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Functional analysis

On the structure of invariant Banach limits

Sur la structure des limites de Banach invariantes

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ABSTRACT

A functional B on the space of bounded real sequences ℓ_∞ is said to be a Banach limit if $B \geqslant 0$, $B(1,1,\ldots) = 1$ and $B(Tx) = B(x)$ for every $x = (x_1, x_2, \ldots) \in \ell_\infty$, where T is a translation operator. The set of all Banach limits $\mathfrak{B}$ is a closed convex set on the unit sphere of ℓ_∞^*. Let C be Cesàro operator $(Cx)_n = \frac{1}{n}\sum_{k=1}^n x_k$, $n = 1, 2, \ldots$ Denote $\mathfrak{B}(C) = \{B \in \mathfrak{B} : B = BC\}$.
The cardinality of the set of extreme points $\operatorname{ext}\mathfrak{B}(C)$ is $2^{\mathfrak{c}}$, where $\mathfrak{c}$ is the cardinality of continuum. A subspace generated by any countable collection from $\operatorname{ext}\mathfrak{B}(C)$ is isometric to ℓ_1. For given $B \in \mathfrak{B}$, $r \in (0, 2]$, we denote

$$S_{B,r} = \{D \in \mathfrak{B} : \|D - B\|_{\ell_\infty^*} = r\}.$$

We prove that $B \in \operatorname{ext}\mathfrak{B}$ if and only if the sphere $S_{B,r}$ is convex for every $r \in (0, 2)$.

RÉSUMÉ

Une forme linéaire B sur l'espace ℓ_∞ des suites bornées est appelée une limite de Banach si $B \geqslant 0$, $B(1,1,\ldots) = 1$ et $B(Tx) = B(x)$ pour tout $x = (x_1, x_2, \ldots) \in \ell_\infty$, T désignant l'opérateur de translation. L'ensemble $\mathfrak{B}$ des limites de Banach est un sous-ensemble convexe fermé de la shère unité de ℓ_∞^*. Soit C l'opérateur de Cesàro, $(Cx)_n = \frac{1}{n}\sum_{k=1}^n x_k$, $n = 1, 2, \ldots$ Posons $\mathfrak{B}(C) = \{B \in \mathfrak{B} : B = BC\}$.
La cardinalité de l'ensemble des points extrémaux $\operatorname{ext}\mathfrak{B}(C)$ est $2^{\mathfrak{c}}$, où $\mathfrak{c}$ désigne la cardinalité du continuum. Un sous-espace engendré par une famille dénombrable de $\operatorname{ext}\mathfrak{B}(C)$ est isométrique à ℓ_1. Étant donnés $B \in \mathfrak{B}$ et $r \in (0, 2]$, notons

$$S_{B,r} = \{D \in \mathfrak{B} : \|D - B\|_{\ell_\infty^*} = r\}.$$

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Nous montrons que $B \in \operatorname{ext}\mathfrak{B}$ si et seulement si la sphère $S_{B,r}$ est convexe pour tout $r \in (0,2)$.

Version française abrégée

Nous introduisons la notion d'opérateur $\mathfrak{B}$-propre et étudions les propriétés de l'ensemble des limites de Banach invariantes par un opérateur $\mathfrak{B}$-propre. Nous donnons un critère aisément vérifiable pour qu'un opérateur soit $\mathfrak{B}$-propre.

Proposition 0.1. *Un opérateur linéaire borné W agissant sur ℓ_∞ est $\mathfrak{B}$-propre si et seulement si la suite $W\mathbb{1}$ est presque convergente vers 1, $q(Wx) \geqslant 0$ pour tout $x \geqslant 0$ et l'espace ac_0 est W-invariant.*

Il est facile de voir que l'opérateur de Cesàro

$$(Cx)_n = \frac{1}{n}\sum_{k=1}^{n} x_k, \quad n \in \mathbb{N}$$

et l'opérateur de dilatation

$$\sigma_n(x_1, x_2, \ldots) = (\underbrace{x_1, x_1, \ldots, x_1}_{n}, \underbrace{x_2, x_2, \ldots, x_2}_{n}, \ldots), \quad n \in \mathbb{N},$$

vérifient les hypothèses de la proposition ci-dessus ; ce sont par conséquent des opérateurs $\mathfrak{B}$-propres.

