in [1]) to improve its accuracy, inspired by the *spectral filtering technique* [4], generally following the use of the SVD. Essentially, we correct the *consistency* error of the RB–VV approximation $\tilde{u}_N^{vv}(\mu) = \sum_{k=1}^N \beta_k(\mu)\tilde{\zeta}_k$, i.e. the fact that

$$\tilde{u}_N^{vv}(\mu^i) \neq u_h(\mu^i) \qquad \forall \mu^i \in S_N = \{\mu^1, \ldots, \mu^N\}. \tag{14}$$

Hence, we express the N snapshots over the reduced basis as $u_h(\mu^i) = \sum_{n=1}^N \alpha_n^i \tilde{\zeta}_n$, for any $i = 1, \ldots, N$, and define the matrix $\mathbb{B}$ of components

$$(\mathbb{B})_{in} = \alpha_n^i = (u_h(\mu^i), \zeta_n)_V.$$

Then, we solve the RB problem (12) for each $\mu = \mu^i \in S_N$ and get $\tilde{u}_N^{vv}(\mu^i) = \sum_{k=1}^N \beta_k(\mu^i)\tilde{\zeta}_k$, from which we can define the matrix

$$(\mathbb{B}_R)_{ik} := \beta_k(\mu^i).$$

All these computations can be performed *offline*, since they do not depend on the actual parameter value μ.

Finally, the *rectified* solution $u_N^{vv}(\mu) = \sum_{j=1}^N \beta_j^R(\mu)\tilde{\zeta}_j$ is computed online from $\tilde{u}_N^{vv}(\mu) = \sum_{j=1}^N \beta_j(\mu)\tilde{\zeta}_j$, the solution to problem (12), by computing the new coordinates as $\boldsymbol{\beta}^R(\mu) = \mathbb{B}\mathbb{B}_R^{-1}\boldsymbol{\beta}(\mu)$. Also the matrix $\mathbb{B}\mathbb{B}_R^{-1}$ can be computed once and for all, the rectification requiring a simple multiplication by $\boldsymbol{\beta}(\mu)$ to be performed online for any new $\mu \in \mathcal{D}$. A similar rectification strategy has been also exploited in [6].

Heuristically, the rectification relies on an interpolation process, since $\tilde{u}_N^{vv}(\mu)$ coincides with (the spectral approximation over the reduced basis of) $u_h(\mu)$ for any $\mu = \mu^i$, $i = 1, \ldots, N$; this could explain, at some extent, why this post-processing helps to improve (substantially) the results, as shown in Sect. 4.

