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Generalized adding machines and Julia sets

Odomètre stochastique et ensembles de Julia

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ABSTRACT

We define stochastic adding machines based on Cantor Systems of numeration. Our aim here is to compute the parts of spectra of the transition operators associated with these stochastic adding machines in different classical Banach spaces. We show that these spectra are connected to fibered Julia sets.

RÉSUMÉ

Nous définissons l'odomètre stochastique associé à un système de numération de Cantor. Nous calculons les parties du spectre de l'opérateur de transition associé à cet odométre dans différents espaces de Banach classiques. Nous montrons que le spectre est lié aux ensembles de Julia fibrés.

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1. Introduction

Let us fix a sequence $(d_i)_{i\geq 0} = \bar{d}$ of positive integers such that $d_0 = 1$ and $d_i \geq 2$ for $i \geq 1$. It is known [4,6] by using a greedy algorithm that any non-negative integer n can be written in a unique way as $n = \sum_{j=1}^{k_n} a_j(n) q_{j-1}$, where $k_n \in \mathbb{Z}_+$, $q_{j-1} = d_0 \dots d_{j-1}$ and $a_j(n) \in \{0, \dots, d_j - 1\}$ for all $j \geq 1$. This is referred to as the Cantor System of Numeration associated with $\bar{d}$. A classical example is the base-10 expansion, where $d_i = 10$ for all $i \geq 1$.

It is classical that there exists an algorithm, or adding machine in our case, that maps the digits of n into those of $n+1$. This algorithm is defined as follows: put $\zeta_n = \min\{j \geq 1 : a_j(n) \neq d_j - 1\}$, then

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$$a_j(n+1) = \begin{cases} 0 & , \ j < \zeta_n \,, \\ a_j(n)+1 & , \ j = \zeta_n \,, \\ a_j(n) & , \ j > \zeta_n \,. \end{cases}$$

Now, we define the stochastic adding machine based on the following: Suppose that at the j-th step of the adding machine algorithm with some given probability, independently of any other step, we lose information about the counter that records the number of steps already performed by the algorithm and it stops. This implies that the outcome of the adding machine is a random variable. Formally, we fix a sequence $(\xi_j)_{j\geq 1}$ of independent random variables such that ξ_j has Bernoulli distribution with parameter p_j, where $(p_j)_{j\geq 1} = \bar{p}$ is a sequence of strictly positive probabilities. Here $\xi_j = 0$ means that the j-th step of the algorithm is not allowed to be performed. So define the random time $\tau = \inf\{j : \xi_j = 0\}$. Then the Adding Machine with Fallible Counter algorithm associated with $(\bar{d}, \bar{p})$, AMFC$_{\bar{d},\bar{p}}$, is defined by applying the adding machine algorithm to n and stopping at the step $\tau \wedge \zeta_n$ (this means that steps $j \geq \tau$ are not performed when $\tau < \zeta_n$).

Fix an initial, possibly random, state $X(0) \in \mathbb{Z}_+$. We apply recursively the AMFC$_{\bar{d},\bar{p}}$ algorithm to its successive outcomes starting at $X(0)$ and using independent sequences of Bernoulli random variables at different times. These random sequences are associated with the same fixed sequence of probabilities $(p_j)_{j\geq 1}$. In this way, we generate a discrete time-homogeneous Markov chain $(X(t))_{t\geq 0}$, where $X(t)$ represents the outcome after t successive applications of the AMFC$_{\bar{d},\bar{p}}$ algorithm, which we call the AMFC$_{\bar{d},\bar{p}}$ stochastic machine.

The stochastic machine was introduced in the literature by Killeen and Taylor in [5] for the case $d_j = 2$, $p_j = p \in (0,1)$ for all $j \geq 1$. Among other things, the authors show that the spectrum of the transition operator of the stochastic machine acting on l^∞ is the filled-in Julia set of the quadratic polynomial $\frac{z^2-(1-p)}{p}$. Further spectral properties of the same transition operator and its dual acting on c_0, c, l^α, $\alpha \geq 1$, are considered by El Abdalaoui and Messaoudi in [1].

In [7], Messaoudi, Sester and Valle have introduced the stochastic machines associated with nonconstant sequences $\bar{p}$, and $\bar{d} \equiv d$ constant. It is shown that the spectrum of its transition operator acting on l^∞ is equal to the filled-in fibered Julia set E associated with a sequence of polynomial maps.