Nous généralisons des résultats de [11] en montrant que, pour tout opérateur $\mathfrak{B}$-propre W qui, soit préserve les intervalles, soit est positif et contractant, et pour toute suite $\{B_n\}$ de points extrêmaux distincts de l'ensemble des limites de Banach W-invariantes, la fermeture $[B_n]$ de l'espace vectoriel engendré par cette suite est isométriquement isomorphe à ℓ_1.

Désignant par $\mathfrak{B}(C)$ l'ensemble des limites de Banach invariantes par l'opérateur de Cesàro, on démontre les résultats suivants.

Théorème 0.2. *On a*

$$\mathfrak{B}(C) \subset \mathfrak{B} \setminus \operatorname{conv}^n(\operatorname{ext}\mathfrak{B}),$$

où $\operatorname{conv}^n(\operatorname{ext}\mathfrak{B})$ désigne la fermeture de l'enveloppe convexe de $\operatorname{ext}\mathfrak{B}$ pour la topologie de la norme de l_∞^*.

Théorème 0.3. *La cardinalité de $\operatorname{ext}\mathfrak{B}(C)$ est égale à 2^c.*

Nous terminons cette note par une caractérisation des points extrémaux des limites de Banach en termes de convexité des sphères centrées en celles-ci :

Théorème 0.4.

(i) *Pour tout $B \in \mathfrak{B}$, la sphère $S_{B,2}$ est un sous-ensemble convexe de $\mathfrak{B}$;*
(ii) *étant donné $B \in \mathfrak{B}$, la sphère $S_{B,r}$ est convexe pour tout $r \in (0,2)$ si et seulement si $B \in \operatorname{ext}\mathfrak{B}$;*
(iii) *pour tout $r \in (0,2)$ il existe $B \in \mathfrak{B}$ tel que la sphère $S_{B,2}$ n'est pas convexe.*

English version

1. Introduction

Throughout the paper, we denote by ℓ_∞ the space of all bounded real sequences $x = (x_1, x_2, \ldots)$ equipped with the norm

$$\|x\|_{\ell_\infty} = \sup_{n\in\mathbb{N}} |x_n|,$$

and the usual partial order. Here $\mathbb{N}$ stands for the set of natural numbers.

A linear functional $B \in \ell_\infty^*$ is said to be a Banach limit (some authors use the term Banach–Mazur limit or extended limits) if

(1) $B \geqslant 0$, that is $Bx \geqslant 0$ for every $x \geqslant 0$,
(2) $B\mathbb{1} = 1$, where $\mathbb{1} = (1, 1, \ldots)$,
(3) $B(Tx) = B(x)$ for every $x \in \ell_\infty$, where T is a translation operator, that is $T(x_1, x_2, \ldots) = (0, x_1, x_2, x_3, \ldots)$.

The existence of Banach limits was established by S. Mazur and then the proof appeared in the book of S. Banach [3]. We denote the set of all Banach limits by $\mathfrak{B}$. It follows from the definition that $\mathfrak{B}$ is a closed convex set on the unit sphere of ℓ_∞^*, $\liminf_{n\to\infty} x_n \leqslant Bx \leqslant \limsup_{n\to\infty} x_n$ for every $x = (x_1, x_2, \ldots) \in \ell_\infty$, $B \in \mathfrak{B}$. In particular, $Bx = \lim_{n\to\infty} x_n$ for every convergent sequence $x \in \ell_\infty$.

