Partial differential equations

# Note to the problem of the asymptotic behavior of a viscous incompressible flow around a rotating body 

# Remarque sur le problème du comportement asymptotique de l'écoulement d'un fluide incompressible autour d'un corps rigide en rotation 

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## A R T I C L E IN F O

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#### Abstract

We consider weak solutions to the stationary Navier-Stokes system with Oseen and rotational terms, in an exterior domain. We are interested in the leading term for the velocity field and its gradient. Moreover, we deal with the asymptotic behavior at infinity. We proved that the velocity may be split, within constants, into the first column of the fundamental solution to the Oseen system, plus a remainder term decaying pointwise near infinity at a rate which is higher than the decay rate of the Oseen tensor. This result improves the theory proposed by M. Kyed [Asymptotic profile of a linearized flow past a rotating body, Q. Appl. Math. 71 (2013) 489-500; On the asymptotic structure of a NavierStokes flow past a rotating body, J. Math. Soc. Jpn. 66 (2014) 1-16].


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## Ré S U M É

Nous considérons des solutions faibles du système de Navier-Stokes stationnaire avec un terme d'Oseen et des termes rotationnels dans un domaine extérieur. Notre intérêt se porte sur la partie principale d'un développement asymptotique de la vitesse et de son gradient. Nous montrons que la vitesse peut être scindée, à des constantes près, en la première colonne de la solution fondamentale du système d'Oseen («tenseur d'Oseen»), plus un reste qui décroît ponctuellement dans un voisinage d'infini, à un taux qui est plus élevé que le taux de décroissance du tenseur d'Oseen. Ce résultat améliore la théorie présentée par M. Kyed [Asymptotic profile of a linearized flow past a rotating body, Q. Appl. Math. 71 (2013) 489-500; On the asymptotic structure of a Navier-Stokes flow past a rotating body, J. Math. Soc. Jpn. 66 (2014) 1-16].
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## 1. Introduction

We shall consider the following problem. Let $\mathfrak{D} \subset \mathbb{R}^{3}$ be an open bounded set. Suppose this set describes a rigid body moving with constant nonzero translational and angular velocity in an incompressible viscous fluid. Then the flow around this body with respect to a frame attached to this body is governed by the following set of non-dimensional equations (see [7]):

$$
\begin{equation*}
-\Delta u+\tau \partial_{1} u+\tau(u \cdot \nabla) u-(\omega \times x) \cdot \nabla u+\omega \times u+\nabla \pi=f, \operatorname{div} u=0 \tag{1}
\end{equation*}
$$

in the exterior domain $\overline{\mathfrak{D}}^{c}:=\mathbb{R}^{3} \backslash \overline{\mathfrak{D}}$, supplemented by a decay condition at infinity,

$$
\begin{equation*}
u(x) \rightarrow 0 \text { for }|x| \rightarrow \infty \tag{2}
\end{equation*}
$$

and suitable boundary conditions on $\partial \mathfrak{D}$.
In (1) and (2), the functions $u: \overline{\mathfrak{D}}^{c} \mapsto \mathbb{R}^{3}$ and $\pi: \overline{\mathfrak{D}}^{c} \mapsto \mathbb{R}$ are the unknown relative velocity and pressure field of the fluid, respectively, whereas the function $f: \overline{\mathfrak{D}}^{c} \mapsto \mathbb{R}^{3}$ stands for a prescribed volume force acting on the fluid. The vector $\tau(-1,0,0)$ represents the uniform velocity of the flow at infinity or the velocity of the body, depending on the physical situation, and $\omega:=\varrho \cdot(1,0,0)$ corresponds to the constant angular velocity of the body. In particular, the translational and angular velocity vectors are parallel. From a physical point of view, this assumption is natural for a steady flow. The parameters $\tau \in(0, \infty)$ and $\varrho \in \mathbb{R} \backslash\{0\}$ are dimensionless quantities that can be identified with the Reynolds and Taylor numbers, respectively. They will be considered as fixed, like the domain $\mathfrak{D}$.

We are interested in "Leray solutions" to (1), (2), that is weak solutions characterized by the conditions $u \in L^{6}\left(\overline{\mathfrak{D}}^{c}\right)^{3} \cap$ $W_{\text {loc }}^{1,1}\left(\overline{\mathfrak{D}}^{c}\right)^{3}, \nabla u \in L^{2}\left(\overline{\mathfrak{D}}^{c}\right)^{9}$ and $\pi \in L_{\text {loc }}^{2}\left(\overline{\mathfrak{D}}^{c}\right)$.