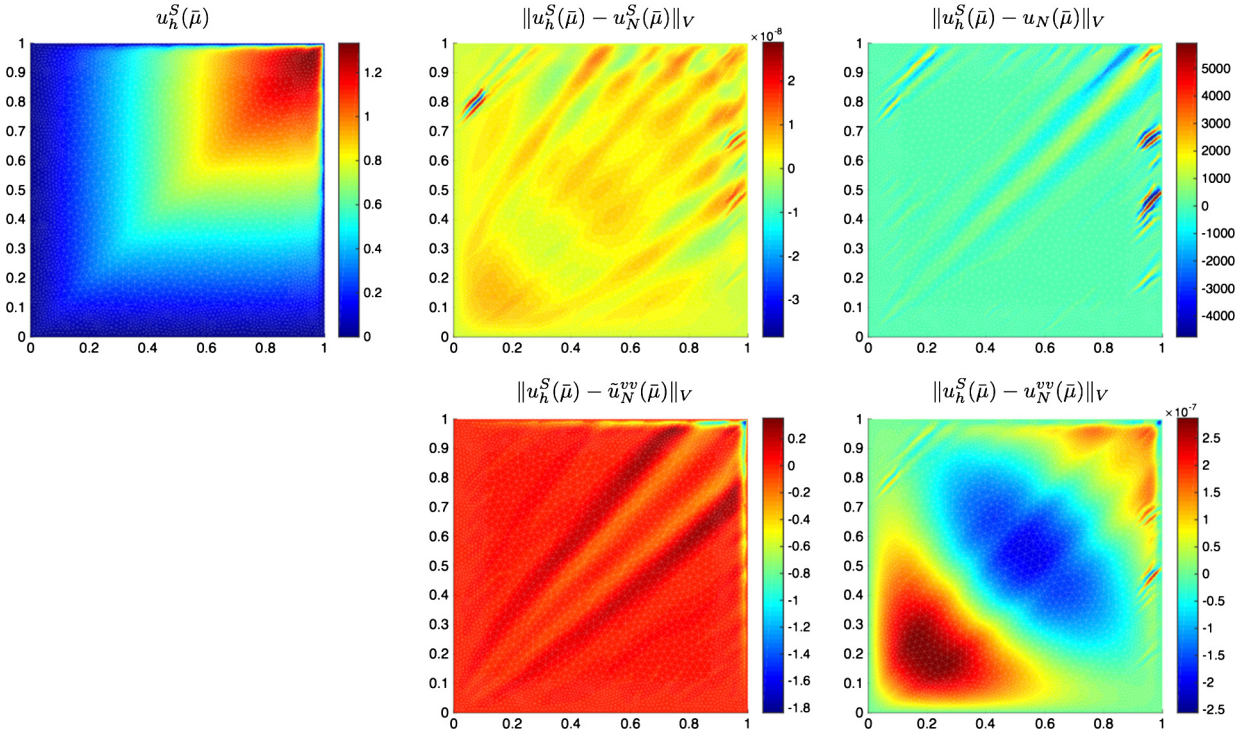

Fig. 1. Top: SUPG solution $u_h^S(\bar{\mu})$, errors $\|u_h^S(\bar{\mu}) - u_N^S(\bar{\mu})\|_V$, $\|u_h^S(\bar{\mu}) - u_N(\bar{\mu})\|_V$ for the approximations G–RB with stabilization and G–RB without stabilization. Bottom: errors $\|u_h^S(\bar{\mu}) - \tilde{u}_N^{vv}(\bar{\mu})\|_V$, $\|u_h^S(\bar{\mu}) - u_N^{vv}(\bar{\mu})\|_V$ for the approximations RB–VV with spectral vanishing viscosity, without and with rectification, respectively. Solutions are reported for the case $\bar{\mu} = 10^{-6}$; errors are of order 10^{-8}, 10^3, 10^{-1} and 10^{-7}, respectively.