G.G. Lorentz [8] proved that for every $x \in \ell_\infty$, $a \in \mathbb{R}^1$, the equality $Bx = a$ holds for all $B \in \mathfrak{B}$ if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k = a$$

uniformly in $m \in \mathbb{N}$. In this case, x_k is said to be almost convergent to a. The set of all almost convergent sequences is denoted by ac. For example, $B\left((-1)^k\right) = 0$ for all $B \in \mathfrak{B}$. Lorentz's result was strengthened by L. Sucheston [13], who proved that for every $x \in \ell_\infty$, one has

$$\{Bx : B \in \mathfrak{B}\} = [q(x), p(x)],$$

where

$$q(x) = \lim_{n\to\infty} \inf_{m\in\mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k, \quad p(x) = \lim_{n\to\infty} \sup_{m\in\mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k.$$

Two years after Lorentz's fundamental paper [8] was published, an interesting paper by W. Eberlein [7] appeared. In this paper, he established the existence of Banach limits invariant under Hausdorff transformations. In [6,12], analogous results were proved for the dilation operator and the Cesàro operator. Eberlein's approach was further extended in [10].

Let Γ denote the set of all linear operators $H \in \mathcal{L}(\ell_\infty)$ satisfying the following conditions:

(i) $H \geqslant 0$ and $H\mathbb{1} = 1$,
(ii) $Hc_0 \subset c_0$,
(iii) $\limsup_{j\to\infty} (A(I-T)x)_j \geqslant 0$ for all $x \in \ell_\infty$, $A \in R(H)$, where $R(H) = \operatorname{conv}\{H^k, k \geqslant 0\}$.

For example, conditions (i), (ii), (iii) are satisfied by the Cesàro operator

$$(Cx)_n = \frac{1}{n} \sum_{k=1}^{n} x_k, \quad n \in \mathbb{N}$$

and the dilation operator

$$\sigma_n(x_1, x_2, \ldots) = (\underbrace{x_1, x_1, \ldots, x_1}_{n}, \underbrace{x_2, x_2, \ldots, x_2}_{n}, \ldots), \quad n \in \mathbb{N}.$$

It was proved in [10] that, for every $H \in \Gamma$, there exists $B \in \mathfrak{B}$ such that $Bx = BHx$ for all ℓ_∞. Denote the set of Banach limits invariant under H by $\mathfrak{B}(H)$. Clearly, $\mathfrak{B}(H)$ is a closed convex subset of $\mathfrak{B}$. Under some additional assumptions on $H \in \Gamma$, the diameter of $\mathfrak{B}(H)$, that is the number $\sup_{B_1, B_1 \in \mathfrak{B}(H)} \|B_1 - B_2\|_{\ell_\infty^*}$, coincide with the diameter of $\mathfrak{B}$ and is equal to 2. In particular, the operators C and σ_n satisfy the assumptions mentioned above. Also, invariant Banach limits were considered in [4,11]. The present paper continues the study of the sets $\mathfrak{B}(C)$ and $\mathfrak{B}(\sigma_n)$.

2. Main section

By Krein–Milman's theorem, the set $\mathfrak{B}$ is compact in $\sigma(\ell_\infty^*, \ell_\infty)$ topology and $B = \overline{\operatorname{conv}} \operatorname{ext} \mathfrak{B}$, where $\operatorname{ext} \mathfrak{B}$ is the set of extreme points of $\mathfrak{B}$ and the closure is taken in weak-* topology of the space ℓ_∞^*. Similar facts hold for the sets $\mathfrak{B}(C)$ and $\mathfrak{B}(\sigma_n)$, that is,

$$\mathfrak{B}(C) = \overline{\operatorname{conv}} \operatorname{ext} \mathfrak{B}(C),$$

$$\mathfrak{B}(\sigma_n) = \overline{\operatorname{conv}} \operatorname{ext} \mathfrak{B}(\sigma_n).$$

Consider a more general case.

