0. Introduction

Let $X$ be a compact complex orbifold of dimension $n$. That is, $X$ is a complex space endowed with the following property: each point $p \in X$ possesses a neighborhood, which is the quotient $\tilde{U}/G_p$, where $\tilde{U}$ is a complex manifold, say of dimension $n$, and $G_p$ is a properly discontinuous finite group of automorphisms of $\tilde{U}$, so that locally we have a quotient map $(\tilde{U}, \tilde{p}) \rightarrow (\tilde{U}/G_p, p)$. See [1].

Let $\eta(X)$ be the complex Lie algebra of all holomorphic vector fields of $X$. Choose any Hermitian metric $h$ on $X$ and let $\nabla$ and $\Theta$ be the Hermitian connection and the curvature form with respect to $h$, respectively. Let $\xi$ be a global holomorphic vector field on $X$ and consider the operator

$$L(\xi):= [\xi, \cdot] - \nabla_\xi (\cdot) : T^{1,0}X \longrightarrow T^{1,0}X.$$
Let $\phi$ be an invariant polynomial of degree $n + k$; the Futaki–Morita integral invariant is defined by

$$f_\phi(\xi) = \int_X \tilde{\phi}(L(\xi), \ldots, L(\xi), \frac{i}{2\pi} \Theta, \ldots, \frac{i}{2\pi} \Theta),$$

where $\tilde{\phi}$ denotes the polarization of $\phi$. Morita and Futaki proved in [6] that the definition of $f_\phi(\xi)$ does not depend on the choice of the Hermitian metric $h$. It is well known that the Futaki–Morita integral invariant can be calculated via a Bott-type residue formula for non-degenerated holomorphic vector fields, see [5–7] and [4] in the orbifold case. In this work, we prove a residue formula for holomorphic vector fields with isolated and possibly degenerated singularities in terms of Grothendieck’s residues (see [8, Chapter 5]).

**Theorem 1.** Let $\xi \in \eta(X)$ a holomorphic vector field with only isolated singularities, then

$$(n + k) \frac{n}{n} f_\phi(\xi) = (-1)^k \sum_{p \in \text{Sing}(\xi)} \frac{1}{\#G_p} \text{Res}_p \left\{ \frac{\phi(J\tilde{\xi}) d\tilde{\xi}_1 \wedge \cdots \wedge d\tilde{\xi}_n}{\tilde{\xi}_1 \cdots \tilde{\xi}_n} \right\},$$

where, given $p$ such that $\xi(p) = 0$ and $(\tilde{U}, \tilde{p}) \to (\tilde{U}/G_p, p)$ denotes the projection: $\tilde{\xi} = \pi_p^* \xi$, $J\tilde{\xi} = \left( \frac{\partial \tilde{\xi}_i}{\partial \tilde{z}_j} \right)_{1 \leq i, j \leq n}$ and

$$\text{Res}_p \left\{ \frac{\phi(J\tilde{\xi}) d\tilde{\xi}_1 \wedge \cdots \wedge d\tilde{\xi}_n}{\tilde{\xi}_1 \cdots \tilde{\xi}_n} \right\}$$

is Grothendieck’s point residue and $(\tilde{z}_1, \ldots, \tilde{z}_n)$ is a germ of the coordinate system on $(\tilde{U}, \tilde{p})$.

We note that such residue can be calculated using Hilbert’s Nullstellensatz and Martinelli’s integral formula. In fact, if $\tilde{z}_i^{a_i} = \sum_{j=1}^{n} b_{ij} \tilde{\xi}_j$, then (see [11])

$$\text{Res}_p \left\{ \frac{\phi(J\tilde{\xi}) d\tilde{\xi}_1 \wedge \cdots \wedge d\tilde{\xi}_n}{\tilde{\xi}_1 \cdots \tilde{\xi}_n} \right\} = \frac{1}{\prod_{i=1}^{n} (a_i - 1)!} \left( \frac{\partial^n}{\partial z_1^{a_1} \cdots \partial z_n^{a_n}} \text{Det}(b_{ij}) \phi(J\tilde{\xi}) \right) (\tilde{p}).$$

Moreover, note that if $p \in \text{Sing}(\xi)$ is a non-degenerated singularity of $\xi$, then

$$\text{Res}_p \left\{ \frac{\phi(J\tilde{\xi}) d\tilde{\xi}_1 \wedge \cdots \wedge d\tilde{\xi}_n}{\tilde{\xi}_1 \cdots \tilde{\xi}_n} \right\} = \frac{\phi(J\tilde{\xi}(\tilde{p}))}{\text{Det}(J\tilde{\xi}(\tilde{p}))}.$$

**Theorem 1** allows us to calculate the Morita–Futaki invariant for holomorphic vector fields with possible degenerated singularities. For instance, in a recent work [9], the Futaki–Bott residue for vector fields with degenerated singularities, on the blowup of Kähler surfaces, was calculated by Li and Shi. Compare the equation (1) with Lemma 3.6 of [9].