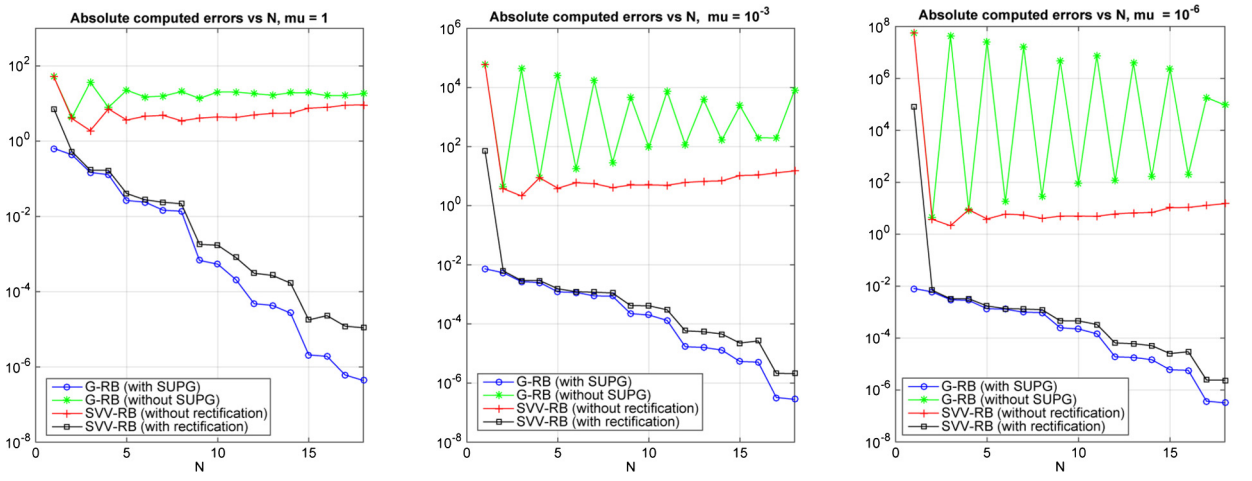

Fig. 2. Errors $\|u_h^S(\bar{\mu}) - u_N^S(\bar{\mu})\|_V$ (in blue), $\|u_h^S(\bar{\mu}) - u_N(\bar{\mu})\|_V$ (in green), $\|u_h^S(\bar{\mu}) - \tilde{u}_N^{vv}(\bar{\mu})\|_V$ (in red), $\|u_h^S(\bar{\mu}) - u_N^{vv}(\bar{\mu})\|_V$ (in black) for different $N = 1, \ldots, N_{\max} = 18$, for $\bar{\mu} = 1$ (left), $\bar{\mu} = 10^{-3}$ (center), $\bar{\mu} = 10^{-6}$ (right).

4. Numerical results

We now compare the approximations obtained through *(i)* the high-fidelity SUPG method, solving (4); *(ii)* the SUPG–RB method, solving (8); *(iii)* the RB method, solving the pure Galerkin problem (7) without any *online* stabilization term; *(iv)* the RB–VV method (12), relying on the online spectral vanishing viscosity and, finally, *(v)* the RB–VV method with the rectification post-processing. These five approximations are denoted, for any $\mu \in \mathcal{P}$, by $u_h^S(\mu)$, $u_N^S(\mu)$, $u_N(\mu)$, $\tilde{u}_N^{vv}(\mu)$, $u_N^{vv}(\mu)$, respectively. We set $f = 1$ and we build the RB space V_N by relying on a Singular Value Decomposition approach (or POD like that are somehow expensive) or on a greedy algorithm: at each step n, this latter algorithm retains the snapshot $u_h^S(\mu^n)$ predicted to be worst approximated by the current RB space V_{n-1}; an a posteriori error bound in the spirit of [12] or [3] is used as an error indicator in this respect (the optimality of the greedy approach being related to the robustness and

efficiency of the a posteriori estimator which is not the subject of this note). For the case at hand, by imposing a tolerance of 10^{-5} on the relative error bound, the greedy algorithm selects $N_{\max} = 18$ basis functions; the space $\tilde{V}_N$ is then obtained by rotating the basis function of V_N. A latin hypercube sampling on $\eta \in [0, 6]$ allows us to define the training sample for $\mu = 10^{-\eta}$ required to run the (weak) greedy algorithm; see, e.g., [10] for further details.

Then, the accuracy of the RB approximations is tested for $N = 1, \ldots, N_{\max}$ and different values of $\mu \in \mathcal{P}$; we report in Fig. 1 the errors (in the V-norm[1]) between $u_h^S(\bar{\mu})$ and the four RB approximations introduced above, for $\bar{\mu} = 10^{-6}$, corresponding to the smallest viscosity coefficient in $\mathcal{P}$; then, we report in Fig. 2 the behavior, with respect to N, of the errors (in the V-norm) in three different cases, $\mu = 1, 10^{-3}, 10^{-6}$.

Clearly, the pure Galerkin RB approximation (without SUPG stabilization terms) is not stable, whereas the SUPG–RB method is stable and accurate over the whole parametric range, showing an exponential error decay – this indeed confirms theoretical results dealing with elliptic parametrized PDEs in case of parametric analytic regularity (see, e.g., [10]).

Applying the rectification method to the G–RB solution does not cure the instability, and for this reason the results have not been reported: indeed, the rectification method needs as many constraints as possible to be effective – i.e., N should be as large as possible – but this usually entails worse and worse G–RB approximations. The spectral vanishing viscosity method, for which here we take $c = 10^{-2}$, $\bar{N}_1 = \lfloor N/3 \rfloor$ and $\bar{N}_2 = \lfloor 2N/3 \rfloor$, yields a stable solution, but not accurate enough. A further post-processing relying on the proposed rectification method allows us to recover the accuracy in this case, too, yielding an error decay indeed very close to the one provided by the SUPG–RB method.

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[1] Note that the V-norm is natural here since we are considering a problem with an outflow layer; in more general situations with inner layers where the crosswind derivative may dominate, the advective derivative norm may be more appropriate.