Definition 2.1. *A linear bounded operator W acting on ℓ_∞ is called $\mathfrak{B}$-proper if its adjoint W^* maps the set $\mathfrak{B}$ into itself, that is, $W^*\mathfrak{B} \subseteq \mathfrak{B}$.*

If W is $\mathfrak{B}$-proper, then by the Brouwer–Schauser–Tychonoff Theorem [2, Corollary 17.56], the set $\mathfrak{B}(W)$ is non-empty. Since it is easily seen to be compact and convex, by the Krein–Milman theorem the set $\operatorname{ext}\mathfrak{B}(W)$ is non-empty too. We state an easily verifiable criterion for an operator to be $\mathfrak{B}$-proper.

Proposition 2.2. *A linear bounded operator W acting on ℓ_∞ is $\mathfrak{B}$-proper if and only if the sequence $W\mathbb{1}$ is almost converging to 1, $q(Wx) \geqslant 0$ for all $x \geqslant 0$, and the space ac_0 is W-invariant.*

It is easy to see that the operators σ_n and C satisfy the assumptions of the proposition above and, thus, are $\mathfrak{B}$-proper operators.

In the next theorem, we consider conditions that guarantee that two distinct functionals $B, D \in \operatorname{ext}\mathfrak{B}(W)$ are disjoint in the Banach lattice ℓ_∞^*, that is, $B \wedge D = 0$. Recall that a positive operator W acting in a Banach lattice E is called (see, e.g., [9, Definition 1.4.18]) *interval preserving* (*almost interval preserving*, resp.) if $W[0, x] = [0, Wx]$ ($W[0, x]$ is dense in $[0, Wx]$, resp.) for all $x \in E^+$. An operator W is called a *lattice homomorphism* (see, e.g., [9, Definition 1.3.10]) if it preserves the lattice operations. By Ando's Theorem [9, Theorem 1.4.19], an operator W acting in a Banach lattice E is almost interval preserving if and only if W^* is a lattice homomorphism. Clearly, the translation operator T is interval preserving. Thus, an operator T^* is a lattice homomorphism. In particular, for $B, D \in \mathfrak{B}$, we have $T^*(B \wedge D) = B \wedge D$. Hence, $B \wedge D > 0$ implies $\frac{B\wedge D}{\|B\wedge D\|_{\ell_\infty^*}} \in \mathfrak{B}$ (this also follows from the fact that T is non-expansive, that is $\|T\|_{\mathcal{L}(\ell_\infty)} \leqslant 1$). On the other hand, the dilation operator σ_n, $n \geqslant 2$ and the Cesàro operator C are not interval preserving. However, the generalised Cesàro operator $Q : \ell_\infty \to \ell_\infty$, defined by the formula

$$Qx = (x_1, \frac{x_2 + x_3}{2}, \frac{x_4 + x_5 + x_6}{3}, \ldots)$$

is interval preserving.

Theorem 2.3. *Let W be a $\mathfrak{B}$-proper operator and let $B, D \in \operatorname{ext}\mathfrak{B}(W)$, $B \neq D$. The functional $B \wedge D$ is W-invariant if and only if $B \wedge D = 0$. In particular, if either*

(i) *W is interval preserving or*
(ii) *W is positive and non-expansive,*

then $B \wedge D = 0$ for all distinct $B, D \in \operatorname{ext}\mathfrak{B}(W)$. In this case $\|B - D\|_{l_\infty^*} = 2$.

The dilation operator σ_n and the Cesàro operator C are positive and non-expansive. Thus, the part (i) of the preceding theorem implies the following result.

Corollary 1. *If either $B, D \in \operatorname{ext}\mathfrak{B}(C)$ or $B, D \in \operatorname{ext}\mathfrak{B}(\sigma_n)$ and $B \neq D$, then $B \wedge D = 0$.*

We will need the following lemma.

Lemma 2. *For every disjoint bounded sequence $\{x_n\}$ in AL-space E such that $x_n \neq 0$ for all n, the closure $[x_n]$ of a linear hull of the set $\{x_1, x_2, \ldots\}$ is isometric to l_1. If, additionally $x_n \geqslant 0$ for all n, then this isometry can be chosen to be an order isometry.*

The preceding lemma and Theorem 2.3 imply the following result.