Futaki showed in [5] that if $X$ admits a Kähler–Einstein metric, then $f_{C_1^{n+1}} = 0$, where $C_1 = Tr$ denotes the trace, i.e., the first elementary symmetric polynomial. Taking $\phi = C_1^{n+1} = Tr^{n+1}$, we obtain the following corollary of **Theorem 1**.

**Corollary 2.** Let $\xi \in \eta(X)$ with only isolated singularities, then

$$f_{C_1^{n+1}}(\xi) = \frac{1}{(n + 1)^2} \sum_{p \in \text{Sing}(\xi)} \frac{1}{\#G_p} \text{Res}_p \left\{ \frac{Tr^{n+1}(J\tilde{\xi}) d\tilde{\xi}_1 \wedge \cdots \wedge d\tilde{\xi}_n}{\tilde{\xi}_1 \cdots \tilde{\xi}_n} \right\}.$$

This result generalizes the Proposition 1.2 of [4]. It is well known that projective planes are Kähler–Einstein. However, the non-existence of Kähler–Einstein metrics on singular weighted projective planes was shown in previous works; see, for example, [12]. As an application of **Theorem 1**, we will give, in Section 1, a new proof of this fact.

### 1. Non-existence of Kähler–Einstein metric on weighted projective planes

Here we consider weighted complex projective planes with only isolated singularities, which we briefly recall.

Let $w_0, w_1, w_2$ be positive integers two by two co-primes, set $w := (w_0, w_1, w_2)$ and $|w| := w_0 + w_1 + w_2$. Define an action of $C^\times$ in $C^3 \setminus \{0\}$ by

$$C^\times \times C^3 \setminus \{0\} \to C^3 \setminus \{0\}, \quad (\lambda, (z_0, z_1, z_2)) \mapsto (\lambda^{w_0} z_0, \lambda^{w_1} z_1, \lambda^{w_2} z_2)$$

and let $P_w^2 := C^3 \setminus \{0\}/\sim$. The weights are chosen to be pairwise co-primes in order to assure a finite number of singularities and to give $P_w^2$, the structure of an effective, Abelian, compact orbifold of dimension 2. The singular locus is:
Theorem 2. The singular set \( \text{Sing}(\mathbb{P}^2_w) = \{ [1 : 0 : 0]_w, [0 : 1 : 0]_w, [0 : 0 : 1]_w \} \).

We have the canonical projection
\[
\pi : \mathbb{C}^3 \setminus \{0\} \longrightarrow \mathbb{P}^2_w \quad (z_0, z_1, z_2) \longmapsto [z_0^w : z_1^w : z_2^w]_w
\]
and the natural map
\[
\varphi_w : \mathbb{P}^n \longrightarrow [z_0 : z_1 : z_2] \longmapsto [z_0^w : z_1^w : z_2^w]_w
\]
of degree \( \deg \varphi_w = w_0w_1w_2 \). The map \( \varphi_w \) is good in the sense of [1, section 4.4], which means, among other things, that \( V \)-bundles behave well under pullback. It is shown in [10] that there is a line \( V \)-bundle \( \mathcal{O}_{\mathbb{P}^2_w}(1) \) on \( \mathbb{P}^2_w \), unique up to isomorphism, such that
\[
\varphi_w^* \mathcal{O}_{\mathbb{P}^2_w}(1) \cong \mathcal{O}_{\mathbb{P}^2_w}(1)
\]
and, by naturality, \( c_1(\varphi_w^* \mathcal{O}_{\mathbb{P}^2_w}(1)) = c_1(\mathcal{O}_{\mathbb{P}^2}(1)) = \varphi_w^* c_1(\mathcal{O}_{\mathbb{P}^2_w}(1)) \), from which we obtain the Chern number
\[
\left[ \mathbb{P}^2_w \right] \sim \left( c_1(\mathcal{O}_{\mathbb{P}^2_w}(1)) \right)^n = \int_{\mathbb{P}^2_w} \left( c_1(\mathcal{O}_{\mathbb{P}^2}(1)) \right)^2 = \frac{1}{w_0w_1w_2}
\]
since
\[
1 = \int_{\mathbb{P}^2} \left( c_1(\mathcal{O}_{\mathbb{P}^2}(1)) \right)^2 = \int_{\mathbb{P}^2} \varphi_w^* \left( c_1(\mathcal{O}_{\mathbb{P}^2_w}(1)) \right)^2 = (\deg \varphi_w) \int_{\mathbb{P}^2_w} \left( c_1(\mathcal{O}_{\mathbb{P}^2_w}(1)) \right)^2.
\]