Motivated by [1] and [7], the aim of this note is the analysis of the different parts of the spectra for the transition operators of the more general AMFC$_{\bar{d},\bar{p}}$ stochastic machines acting in the usual Banach spaces $(l^\alpha, \|\cdot\|_\alpha)$, $1 \leq \alpha \leq \infty$, $(c_0, \|\cdot\|_\infty)$ and $(c, \|\cdot\|_\infty)$ and their connection with fibered Julia sets. All the results described here are strictly included in [8], where the extended proofs are given.

Recall that for $w = (w(n))_{n\geq 0} \in \mathbb{C}^{\mathbb{Z}_+}$, we have:

$$\|w\|_\infty = \sup_{n\geq 0} |w(n)| < \infty \,, \quad \|w\|_q = \Big(\sum_{n\geq 0} |w(n)|^q\Big)^{\frac{1}{q}} \,, \ 1 \leq q < \infty \,,$$

and

$$\begin{aligned} l^\infty &= l^\infty(\mathbb{Z}_+) = \{w \in \mathbb{C}^{\mathbb{Z}_+} : \|w\|_\infty < \infty\} \,, \\ l^q &= l^q(\mathbb{Z}_+) = \{w \in \mathbb{C}^{\mathbb{Z}_+} : \|w\|_q < \infty\} \,, \\ c &= c(\mathbb{Z}_+) = \{w \in l^\infty : w \text{ is convergent}\} \,, \\ c_0 &= c_0(\mathbb{Z}_+) = \{w \in c : \lim_{n\to\infty} w(n) = 0\} \,. \end{aligned}$$

2. Main results

Our first aim in this section is to describe the transition probabilities of $(X(t))_{t\geq 0}$, which we denote $s(n,m) = s_{\bar{p},\bar{d}}(n,m) := P(X(t+1) = m | X(t) = n)$. They can be obtained directly from the description of the chain: for every $n \geq 0$, if $n = a_{k_n} \ldots a_1$ (written in base (q_n)), where $0 \leq a_1 < d_1 - 1$ ($\zeta_n = 1$), then $s(n,m) = p_1$ if $m = n+1$, $s(n,m) = 1 - p_1$ if $m = n$ and $s(n,m) = 0$ otherwise. Now, if $\zeta_n \geq 2$ and $n = a_{k_n} \ldots a_{\zeta_n}(d_{\zeta_n - 1} - 1) \ldots (d_1 - 1)$, then

$$s(n,m) = \begin{cases} (1-p_{r+1}) \prod_{j=1}^r p_j & , \ m = n - \sum_{j=1}^r (d_j - 1) q_{j-1}, \\ & \quad = a_{k_n} \ldots a_{\zeta_n}(d_{\zeta_n - 1} - 1) \ldots (d_{r+1} - 1) 0 \ldots 0, \\ & \quad \text{for some} r \leq \zeta_n - 1, \\ 1 - p_1 & , \ m = n \,, \\ \prod_{j=1}^{\zeta_n} p_j & , \ m = n+1 = a_{k_n} \ldots a_{\zeta_n + 1}(a_{\zeta_n} + 1) 0 \ldots 0 \,, \\ 0 & , \ \text{otherwise} \,. \end{cases} \tag{1}$$

With the transition probabilities, we obtain the countable transition matrix of the AMFC$_{\bar{d},\bar{p}}$ stochastic machine $S = S_{\bar{d},\bar{p}} = (s(n,m))_{n,m\geq 0}$. As an example, the first entries of the matrix S associated with base 3 ($d_j = 3$ for all $j \geq 1$) are given below:

$$\begin{bmatrix} 1-p_1 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1-p_1 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ p_1(1-p_2) & 0 & 1-p_1 & p_1p_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1-p_1 & p_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1-p_1 & p_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & p_1(1-p_2) & 0 & 1-p_1 & p_1p_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-p_1 & p_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p_1 & p_1 & 0 & \cdots \\ p_1p_2(1-p_3) & 0 & 0 & 0 & 0 & 0 & p_1(1-p_2) & 0 & 1-p_1 & p_1p_2p_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that S is doubly stochastic if and only if $\prod_{j=1}^{+\infty} p_j = 0$. In fact, S is stochastic and the sum of coefficients of every column is 1, except the first one, whose sum is $1-\prod_{j=1}^{+\infty} p_j$.

