Functional analysis

Toral and spherical Aluthge transforms of 2-variable weighted shifts

Transformations d’Aluthge torales et sphériques de shifts pondérés à deux variables

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A B S T R A C T
We introduce two natural notions of Aluthge transforms (toral and spherical) for 2-variable weighted shifts and study their basic properties. Next, we study the class of spherically quasinormal 2-variable weighted shifts, which are the fixed points for the spherical Aluthge transform. Finally, we briefly discuss the relation between spherically quasinormal and spherically isometric 2-variable weighted shifts.
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R É S U M É
Nous introduisons deux notions naturelles des transformations d’Aluthge (torales et sphériques) pour les shifts pondérés à deux variables et nous étudions leurs propriétés. Ensuite, nous étudions la classe de shifts pondérés à deux variables sphériques et quasinormaux, qui sont les points fixes pour la transformation d’Aluthge sphérique. Enfin, nous discutons brièvement la relation entre les shifts pondérés à deux variables qui sont sphériquement quasinormaux et ceux qui sont sphériquement isométriques.
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Pour $T \in B(\mathcal{H})$, la décomposition polaire de $T$ est $T = U|T|$, où $U$ est une isométrie partielle et $|T| := \sqrt{T^*T}$. La transformation d’Aluthge de $T$ est l’opérateur $\tilde{T} := |T|^2 U|T|^2$. Cette transformation a été considérée pour la première fois dans la référence [1] afin d’étudier des opérateurs $p$-hyponormal et log-hyponormal. En bref, l’idée sous-jacente à la transformation d’Aluthge est de convertir un opérateur en un autre qui partage avec le premier quelques propriétés spectrales,

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mais qui reste plus proche d’un opérateur normal. Ces dernières années, la transformation d’Aluthge a reçu une attention considérable.

Dans cet article, nous présentons d’abord la décomposition polaire des opérateurs bornés, puis nous étudions deux transformations d’Aluthge (torale et sphérique) de shifts à deux variables \( W_{(\alpha,\beta)} = (T_1, T_2) \). Puisque, a priori, il y a plusieurs notions possibles, nous discutons deux définitions plausibles et les propriétés fondamentales dans la deuxième partie. Notre recherche nous permettra de comparer les deux définitions quant à la façon dont elles généralisent la notion d’une variable. Après avoir discuté quelques propriétés fondamentales de chaque transformation d’Aluthge, nous procéurons à l’étude des deux transformations dans le cas des shifts pondérés à deux variables. Nous considérons des aspects tels que la préservation de l’hyponormalité conjointe, la continuité de norme, la quasi-normalité sphérique et l’isométrie sphérique.

1. Introduction

Pour \( T \in B(\mathcal{H}) \), la polar decomposition de \( T \) est \( T = U|T| \), où \( U \) est une matrice parfaite et \( |T| := \sqrt{T^*T} \). La transformée d’Aluthge de \( T \) est l’opérateur \( \tilde{T} := |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \). Cette transformation fut d’abord définie dans [1] pour étudier les opérateurs hyponormaux. La transformée d’Aluthge de \( T \) est souvent utilisée pour convertir un opérateur en un opérateur à norme plus petite. Roughly speaking, the idea behind the Aluthge transform is to convert an operator into another one that shares with the first one some spectral properties, but it is closer to being a normal operator. In recent years, the Aluthge transform has received substantial attention. Jung, Ko and Pearcy proved in [15] that \( T \) has a nontrivial invariant subspace if and only if \( \tilde{T} \) does. (Since every normal operator has nontrivial invariant subspaces, the Aluthge transform has a natural connection with the invariant subspace problem.)

For a weighted shift \( W_\alpha \equiv \text{shift}\left(\alpha_0, \alpha_1, \cdots\right) \), the Aluthge transform \( \hat{W}_\alpha \) of \( W_\alpha \) is also a unilateral weighted shift, given by \( \hat{W}_\alpha \equiv \text{shift}\left(\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_0\alpha_2}, \cdots\right) \) (see [17]). Nous retenons que \( W_\alpha \) est hyponormal si et seulement si \( \alpha_0 \leq \alpha_1 \leq \cdots \). Si \( W_\alpha \) est hyponormal, alors \( \hat{W}_\alpha \) est aussi hyponormal. Toutefois, la conversée n’est pas en général. Par exemple, si \( W_\alpha \equiv \text{shift}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots\right) \), alors \( W_\alpha \) est clairement non hyponormal, mais \( \hat{W}_\alpha = U_+ \) est sub-normal. (Et dans ce dernier cas, \( U_+ \) dénote la (non-linaire) transformée (ouverte)…) In [17], S.H. Lee, W.Y. Lee, and the second-named author showed that for \( k \geq 2 \), the Aluthge transform, when acting on weighted shifts, does not preserve \( k \)-hyponormality. Finally, G. Exner proved in [12, Example 2.11] that the Aluthge transform of a subnormal weighted shift need not be subnormal.