There exist an Euler type sequence on \( \mathbb{P}^n_w \)
\[
0 \longrightarrow \mathbb{C} \longrightarrow \bigoplus_{i=0}^2 \mathcal{O}_{\mathbb{P}^2_w}(w_i) \longrightarrow T\mathbb{P}^2_w \longrightarrow 0,
\]
where

(i) \( 1 \longmapsto (w_0z_0, w_1z_1, w_2z_2) \).

(ii) \( (P_0, P_1, P_2) \longmapsto \pi_* \left( \sum_{i=0}^2 P_i \frac{\partial}{\partial z_i} \right) \).

It is well known that the non-singular weighted projective planes admit Kähler–Einstein metrics. On the other side, singular weighted projective spaces do not admit Kähler–Einstein metrics, see [12]. We give a simple proof of the non-existence of Kähler–Einstein metrics on singular \( \mathbb{P}^2_{\omega} \) by using Corollary 2.

**Theorem 3.** The singular weighted projective space \( \mathbb{P}^2_{\omega} \) does not admit any Kähler–Einstein metric.

**Proof.** Choose \( a_0, a_1, a_2 \in \mathbb{C}^* \) such that \( a_i w_j \neq a_j w_i \) for all \( i \neq j \). Suppose, without loss of generality, that \( 1 \leq w_0 \leq w_2 < w_1 \). Consider the holomorphic vector field on \( \mathbb{P}^2_{\omega} \), given by
\[
\xi_\omega = \sum_{k=0}^2 a_k Z_k \frac{\partial}{\partial Z_k} \in H^0(\mathbb{P}^2_{\omega}, T\mathbb{P}^2_{\omega}),
\]
where \( (Z_0, Z_1, Z_3) \) denotes the homogeneous coordinate system.

The local expression of \( \xi \) over \( U_i = \{ [Z_0 : Z_1 : Z_3] \in \mathbb{P}^2 : Z_i \neq 0 \} \) is given by
\[
\xi_\omega|_{U_i} = \sum_{k=0}^2 \left( a_k - a_i \frac{w_k}{w_i} \right) Z_k \frac{\partial}{\partial Z_k}.
\]

Therefore, the singular set \( \text{Sing}(\xi_\omega|_{U_i}) \) is reduced to \( \{0\} \) and it is nondegenerate. In general,
\[
\text{Sing}(\xi_\omega) = \{ [1 : 0 : 0]_\omega, [0 : 1 : 0]_\omega, [0 : 0 : 1]_\omega \} = \text{Sing}(\mathbb{P}^2_{\omega}).
\]
It follows from Corollary 2 that
\[
f(\xi_0) = \frac{-1}{32} \sum_{i=0}^{n} \frac{1}{w_i^2}(\sum_{k \neq i} (a_k w_i - a_i w_k))^3 \prod_{k \neq i} (a_k w_i - a_i w_k).
\]

Thus
\[
\zeta(a_0, a_1, a_2) = -32 w_0^2 w_1^2 w_2^2 \prod_{0 \leq i < j \leq 2} (a_i w_j - a_j w_i) f(\xi_0) =
\]
\[
(3w_1^2 w_2^2 w_0 - 3w_1^2 w_2 w_0^2 + 3w_1^2 w_2^2 w_0 + 3w_1^2 w_2^2 w_0 - 3w_1^2 w_2^2 w_1 + 3w_1^2 w_2^2 w_1^2 + 6w_0^2 w_4^2 w_2^2 +
\]
\[
+ 3w_0^2 w_2^2 - 3w_0^2 w_2^2 - 6w_0^2 w_4^2 w_2^2) \cdot a_1 a_2 a_3 + \cdots
\]
is a homogeneous polynomial of degree 4 in the variables \(a_0, a_1, a_2\). Suppose by contradiction that \(\zeta(a_0, a_1, a_2) = 0\). In particular, the coefficient of the monomial \(a_0^2 a_1 a_2^2\) is zero. Thus, we have the following equation
\[
w_2(w_1 w_2 + w_2^2 + w_0^2 + 2w_1 + 2w_0) = w_1(w_1 w_2 + w_1^2 + w_0^2 + 2w_0 + w_1).
\]
This contradicts \(1 \leq w_0 \leq w_2 < w_1\). Thus the non-vanishing of \(\zeta(a_0, a_1, a_2)\) implies that \(f(\xi_0)\) is not zero. Therefore, \(\|\alpha\|_\omega\) does not admit Kähler–Einstein metrics. \(\Box\)

2. Proof of Theorem 1

For the proof, we will use Bott–Chern’s transgression method, see [2] and [3].