From [8] and [3], it follows that the velocity part $u$ of a Leray solution ( $u, \pi$ ) to (1), (2) decays for $|x| \rightarrow \infty$ as expressed by the estimates

$$
\begin{equation*}
|u(x)| \leq C(|x| s(x))^{-1}, \quad|\nabla u(x)| \leq C(|x| s(x))^{-3 / 2} \tag{3}
\end{equation*}
$$

for $x \in \mathbb{R}^{3}$ with $|x|$ sufficiently large, where $s(x):=1+|x|-x_{1}\left(x \in \mathbb{R}^{3}\right)$ and $C>0$ a constant independent of $x$. The factor $s(x)$ may be considered as a mathematical manifestation of the wake extending downstream behind a body moving in a viscous fluid.

By Kyed [10], it was shown that

$$
\begin{equation*}
u_{j}(x)=\gamma E_{j 1}(x)+R_{j}(x), \quad \partial_{l} u_{j}(x)=\gamma \partial_{l} E_{j 1}(x)+S_{j l}(x) \quad\left(x \in \overline{\mathfrak{D}}^{c}, 1 \leq j, l \leq 3\right) \tag{4}
\end{equation*}
$$

where $E: \mathbb{R}^{3} \backslash\{0\} \mapsto \mathbb{R}^{4} \times \mathbb{R}^{3}$ denotes a fundamental solution to the Oseen system

$$
\begin{equation*}
-\Delta v+\tau \partial_{1} v+\nabla \Pi=f, \operatorname{div} v=0 \quad \text { in } \mathbb{R}^{3} \tag{5}
\end{equation*}
$$

The definition of the function $E$ is stated in Section 2. As becomes apparent from this definition, the term $E_{j 1}(x)$ may be expressed explicitly in terms of elementary functions. The coefficient $\gamma$ is also given explicitly, its definition involving the Cauchy stress tensor. The remainder terms $R$ and $S$ are characterized by the relations $R \in L^{q}\left(\overline{\mathfrak{D}}^{c}\right)^{3}$ for $q \in(4 / 3, \infty), S \in L^{q}\left(\overline{\mathfrak{D}}^{c}\right)^{3}$ for $q \in(1, \infty)$. Since it is known from [6, Section VII.3] that $E_{j 1} \mid B_{r}^{c} \notin L^{q}\left(B_{r}^{c}\right)$ for $r>0$, $q \in[1,2]$, and $\partial_{l} E_{j 1} \mid B_{r}^{c} \notin L^{q}\left(B_{r}^{c}\right)$ for $r>0, q \in[1,4 / 3], j, l \in\{1,2,3\}$, the function $R$ decays faster than $E_{j 1}$, and $S_{j l}$ faster than $\partial_{l} E_{j 1}$, in the sense of $L^{q}$-integrability. Thus the equations in (4) may in fact be considered as asymptotic expansions of $u$ and $\nabla u$, respectively. The theory in [10] is valid under the assumption that $u$ verifies the boundary conditions

$$
\begin{equation*}
u(x)=e_{1}+(\omega \times x) \quad \text { for } x \in \partial \mathfrak{D} \tag{6}
\end{equation*}
$$

and $f$ vanishes. Reference [10] does not deal with the pointwise decay of $R$ and $S$.
In Theorem 3.1 below, we derive a pointwise decay of $u$ and $\nabla u$, respectively, which is independent of the boundary conditions, but compared to [10] and as indicated in (4) our leading term is less explicit than the term $\gamma E_{j 1}(x)$ in (4), and, instead of the fundamental solution $E_{j 1}(x)$ to the stationary Oseen system, we use the time integral of the fundamental solution to the evolutionary Oseen system.