Given a sequence of nonnegative integers $\bar{d} = (d_j)_{j\geq 0}$, $d_0 = 1$ and $d_j > 1$ for $j > 1$, and of probabilities $\bar{p} = (p_j)_{j\geq 1}$, $p_j \in (0,1]$, we can show that the $\mathrm{AMFC}_{\bar{d},\bar{p}}$ chain is irreducible if and only if $p_j < 1$ for infinitely many j's. Moreover, when $p_j = 1$ for every $j \geq 1$, we have that the $\mathrm{AMFC}_{\bar{d},\bar{p}}$ stochastic machine is the deterministic shift map $n \mapsto n+1$ on $\mathbb{Z}_+$.

We also have (analogous to [7]) that the $\mathrm{AMFC}_{\bar{d},\bar{p}}$ Markov chain is null recurrent if and only if $\prod_{j=1}^{+\infty} p_j = 0$; otherwise the chain is transient. The proof in [7] is for a constant base, $d_i = d$ for every $i \geq 1$, however the generalization is straightforward.

Let $\Omega \in \{c_0,\ c,\ l^q,\ 1 \leq q \leq \infty\}$. We denote by Ω' the dual of Ω. We also denote by $\sigma(\Omega,\bar{d},\bar{p})$, $\sigma_p(\Omega,\bar{d},\bar{p})$, $\sigma_r(\Omega,\bar{d},\bar{p})$ and $\sigma_c(\Omega,\bar{d},\bar{p})$ respectively the spectrum, point spectrum, residual spectrum and continuous spectrum of $S_{\bar{d},\bar{p}}$ acting as a linear operator on Ω. Recall that λ belongs to $\sigma(\Omega,\bar{d},\bar{p})$ (resp. $\sigma_p(\Omega,\bar{d},\bar{p})$) if $(S-\lambda I)$ is not bijective (resp. not one to one). If $(S-\lambda I)$ is one to one and not onto, then $\lambda \in \sigma_r(\Omega,\bar{d},\bar{p})$ if $(S-\lambda I)(\Omega)$ is not dense in Ω, otherwise, we say that $\lambda \in \sigma_c(\Omega,\bar{d},\bar{p})$.

For all $j \geq 1$, let $f_j(z) := \left(\frac{z-(1-p_j)}{p_j}\right)^{d_j}$ and $\tilde{f}_j := f_j \circ \ldots \circ f_1$. Also set $\tilde{f}_0$ as the identity function on $\mathbb{C}$ and $E_{\bar{d},\bar{p}} := \Big\{z \in \mathbb{C} : (\tilde{f}_j(z))_{j\geq 0} \text{ is bounded}\Big\}$. The set $\partial E_{\bar{d},\bar{p}}$ is what is called a fibered Julia set and $E_{\bar{d},\bar{p}}$ the filled-in fibered Julia set associated with the sequence of maps $(f_j)_{j\geq 1}$, see [9].

Theorem 2.1. *Let $\bar{p} = (p_i)_{i\geq 0} \in (0,1]^{\mathbb{N}}$ and $\bar{d} = (d_i)_{i\geq 0}$ where $d_0 = 1$ and $(d_i)_{i\geq 1} \in \{2,3,4,\ldots\}^{\mathbb{N}}$. Let $\Omega \in \{c_0,\ c,\ l^\alpha,\ 1 \leq \alpha \leq \infty\}$. Then the spectrum of the $\mathrm{AMFC}_{\bar{d},\bar{p}}$ transition operator acting on Ω is equal to the fibered Julia set $E_{\bar{d},\bar{p}}$, i.e. $\sigma(\Omega,\bar{d},\bar{p}) = E_{\bar{d},\bar{p}}$.*