In this article, we first introduce the polar decompositions of a commuting pair of bounded operators and study two Aluthge transforms (toral and spherical) of 2-variable weighted shifts \( W_{(\alpha,\beta)} = (T_1, T_2) \). Since a priori there are several possible notions, we discuss two plausible definitions and their basic properties in Section 2. Our research will allow us to compare both definitions in terms of how well they generalize the 1-variable notion. After discussing some basic properties of each Aluthge transform, we proceed to study both transformations in the case of 2-variable weighted shifts. We consider such topics as the preservation of joint hyponormality, norm continuity, spherical quasinormality, and spherical isometries.

We first recall that a commuting pair is subnormal if and only if it is \( k \)-hyponormal for every \( k \geq 1 \) (see [7, Theorem 2.3]). For \( i = 1, 2 \), let us consider the polar decomposition \( T_i = U_i|T_i| \) of \( T_i \). Then, for a 2-variable weighted shift \( W_{(\alpha,\beta)} = (T_1, T_2) \), we define the toral Aluthge transform of \( W_{(\alpha,\beta)} \) as follows: \( \hat{W}_{(\alpha,\beta)} = (\hat{T}_1, \hat{T}_2) := (|T_1|^2 U_1|T_1|^2, |T_2|^2 U_2|T_2|^2) \). As we will see in Proposition 2.1, the commutativity of \( \hat{W}_{(\alpha,\beta)} \) does not automatically follow from the commutativity of \( W_{(\alpha,\beta)} \).

There is a second plausible definition of the Aluthge transform, which uses a joint polar decomposition. Assume that we have a decomposition of the form \( (T_1, T_2) = (U_1P, U_2P) \), where \( P := \sqrt{T_1^*T_1 + T_2^*T_2} \) and \( \ker U_1 \cap \ker U_2 = \ker P \). Noter \( \hat{W}_{(\alpha,\beta)} = (\hat{T}_1, \hat{T}_2) := \left( \sqrt{P}U_1\sqrt{P}, \sqrt{P}U_2\sqrt{P} \right) \). Nous référerons à \( \hat{W}_{(\alpha,\beta)} \) comme la transformée sphérique d’Aluthge de \( W_{(\alpha,\beta)} \). Même si \( \hat{T}_1 = \sqrt{P}U_1\sqrt{P} \) n’est pas la transformée d’Aluthge de \( T_1 \), nous observons que \( Q := \sqrt{U_1^*U_1 + U_2^*U_2} \) est (une) parité isométrie; pour, \( PQ^2P = P^2 \), de laquelle nous suivons qu’il est isométrique sur le polaire.

We will prove in Section 2 that this particular definition of the Aluthge transform preserves commutativity, and that it also behaves well in terms of hyponormality, for a large class of 2-variable weighted shifts. There is also another useful aspect of the spherical Aluthge transform, which we now mention. If we consider the fixed points of this transform acting on 2-variable weighted shifts, we obtain an appropriate generalization of the concept of quasinormality. More precisely, if a 2-variable weighted shift \( W_{(\alpha,\beta)} = (T_1, T_2) \) satisfies \( \hat{W}_{(\alpha,\beta)} = (\hat{T}_1, \hat{T}_2) \), then \( T_1^*T_1 + T_2^*T_2 \) is, up to scalar multiple, a spherical isometry. (We recall that when a commuting pair \( T = (T_1, T_2) \) is called a spherical isometry if \( P^2 = T_1^*T_1 + T_2^*T_2 = I \) [111].) It follows that we can then study some properties of the spherical Aluthge transform using well-known results about spherical isometries. In this paper, we also focus on the following basic problem.

Problem 1.1.

(i) For \( k \geq 1 \), if \( W_{(\alpha,\beta)} \) is \( k \)-hyponormal, does it follow that the toral Aluthge transform \( \hat{W}_{(\alpha,\beta)} \) \( k \)-hyponormal? What about the case of the spherical Aluthge transform? When does either Aluthge transform preserve hyponormality?

(ii) When do we have the equality \( \hat{W}_{(\alpha,\beta)} = \hat{W}_{(\alpha,\beta)} \)?