Let \(p_1, \ldots, p_m\) be the zeros of \(\xi\). Let \(\{U_\beta\}\) be an open cover orbifold of \(X\) (\(\varphi_\beta : \tilde{U}_\beta \to U_\beta \subset X\) coordinate map). Suppose that \(\{U_\beta\}\) is a trivializing neighborhood for the holomorphic tangent orbibundle \(TX\) (see [1, section 2.3]) of \(X\) and that we have disjoint neighborhoods \(U_\alpha\) with \(p_\alpha \in U_\alpha\) and \(p_\alpha \not\in U_\beta\) if \(\alpha \not= \beta\). On each \(\tilde{U}_\alpha\), take local coordinates \(\tilde{z}^\alpha = (\tilde{z}_1^\alpha, \ldots, \tilde{z}_n^\alpha)\) and the holomorphic frame \(\{\tilde{a}_1^\alpha, \ldots, \tilde{a}_n^\alpha\}\) of \(TX\). Thus, we have a local representation
\[
\tilde{\xi}^\alpha = \sum \tilde{z}_i^\alpha \frac{\partial}{\partial \tilde{z}_i^\alpha},
\]
where \(\tilde{z}_i^\alpha\) are holomorphic functions in \(\tilde{U}_\alpha\), \(1 \leq i \leq n\). Let \(h_0\) the Hermitian metric in \(\tilde{U}_\alpha\) defined by \(\langle \partial / \partial \tilde{z}_i^\alpha, \partial / \partial \tilde{z}_j^\alpha \rangle = \delta_{ij}\).

Also consider \(\tilde{U}_\alpha' \subset \tilde{U}_\alpha\) and \(U_\alpha' = \varphi_\alpha(\tilde{U}_\alpha')\) for each \(\alpha\). Take a Hermitian metric \(h_0\) in any \(X\setminus \cup_{\alpha}(p_\alpha)\) and \(\{\rho_0, \rho_\alpha\}\) a partition of unity subordinate to the cover \(X\setminus \cup_{\alpha}(p_\alpha)\). Define a Hermitian metric \(h = \rho_0 h_0 + \sum \rho_\alpha h_0\) in \(X\). Then we have that for every \(\alpha\), the metric curvature \(\Theta = 0\) in \(U_\alpha'\).

Consider the matrix of the metric connection \(\nabla\) in the open \(\tilde{U}^\beta\) given by \(\theta^\beta = (\sum_k \Gamma_{ik}^\beta \tilde{d}z_k^\beta)\).

The local expression of \(L(\xi)\) is given by \(\tilde{E}^\beta = (\tilde{e}^\beta_{ij})\) such that
\[
\tilde{e}^\beta_{ij} = -\frac{\partial \tilde{Z}^\beta_{ij}}{\partial \tilde{Z}^\alpha} - \sum_k \Gamma_{ik}^\beta \tilde{z}_k^\alpha,
\]
see [2] and [8]. We indicate by \(A^{p,q}(X)\) the vector space of complex-valued \((p+q)\)-forms on \(X\) of type \((p,q)\). Define
\[
\phi_r := \binom{n + k}{r} \phi(E, \ldots, E, \Theta, \ldots, \Theta) \in A^{r-r}(X) \quad r = 0, \ldots, n.
\]
Let \(\omega \in A^{1,0}(X)\) in \(X\setminus \text{Sing}(\xi)\), with \(\omega(\xi) = 1\). Following Bott’s idea (see [2]), it is sufficient to show that there exists \(\psi\) such that \(i(\xi)\tilde{\partial} \psi + \phi_r = 0\) on \(X\setminus \text{Sing}(\xi)\). We take \(\psi = \sum_{r=0}^{n-1} \psi_r\) such that
\[
\psi_r = \omega \wedge (\tilde{d} \omega)^{n-r-1} \wedge \phi_r \in A^{n,n-1}(X) \quad r = 0, \ldots, n - 1.
\]
The following formulas hold (see [2] or [8]):

a) \(\tilde{\partial} \Theta = 0\), \(\tilde{\partial} E = i(\xi) \Theta\);

b) \(\tilde{\partial} \phi_r = i(\xi) \phi_{r+1}, r = 0, \ldots, n + 1\);

c) \(i(\xi) \tilde{\partial} \omega = 0\).