Outline of the proof. Using the auto-similarity of the operator S, we can prove that for $v = (v_n)_{n\geq 0} \in \mathbb{C}^\infty$ and $\lambda \in \mathbb{C}$, we have that $Sv = \lambda v$, if and only if, for all $n \geq 1$, $v_n = a_n v_0$ where $a_n = \prod_{r=1}^{k_n} (\iota_\lambda(r))^{b_r(n)}$ for all $n \geq 1$, $b_r(n)$ is the r digit of n in its expansion in base $(q_i)_{i\geq 0}$ and $\iota_\lambda(r) = \iota(r)$ satisfies:

$$\iota(1) = \frac{\lambda}{p_1} - \frac{1-p_1}{p_1} \text{ and } \iota(j+1) = \frac{\iota(j)^{d_j}}{p_{j+1}} - \frac{1-p_{j+1}}{p_{j+1}},\ \forall j > 1. \tag{2}$$

Now we consider first the case $\Omega = l^\infty$. Suppose that $\lambda \in E_{\bar{d},\bar{p}}$, then using triangle inequality, we can prove that $|\tilde{f}_n(\lambda)| \leq 1$ for all $n \geq 1$. Since $\iota_\lambda(n) = (h_n \circ \tilde{f}_{n-1})(\lambda)$ where $h_n(z) = \frac{z}{p_n} - \frac{1-p_n}{p_n}$, a simple computation yields that $\sup_n |\iota_\lambda(n)| \leq 1$ and $\sup_n |a_n| \leq 1$. Hence $\lambda \in \sigma_p(\Omega,\bar{d},\bar{p})$. Therefore, we have proved that $E_{\bar{d},\bar{p}} \subset \sigma_p(\Omega,\bar{d},\bar{p})$.

For the other inclusion, we first need to introduce a notation. Let $\bar{d}_j = (d_{i+j})_{i\geq 1}$ and define $\bar{p}_j$ analogously. Now consider $\tilde{S}_{\bar{d},\bar{p}} := \frac{S_{\bar{d},\bar{p}}-(1-p_1)I}{p_1}$, which is also a stochastic operator acting on $\mathbb{Z}_+$. We can prove that $\tilde{S}_{\bar{d},\bar{p}}$ is associated with a irreducible Markov chain with period d_1. Thus $\tilde{S}_{\bar{d},\bar{p}}^{d_1}$ has d_1 communication classes (see [8] and [7] for the case $d_i = d$ for all $i > 1$). It is straightforward to verify that the communication classes of $\tilde{S}_{\bar{d},\bar{p}}^{d_1}$ are $\{j \in \mathbb{N}\ :\ j = n \bmod d_1\}$, $0 \leq n \leq d_1 - 1$. Furthermore, $\tilde{S}_{\bar{d},\bar{p}}^{d_1}$ acts on each of these classes as a copy of $S_{\bar{d}_2,\bar{p}_2,}$. Therefore, the spectrum of $\tilde{S}_{\bar{d},\bar{p}}^{d_1}$ is equal to the spectrum of $S_{\bar{d}_2,\bar{p}_2}$. Since, $\tilde{S}_{\bar{d},\bar{p}}^{d_1} = \tilde{f}_1(S_{\bar{d},\bar{p}})$, by the Spectral Mapping Theorem, we have that $\tilde{f}_1\big(\sigma(\Omega,\bar{d},\bar{p})\big) = \sigma(\Omega,\bar{d}_2,\bar{p}_2)$. By induction, we have that $\tilde{f}_{j+1}\big(\sigma(\Omega,\bar{d},\bar{p})\big) = \sigma(\Omega,\bar{d}_{j+1},\bar{p}_{j+1})$, for every $j \geq 1$. Since $S_{\bar{d}_{j+1},\bar{p}_{j+1}}$ is a stochastic operator, its spectrum is a subset of $\overline{\mathbb{D}(0,1)}$. Therefore $|\tilde{f}_{j+1}(\lambda)| \leq 1$, for every j and $\lambda \in \sigma(\Omega,\bar{d},\bar{p})$. This implies that $\sigma(\Omega,\bar{d},\bar{p}) \subset E_{\bar{d},\bar{p}}$. Observe that the last inclusion is also true for all $\Omega \in \{c_0,\ c,\ l^q,\ 1 \leq q < \infty\}$.