(iii) Is the toral Aluthge transform \( (T_1, T_2) \rightarrow (\hat{T}_1, \hat{T}_2) \) continuous in the uniform topology? Similarly, does continuity hold for the spherical Aluthge transform?
Proposition 2.1.

(i) For \( W_{(\alpha, \beta)} \), we have: (a) \( \tilde{W}_{(\alpha, \beta)} \) is a commuting pair if and only if
\[
\alpha(k_1, k_2 + 1) = \alpha(k_1 + 1, k_2 + 2) \quad \text{for all } k_1, k_2 \geq 0
\]
(with similar conditions holding for the weight sequence \( \{\tilde{\beta}(k_1, k_2)\} \)); and (b) \( \tilde{W}_{(\alpha, \beta)} \) is always a commuting pair.

(ii) If \( W_{(\alpha, \beta)} \) is a commuting pair of hyponormal operators, so is \( \tilde{W}_{(\alpha, \beta)} \).

We next show that there exists a subnormal \( W_{(\alpha, \beta)} \) such that \( \tilde{W}_{(\alpha, \beta)} \) is not hyponormal; we also prove that there exists a non-hyponormal \( W_{(\alpha, \beta)} \) such that \( \tilde{W}_{(\alpha, \beta)} \) is hyponormal. We start with some definitions. The core \( c(W_{(\alpha, \beta)}) \) of \( W_{(\alpha, \beta)} \) is the restriction of \( W_{(\alpha, \beta)} \) to the invariant subspace \( M \cap N \), where \( M = \{e_{k_1, k_2} \in \ell^2(\mathbb{Z}^2_+): k_1 \geq 0 \text{ and } k_2 \geq 1\} \) and \( N = \{e_{k_1, k_2} \in \ell^2(\mathbb{Z}^2_+): k_1 \geq 1 \text{ and } k_2 \geq 0\} \).

Given a two-variable unilateral weighted shift \( W_w \), consider the 2-variable weighted shift \( \Theta(W_w) \equiv W_{(\alpha, \beta)} \) on \( \ell^2(\mathbb{Z}^2_+) \) given by the double-indexed weight sequences \( \alpha(k_1, k_2) = \beta(k_1, k_2) = \omega_{k_1 + k_2} \) for \( k_1, k_2 \geq 0 \). The shift \( \Theta(W_w) \) can be represented by the weight diagram in Fig. 1(ii).

Lemma 2.2. \((8)\) Let \( \Theta(W_w) \) be given by Fig. 1(ii), and let \( k \geq 1 \). \( W_w \) is \( k \)-hyponormal if and only if \( \Theta(W_w) \) is (jointly) \( k \)-hyponormal.

We next show that hyponormality is not stable under the toral Aluthge transform; this requires \([9, \text{Theorem 1.3 and Proposition 2.9}]\).

Proposition 2.3. For \( 0 < x, y < 1 \), let \( W_{(\alpha, \beta)} \) be the 2-variable weighted shift in Fig. 1(i), where \( \omega_0 = \omega_1 = \omega_2 = \cdots = 1, \alpha_{(0,0)} = \beta_{(0,0)} = \alpha_{(0,k_2)} = \beta_{(k_1,0)} = y(k_1, k_2 \geq 1) \) and all remaining weights are equal to 1. Then, we have:

(i) \( W_{(\alpha, \beta)} \) is subnormal if and only if \( x \leq s(y) : = \sqrt{\frac{1}{1 - y^2}} \).

(ii) \( W_{(\alpha, \beta)} \) is hyponormal if and only if \( x \leq h(y) : = \frac{1 + y^2}{2} \).

(iii) \( \tilde{W}_{(\alpha, \beta)} \) is hyponormal if and only if \( x \leq CA(y) : = \frac{1 + y^2}{1 - y^2} \).

(iv) \( \tilde{W}_{(\alpha, \beta)} \) is hyponormal if and only if \( x \leq PA(y) : = \frac{2(1 + y^2 - y^4)}{(1 + \sqrt{1 + y^2})(1 + y^2 - y^2)} \).

Clearly, \( s(y) \leq h(y) \leq PA(y) \) and \( CA(y) \leq h(y) \) for all \( 0 < y < 1 \), while \( CA(y) < s(y) \) on \( (0, q) \) and \( CA(y) > s(y) \) on \( (q, 1) \), where \( q \cong 0.52138 \). Thus, \( W_{(\alpha, \beta)} \) is hyponormal but \( \tilde{W}_{(\alpha, \beta)} \) is not hyponormal if \( 0 < CA(y) < x \leq s(y) \), and \( \tilde{W}_{(\alpha, \beta)} \) is hyponormal but \( W_{(\alpha, \beta)} \) is not hyponormal if \( 0 < h(y) < x \leq PA(y) \).