Corollary 3. *Let W be a $\mathfrak{B}$-proper operator and let $\{B_n\}$ be a sequence of distinct elements from $\operatorname{ext}\mathfrak{B}(W)$. If $W^*(B_i \wedge B_j) = B_i \wedge B_j$ for all i, j, then the space $[B_n]$ is isometrically isomorphic to ℓ_1.*

In particular, if W satisfies either condition (i) or (ii) of Theorem 2.3, then $[B_n]$ is isometrically isomorphic to ℓ_1 for every sequence $\{B_n\}$ of distinct elements from $\operatorname{ext}\mathfrak{B}(W)$.

Sets $\operatorname{ext}\mathfrak{B}$ and $\operatorname{ext}\mathfrak{B}(C)$ are disjoint. Moreover, we have the following result.

Theorem 2.4. *One has*

$$\mathfrak{B}(C) \subset \mathfrak{B} \setminus \operatorname{conv}^n(\operatorname{ext}\mathfrak{B}),$$

where $\operatorname{conv}^n(\operatorname{ext}\mathfrak{B})$ denotes the closure of a convex hull of $\operatorname{ext}\mathfrak{B}$ in norm topology of l_∞^*.

The proof of Theorem 2.4 follows directly from [11, Theorem 14].

The cardinality of the set $\operatorname{ext}\mathfrak{B}$ is equal to 2^c, where c is the cardinality of the continuum [5]. We complement this result.

Theorem 2.5. *The cardinality of* $\operatorname{ext}\mathfrak{B}(C)$ *equals to* 2^c.

It follows from Theorems 2.3 and 2.5 that the cardinalities of extreme points and between any two extreme points of $\mathfrak{B}$ and of $\mathfrak{B}(C)$ are the same. Loosely speaking, $\mathfrak{B}$ and $\mathfrak{B}(C)$ are simplices of dimension 2^c. However, $\mathfrak{B}(C) \subset \mathfrak{B}$, and Theorem 2.4 describes the location of $\mathfrak{B}(C)$ in $\mathfrak{B}$.

Interrelations between sets $\mathfrak{B}(\sigma_n)$ for different $n \in \mathbb{N}$, $n \geqslant 2$ were considered in [1]. Now we consider the relation between $\mathfrak{B}(\sigma_n)$ and $\mathfrak{B}(C)$. Note that it was proved in [6] that there exists $B \in \mathfrak{B}(C)$ such that $B \in \mathfrak{B}(\sigma_n)$ for all $n \in \mathbb{N}$.

Theorem 2.6. *If* $m \in \mathbb{N}$, $m \geqslant 2$, *then there exists* $B \in \mathfrak{B}(\sigma_m)$ *such that* $B \overline{\in} \mathfrak{B}(C)$.

We finish this note with a result concerning the spheres in $\mathfrak{B}$. Clearly, in every Banach space, every sphere of strictly positive radius is a non-convex set. For given $B \in \mathfrak{B}$, $r \in (0, 2]$, we denote

$$S_{B,r} = \{D \in \mathfrak{B} : \|D - B\|_{\ell_\infty^*} = r\}.$$

Clearly, $S_{B,r}$ is a non-empty subset of $\mathfrak{B}$ for all $B \in \mathfrak{B}$, $r \in (0, 2]$.

Theorem 2.7.

(i) *For every* $B \in \mathfrak{B}$ *a sphere* $S_{B,2}$ *is a convex subset of* $\mathfrak{B}$*;*
(ii) *for given* $B \in \mathfrak{B}$, *a sphere* $S_{B,r}$ *is convex for every* $r \in (0, 2)$ *if and only if* $B \in \operatorname{ext}\mathfrak{B}$*;*
(iii) *for every* $r \in (0, 2)$, *there exists* $B \in \mathfrak{B}$ *such that a sphere* $S_{B,2}$ *is non-convex.*

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