Now, suppose that $\Omega \in \{c_0, c\}$. Since $(\Omega')' = l^\infty$, by duality and Phillips Theorem (see [10]), we obtain

$$\sigma(\Omega,\bar{d},\bar{p}) = \sigma'(l^1,\bar{d},\bar{p}) = \sigma(l^\infty,\bar{d},\bar{p}) = E_{\bar{d},\bar{p}}.$$

Now, if $\Omega = l^q$, $q \geq 1$, then we can prove that if $\lambda \in E_{\bar{d},\bar{p}} \setminus \sigma_p(\Omega, \bar{d}, \bar{p})$, $v^{(n)} = (v_k^{(n)})_{k\geq 0}$ where $v_k^{(n)} = a_k$ for $k < n$ and $v_k^{(n)} = 0$ for $k \geq n$ (a_k defined above), then $(S - \lambda I)(v^{(n)})/||v^{(n)}||_q$ converges to 0 (in l^q) as n goes to infinity. Hence λ belongs to the approximate point spectrum of S and we are done. $\square$

2.1. Residual spectrum

Theorem 2.2. *1. For $\Omega \in \{c_0, c, l^q, 1 < q \leq \infty\}$, we have that the residual spectrum of the* AMFC$_{\bar{d},\bar{p}}$ *transition operator acting on Ω is empty, i.e. $\sigma_r(\Omega, \bar{d}, \bar{p}) = \emptyset$.*

2. For $\Omega = l^1$, if $\prod_{i=1}^{\infty} p_i = 0$, then $\sigma_r(\Omega, \bar{d}, \bar{p})$ contains a countable subset X of the boundary of $E_{\bar{d},\bar{p}}$. Precisely $X = \bigcup_{n=1}^{+\infty} \tilde{f}_n^{-1}\{1\} \setminus \bigcup_{n=1}^{+\infty} \tilde{f}_n^{-1}\{0\}$.

Sketch of the proof. Fix the space $\Omega \in \{c_0, c, l^q, q > 1\}$. From classical operator theory, we have that the residual spectrum of S is a subset of the point spectrum of the adjoint operator S' on the dual space Ω'. Now if $w = (w_n)_{n\geq 1} \in \Omega'$ is such that $w \neq 0$ and $S'w = \lambda w$, then we can prove (see [8]) that for all $n \geq 1$, $w_n = \frac{1}{a_n} w_0$ where a_n is defined in the proof of Theorem 2.1. Since $w \in l^1$ if $\Omega \in \{c_0, c\}$ and $w \in l^{\frac{q}{q-1}}$ if $\Omega = l^q$, $q > 1$, we deduce that $\lim |a_n| = +\infty$. Hence $\lim |\iota(n)| = \lim |a_{q_n}| = +\infty$, this is a contradiction since $\lambda \in \sigma(S)$, thus $\sigma_r(\Omega, \bar{d}, \bar{p})$ is empty.

Now, assume that $\Omega = l^1$ and $\prod_{i=1}^{\infty} p_i = 0$. From usual results in operator theory (see [3]), we know that $\sigma_r(l^1, \bar{d}, \bar{p}) \subset \sigma_p'(l^\infty, \bar{d}, \bar{p}) \subset \sigma_r(l^1, \bar{d}, \bar{p}) \cup \sigma_p(l^1, \bar{d}, \bar{p})$. Since $\prod_{i=1}^{\infty} p_i = 0$, then we can prove (see [8]) that $\sigma_p(l^1, \bar{d}, \bar{p}) = \emptyset$. Thus $\sigma_r(l^1, \bar{d}, \bar{p}) = \sigma_p'(l^\infty, \bar{d}, \bar{p})$. We deduce that

$$\sigma_r'(l^\infty, \bar{d}, \bar{p}) \subset \left\{\lambda \in \mathbb{C} : (1/a_n)_{n\geq 1} \text{ is bounded} \right\}.$$

With this, we obtain that $\sigma_r(l^1, \bar{d}, \bar{p}) \subset X$. $\square$

Remark 2.1. We can prove (see [8]) that: if $\prod_{i=1}^{\infty} p_i = 0$, $(d_n)_{n\geq 0}$ is bounded and $\lim sup\ p_n < 1$, then $\sigma_r(l^1, \bar{d}, \bar{p}) = \bigcup_{n=0}^{+\infty} \tilde{f}_n^{-1}\{1\} \setminus \bigcup_{n=1}^{+\infty} \tilde{f}_n^{-1}\{0\}$.