Next, we consider the invariance of hyponormality under the two Aluthge transforms. We describe a large class of 2-variable weighted shifts for which both transforms coincide. By Fig. 1(ii) and (iii), and direct calculations, we observe that \( \Theta(W_w) \equiv \tilde{\Theta}(W_w) \). We now have:

Theorem 2.4. If \( \Theta(W_w) \) is hyponormal, then \( \tilde{\Theta}(W_w) \) and \( \Theta(W_w) \) are both hyponormal.
Remark 2.5. As in the 1-variable case, we use $W_{(\alpha,\beta)} = \text{shift} \left( \frac{1}{2}, \frac{1}{2}, 2, \cdots \right)$ to build an example of a 2-variable weighted shift which is not subnormal, but whose Aluthge transforms are both subnormal.

Theorem 2.6. For $W_{(\alpha,\beta)}$, $\tilde{W}_{(\alpha,\beta)} = \tilde{W}_{(\alpha,\beta)}$ if and only if $W_{(\alpha,\beta)}$ is given by Fig. 2(i). Furthermore, if shift $(a, b, c, \cdots)$ is subnormal, then $W_{(\alpha,\beta)}$ is subnormal with Berger measure $\mu = \nu = \delta \left( 1, \sqrt{\frac{n}{2}} \right)$, where $\nu$ is the diagonal Berger measure of the 2-variable weighted shift given by Fig. 2(ii) (see [8]) and * is the convolution product (see [16]).

We now turn our attention to the continuity properties of the toral (resp. spherical) Aluthge transform of a commuting pair. Since the continuity of the toral Aluthge transform is straightforward, we focus on the spherical case. The following result is well known: for a single operator $T \in \mathcal{B}(H)$, the Aluthge transform map $T \rightarrow \tilde{T}$ is $(\|\cdot\|, \|\cdot\|)$-continuous on $\mathcal{B}(H)$ [10]. We extend this to the multivariable case.

Lemma 2.7. (cf. [10, Lemma 2.1]) Let $T = (T_1, T_2)$ be a pair of commuting operators, and let $(T_1, T_2) = (U_1 P, U_2 P)$ be its joint polar decomposition; recall that $P = \sqrt{T_1^* T_1 + T_2^* T_2}$. For $n \in \mathbb{N}$ and $t > 0$, let $f_n(t) := \sqrt{\max \left( \frac{1}{n}, t \right)}$ and let $A_n := f_n(T)$. Then we have:

1. $\|A_n\| \leq \max \left( n^{-\frac{1}{2}}, \|P\|^{-\frac{1}{2}} \right)$;
2. $\|P A_n^{-1} - P\| \leq n^{-\frac{1}{2}}$;
3. $\|A_n - P\| \leq \frac{1}{2} n^{-\frac{1}{2}}$;
4. $\|PA_n - P\| \leq \frac{1}{2} n^{-\frac{1}{2}}$;
5. $\|P_n A_n^{-1} - P_n\| \leq \frac{1}{2} n^{-\frac{1}{2}}$.

From Lemma 2.7, we obtain:

Lemma 2.8. (cf. [10, Lemma 2.2]) Given $R \geq 1$ and $\epsilon > 0$, there are real polynomials $p$ and $q$ such that for every commuting pair $T = (T_1, T_2)$ with $\|T_i\| \leq R$ ($i = 1, 2$), we have

$$\|p^2 U_1 p^2 - p \left( T_1^* T_1 + T_2^* T_2 \right) T_1 q \left( T_1^* T_1 + T_2^* T_2 \right)\| < \epsilon.$$

By Lemmas 2.7 and 2.8, we have:

Theorem 2.9. The spherical Aluthge transform $(T_1, T_2) \rightarrow (\tilde{T}_1, \tilde{T}_2)$ is $(\|\cdot\|, \|\cdot\|)$-continuous on $\mathcal{B}(H)$.