Let us prove b): since \(\tilde{\partial} \Theta = 0\) and \(\tilde{\partial} E = i(\xi) \Theta\), we have
\[
\tilde{\partial} \phi_r = \binom{n + k}{r} \sum_{i=1}^{n+k-r} \tilde{\phi}(E, \ldots, i(\xi) \Theta, \ldots, E, \Theta, \ldots, \Theta) = i(\xi) \phi_{r+1}.
\]
Therefore, a), b) and c) implies that on \( X \setminus \text{Sing}(\xi) \) we get
\[
i(\xi)(\bar{\partial} \psi + \phi_n) = 0.
\]
Therefore, \( d\psi = \bar{\partial} \psi = -\phi_n \) on \( X \setminus \text{Sing}(\xi) \). Thus, by the Satake–Stokes Theorem, we have
\[
\left( \frac{n+k}{n} \right) f_\phi(\xi) = \left( \frac{i}{2\pi} \right)^n \int_X \phi_n = \left( \frac{i}{2\pi} \right)^n \lim_{\epsilon \to 0} \int_{X \setminus \cup \Omega B_\epsilon(p_\alpha)} \phi_n = -\left( \frac{i}{2\pi} \right)^n \lim_{\epsilon \to 0} \int_{X \setminus \cup \Omega B_\epsilon(p_\alpha)} d\psi = \left( \frac{i}{2\pi} \right)^n \lim_{\epsilon \to 0} \sum_{\alpha} \int_{\partial B_\epsilon(p_\alpha)} \psi^\alpha,
\]
where is \( B_\epsilon(p_\alpha) = B_\epsilon(\tilde{p}_\alpha)/G_\alpha \) and \( B_\epsilon(\tilde{p}_\alpha) \) is an Euclidean ball centered at \( \tilde{p}_\alpha \) such that \( \nu_\alpha \subset U_\alpha' \). Since our metric is Euclidean in \( B_\epsilon(\tilde{p}_\alpha) \), its connection is zero and
\[
\tilde{E}_{ij}^\alpha = -\bar{\partial} \xi^\alpha_i/\bar{\partial} z^j.
\]
Therefore, \( \tilde{E}_{ij}^\alpha = -\bar{\partial} \xi^\alpha_i/\bar{\partial} z^j \).

Now, by our choice of metric, \( \Theta \) and hence \( \psi_r \), for \( r > 0 \), vanishes identically in \( B_\epsilon(\tilde{p}_\alpha) \). Then, we have
\[
\tilde{\psi}^\alpha = \tilde{\psi}_j^\alpha = \psi \wedge (\bar{\partial} \psi)^{-1} \phi(\tilde{E}^\alpha) = (-1)^{n+k} \omega \wedge (\bar{\partial} \psi)^{-1} \phi(j\tilde{E}^\alpha)
\]
on \( B_\epsilon(\tilde{p}_\alpha) \). Therefore,
\[
\tilde{\psi}^\alpha = (-1)^{k+1} \omega \wedge (\bar{\partial} \psi)^{-1} \phi(j\tilde{E}^\alpha).
\]
Consider the map \( \Phi : \mathbb{C}^n \to \mathbb{C}^{2n} \) given by \( \Phi(\bar{z}) = (\bar{z} + \xi(\bar{z}), \bar{z}) \). There is a \((2n, 2n-1)\) closed form \( \beta_n \) in \( \mathbb{C}^{2n}\setminus\{0\} \) (the Bochner–Martinelli kernel) such that
\[
\Phi^* \beta_n = \left( \frac{i}{2\pi} \right)^n \omega \wedge (\bar{\partial} \psi)^{-1} \phi(j\tilde{E}^\alpha).
\]
Finally, if we substitute (3) and (4) into (2), and by using Martinelli’s formula ([8, p. 655])
\[
\int_{\partial B_\epsilon(\tilde{p}_\alpha)} \phi(j\tilde{E}^\alpha) \Phi^* \beta_n = \text{Res} \rho_n \left\{ \frac{\phi(j\tilde{E}^\alpha)}{\tilde{E}_1 \cdots \tilde{E}_n} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \right\}
\]
we obtain
\[
\left( \frac{n+k}{n} \right) f_\phi(\xi) = (-1)^{k} \sum_{\alpha} \frac{1}{\#G_\alpha} \text{Res} \rho_n \left\{ \frac{\phi(j\tilde{E}^\alpha)}{\tilde{E}_1 \cdots \tilde{E}_n} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \right\}.
\]

**Acknowledgements**

We are grateful to Marcio Soares and Marcos Jardim for many stimulating conversations on this subject.

**References**