If $\prod_{i=1}^{\infty} p_i > 0$, then $\sigma_r(l^1, \bar{d}, \bar{p}) \cap \bigcup_{n=1}^{+\infty} \tilde{f}_n^{-1}\{1\} = \emptyset$. In this case, we conjecture that $\sigma_r(l^1, \bar{d}, \bar{p})$ is empty.

2.2. Point spectrum

Theorem 2.3. *The following results hold:*

1. For $\Omega \in \{c_0, l^q, q \geq 1\}$, if $(p_i)_{i\geq 0}$ does not converge to 1, then the point spectrum of the AMFC$_{\bar{d},\bar{p}}$ *transition operator acting on Ω is empty, i.e. $\sigma_p(\Omega, \bar{d}, \bar{p}) = \emptyset$.*

2. If $\lim_{j\to\infty} p_j = 1$, then $\sigma_p(c_0, \bar{d}, \bar{p})$ is not empty. Moreover if $p_j \geq 2(\sqrt{2} - 1)$ for every $j \geq 1$, then $0 \in E_{\bar{d},\bar{p}}$ and $\sigma_p(c_0, \bar{d}, \bar{p})$ is equal to the connected component of $\mathrm{int}(E_{\bar{d},\bar{p}})$ that contains 0.

Sketch of the proof. 1. Suppose that there exists $\lambda \in \sigma_p(\Omega, \bar{d}, \bar{p})$. Since $\Omega \subset l^\infty$, we have that an eigenvector $v = (v_i)_{i\geq 0}$ in Ω, associated with λ, satisfies $v_n = a_n v_0$ for all $n \geq 1$ where a_n is given in Theorem 2.1 Hence $\lim_{n\to\infty} \iota_\lambda(n) = \lim_{n\to\infty} a_{q_n} = 0$. By (2), we deduce that $(p_i)_{i\geq 1}$ converges to 1. Thus if $(p_i)_{i\geq 1}$ does not converge to 1, $\sigma_p(\Omega, \bar{d}, \bar{p}) = \emptyset$.

2. Put $\rho = 2(\sqrt{2} - 1)$ and $O = B(0, \rho/2) \subset \mathbb{C}$, the ball of radius $\rho/2$ centered at 0. We can show the following assertion (see [8]): if $\lim_{j\to\infty} p_j = 1$ and if there exists j_0 such that $\inf_{j\geq j_0} p_j \geq \rho$ and $\iota_\lambda(j_0) \in O$, then $\lim_{j\to\infty} |\iota_\lambda(j)| = 0$. Here is worth to point out where the choice of ρ came from. For $r = \rho/2$ we have that

$$r = \frac{r^2}{\rho} + \frac{1 - \rho}{\rho},$$

then $|\iota(j)| \leq r$ and $p_j \geq \rho$ implies that

$$|\iota(j+1)| \leq \frac{|\iota_\lambda(j)|^{d_j}}{p_{j+1}} + \frac{1 - p_{j+1}}{p_{j+1}} \leq \frac{r^2}{\rho} + \frac{1 - \rho}{\rho} = r.$$

Thus if $|\iota(j)| \leq r$ and $p_j \geq \rho$ for every sufficiently large j, we can keep the sequence $(\iota(j))$ bounded away from one.

Since $\lim_{j\to\infty} |\iota_\lambda(j)| = 0$ implies that $\lambda \in \sigma_p(c_0, \bar{d}, \bar{p})$, we have that $g_j^{-1}(O) \subset \sigma_p(c_0, \bar{d}, \bar{p})$ for $j \geq j_0$, where $g_j(\lambda) = \iota_\lambda(j)$. Thus $\sigma_p(c_0, \bar{d}, \bar{p}) \neq \emptyset$.

Now suppose that $p_j \geq 2(\sqrt{2} - 1)$ for every $j \geq 1$. Therefore $O \subset E_{\bar{d},\bar{p}}$, in particular $0 \in E_{\bar{d},\bar{p}}$. Let V be the connected component of $\mathrm{int}(E_{\bar{d},\bar{p}})$ that contains 0. Note that for any $d \geq 2$ and $p > \rho$, we have that $g(O) \subset O$ where

$$g(z) = \frac{z^d}{p} - \frac{1-p}{p}.$$

Then, $g_n(O) \subset O$ for all integer $n \geq 1$.