We now study the class of spherically quasinormal (resp. spherically isometric) commuting pairs of operators ([2–4, 11, 13, 14]). In the literature, spherical quasinormality of a commuting $n$-tuple $T \equiv (T_1, \cdots, T_n)$ is associated with the commutativity of each $T_i$ with $P^2$. It is not hard to prove that, for 2-variable weighted shifts, this is equivalent to requiring that $W_{(\alpha,\beta)} = (T_1, T_2)$ be a fixed point of the spherical Aluthge transform, that is, $W_{(\alpha,\beta)} = \tilde{W}_{(\alpha,\beta)}$. A straightforward calculation shows that this is equivalent to requiring that each $U_i$ commutes with $P$. In particular, $(U_1, U_2)$ is commuting whenever $(T_1, T_2)$ is commuting. Also, recall from Section 1 that a commuting pair $T$ is a spherical isometry if $P^2 = I$. Thus, in the case of spherically quasinormal 2-variable weighted shifts, we always have $U_1^* U_1 + U_2^* U_2 = I$. In the following result, the key new ingredient is the equivalence of (i) and (ii).

Theorem 2.10. For $W_{(\alpha,\beta)} = (T_1, T_2)$, the following statements are equivalent: (i) $W_{(\alpha,\beta)}$ is spherically quasinormal; (ii) for all $(k_1, k_2) \in \mathbb{Z}_+^2$, $\alpha_{(k_1, k_2)}^2 + \alpha_{(k_1, k_2)}^2 = C > 0$; (iii) $T_1^* T_1 + T_2^* T_2 = C \cdot I$.

Sketch of proof. Briefly stated, our strategy is as follows: by the continuous functional calculus, we can assume that $T_1$ and $T_2$ commute with $P$. It follows that for all $(k_1, k_2) \in \mathbb{Z}_+^2$, $\alpha_{(k_1, k_2)}^2 + \alpha_{(k_1, k_2)}^2 = C$ is constant. Next, we compute $T_1^* T_1 + T_2^* T_2$. \qed
By the proof of Theorem 2.10, we remark that once the zero-th row of $T_1$, call it $W_0$, is given, then the entire 2-variable weighted shift is fully determined.

We briefly pause to recall the construction of Stampfli’s shift $W_{\sqrt{a}, \sqrt{b}, \sqrt{c}}:=(\sqrt{a}, \sqrt{b}, \sqrt{c})^\ast$, where $0 < \sqrt{a} < \sqrt{b} < \sqrt{c}$. From [6], $W_{\sqrt{a}, \sqrt{b}, \sqrt{c}}$ is subnormal, with 2-atomic Berger measure $\varphi = \varphi_0 \varphi_3 + \varphi_1 \varphi_4$, where $\varphi_0 := -\frac{ah(1-h)}{b-a}$, $\varphi_1 := \frac{b(a-h)}{b-a}$, $s_0 := \frac{\varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_0}}{2}$, $s_1 := \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_0}}{2}$, $\rho_0 := \frac{s_1 - a}{s_1 - s_0}$, and $\rho_1 := \frac{a - s_0}{s_1 - s_0}$. We are now ready for

**Theorem 2.11.** Consider a spherically quasinormal $W_{(a, b), (\alpha, \beta)} = (T_1, T_2)$ given by Fig. 1(i), where $W_0 := \text{shift}(\alpha(0,0), \alpha(1,0), \cdots) = W_{\sqrt{a}, \sqrt{b}, \sqrt{c}}$. For given $k = (k_1, k_2) \in \mathbb{Z}_+^2$, we let $\alpha^2_{(k_1,k_2)} + \rho^2_{(k_1,k_2)} = \varphi_3$, where $\varphi_3$ is as above. Then, $W_{(a, b), (\alpha, \beta)}$ is subnormal with Berger measure $\varphi := \frac{1}{s_1 - s_0} \delta_{(s_3, s_4)} + \frac{b-a}{s_1 - s_0} \delta_{(s_3, s_4)}$.

The subnormality of $W_{(a, b), (\alpha, \beta)}$ in Theorem 2.11 is a special case of the following result.

**Lemma 2.12.** ([11]) Any spherical isometry is subnormal.

By Theorem 2.10 and Lemma 2.12, we obtain:

**Corollary 2.13.** Any spherically quasinormal 2-variable weighted shift is subnormal.

**Remark 2.14.**

(i) A. Athavale and S. Podder have recently proved that a commuting spherically quasinormal pair is always subnormal [3, Proposition 2.1]; this provides a different proof of Corollary 2.13.

(ii) In a different direction, let $Q_T(X) := T_2^*XT_1 + T_2^*XT_2$. By induction, it is easy to prove that if $T$ is spherically quasinormal, then $Q_T(I) = (Q_T(I))^n (n \geq 0)$; by [5, Remark 4.6], $T$ is subnormal.

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**References**