It is easy to see that $\{\lambda \in \mathbb{C},\ \lim_{n\to\infty} g_n(\lambda) = 0\} = \bigcup_{n=1}^{+\infty} g_n^{-1}(O)$.

Let z_0 be a critical point of g_n, $n \geq 1$. Since $g_n = h_n \circ \tilde{f}_{n-1}$, then $\tilde{f}'_{n-1}(z_0) = 0$. By the chain rule for derivatives, we deduce that there exists $1 \leq k \leq n-2$ such that $\tilde{f}_k(z_0) = 0$. Hence $g_n(z_0) = h_n \circ f_{n-1} \circ \ldots \circ f_{k+1}(0)$.

Since $\lim_{j\to\infty} p_j = 1$ and $p_i \geq \rho = 2(\sqrt{2}-1)$ for all $i \geq 1$, then we can prove that $z_0 \in g_n^{-1}(O)$ for all $n \geq 1$. Thus we deduce by Riemann–Hurwitz formula, that $g_n^{-1}(O)$ is connected for any integer $n \geq 1$. Since $g_n^{-1}(O)$, $n \geq 1$, is a sequence of increasing sets, we deduce that $\bigcup_{n=1}^{+\infty} g_n^{-1}(O)$ is a connected set. Hence $\{\lambda \in \mathbb{C},\ \lim_{n\to\infty} g_n(\lambda) = 0\} \subset V$.

On the other hand, $(g_n)_{n\geq 1}$ is a uniformly bounded sequence of holomorphic functions defined on an open subset $V \subset int(E_{\bar{d},\bar{p}})$. Hence, we deduce by the Arzelà–Ascoli Theorem that $(g_n)_{n\geq 1}$ is normal in V. That is, there exists a subsequence $(g_{n_k})_{k\geq 1}$ of $(g_n)_{n\geq 1}$ such that g_{n_k} converges to a function g on every compact subset of V.

Since g_n converges uniformly on O to 0, we have that g_n converges uniformly on compact sets in V to $g = 0$. Hence $V \subset \{\lambda \in \mathbb{C},\ \lim_{n\to\infty} g_n(\lambda) = 0\}$ and we are done. □

Remark 2.2. We can prove (see [8]) that if $q \geq 1$ and $\sum_{j=1}^{\infty}(1-p_j)^q = \infty$, then $\sigma_p(l^q, \bar{d}, \bar{p}) = \emptyset$. Moreover, if $(p_i)_{i\geq 0}$ is monotone increasing, $p_j \geq 2(\sqrt{2}-1)$ for every $j \geq 1$ and $\sum_{j=1}^{\infty}(1-p_j)^q < \infty$, then $\sigma_p(l^q, \bar{d}, \bar{p})$ equals the connected component of $\mathrm{int}(E_{\bar{d},\bar{p}})$ that contains 0. In particular, in l^1, transience is equivalent to the fact that $\sigma_p(l^1, \bar{d}, \bar{p})$ is not empty.

2.3. Comparison with other systems of numeration, open problems and other relevant connections

Analogous results to Theorems 2.1 and 2.2 were obtained in [1], in the case $d_n = 2$ for all $n > 1$ and $0 < p_i = p < 1$ for all $i \geq 1$. The extension of these two results for Cantor systems of numeration is hard and non-trivial.

In [7], we studied the case $d_n = d > 2$ for all $n > 1$ and $0 < p_i = p < 1$ for $i \geq 1$. We obtained a result similar to Theorem 2.1 just for $\Omega = l^\infty$.

Theorem 2.3 was not obtained for any other system of numeration. For instance, many of the questions answered here remain open for Fibonacci-type bases also discussed in [1].

Just to mention an important connection, the study of these spectra gives information about the dynamical properties of transition operators acting on separable Banach spaces (see for instance [2]). For example, if the operator S of the AMFC$_{\bar{d},\bar{p}}$ acting on a separable Banach space Ω is topologically transitive, then the point spectrum of the adjoint S' acting in the dual space Ω' is empty, and hence the residual spectrum of S acting on Ω is empty. Another characterization is that if S is topologically transitive, then any connected component of the spectrum intersects the unit circle. However, we do not aim here to the study of the dynamical properties of the transition operators of the AMFC Markov chains.

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