In [5], it was shown that $\mathcal{Z}_{j 1}(x, 0)=E_{j 1}(x)$ for $x \in \mathbb{R}^{3} \backslash\{0\}, 1 \leq j \leq 3$, and $\lim _{|x| \rightarrow \infty}\left|\partial_{x}^{\alpha} \mathcal{Z}_{j k}(x, 0)\right|=O\left((|x| s(x))^{-3 / 2-|\alpha| / 2}\right)$ for $1 \leq j \leq 3, k \in\{2,3\}$ [5, Corollary 4.5, Theorem 5.1]. Thus, setting

$$
\begin{equation*}
\mathfrak{G}_{j}(x):=\sum_{k=2}^{3} \beta_{k} \mathcal{Z}_{j k}(x, 0)+\mathfrak{F}_{j}(x) \quad\left(x \in{\overline{B_{S_{1}}}}^{c}, 1 \leq j \leq 3\right) \tag{7}
\end{equation*}
$$

we may deduce from (16) that

$$
\begin{equation*}
u_{j}(x)=\beta_{1} E_{j 1}(x)+\left(\int_{\partial \Omega} u \cdot n \mathrm{do}_{x}\right) x_{j}\left(4 \pi|x|^{3}\right)^{-1}+\mathfrak{G}_{j}(x) \quad\left(x \in{\overline{B_{S_{1}}}}^{c}, 1 \leq j \leq 3\right) \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{|x| \rightarrow \infty}\left|\partial^{\alpha} \mathfrak{G}(x)\right|=O\left((|x| s(x))^{-3 / 2-|\alpha| / 2} \ln (2+|x|)\right) \quad \text { for } \alpha \in \mathbb{N}_{0}^{3} \text { with }|\alpha| \leq 1 \tag{9}
\end{equation*}
$$

(Theorem 3.2, Corollary 3.3). If we compare how the coefficient $\gamma$ from (4) is defined in [10], and the coefficient $\beta_{1}$ from (21) in [4] (see Theorem 3.1 below), and if we take into account the boundary condition (6) satisfied by $u$ in [10], we see that $\gamma$ and $\beta_{1}$ coincide.

Remark. Asymptotic behaviour of the linearized version of the problem is studied in $[2,3,9]$.

## 2. Notation. Definition of fundamental solutions. Auxiliary results

By \|| we denote the Euclidean norm in $\mathbb{R}^{3}$ and the length $\alpha_{1}+\alpha_{2}+\alpha_{3}$ of a multiindex $\alpha \in \mathbb{N}_{0}^{3}$. Let $e_{1}:=(1,0,0)$. For $r>0$, we set $B_{r}:=\left\{y \in \mathbb{R}^{3}:|y|<r\right\}$. If $A \subset \mathbb{R}^{3}$, we denote $A^{c}:=\mathbb{R}^{3} \backslash A$. Recall the abbreviation $s(x):=1+|x|-x_{1}\left(x \in \mathbb{R}^{3}\right)$ introduced in Section 1.

If $A \subset \mathbb{R}^{3}$ is open, $p \in[1, \infty)$ and $k \in \mathbb{N}$, we write $W^{k, p}(A)$ for the usual Sobolev space of order $k$ and exponent $p$. If $B \subset \mathbb{R}^{3}$ is again an open set, we define $L_{\mathrm{loc}}^{p}(B), W_{\mathrm{loc}}^{k, p}(B)$ as the set of all functions $v: B \mapsto \mathbb{R}$ such that $\left.v\right|_{U} \in L^{p}(U)$ and $\left.v\right|_{U} \in W^{k, p}(U)$, respectively, for any open bounded set $U \subset \mathbb{R}^{3}$ with $\bar{U} \subset B$.

For the Fourier transform $\widehat{g}$ of a function $g \in L^{1}\left(\mathbb{R}^{3}\right)$, we choose the definition $\widehat{g}(\xi):=(2 \pi)^{-3 / 2} \int_{\mathbb{R}^{3}} \mathrm{e}^{-\mathrm{i} \xi x} g(x) \mathrm{d} x\left(\xi \in \mathbb{R}^{3}\right)$. This fixes the definition of the Fourier transform of a tempered distribution as well.

The numbers $\tau \in(0, \infty), \varrho \in \mathbb{R} \backslash\{0\}$ introduced in Section 1 will be kept fixed throughout. Recall that the vector $\omega$ is given by $\omega:=\varrho \cdot e_{1}$. We introduce a matrix $\Omega \in \mathbb{R}^{3 \times 3}$ by setting

$$
\Omega:=\varrho\left(\begin{array}{rrr}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right)
$$

Note that $\omega \times x=\Omega \cdot x$ for $x \in \mathbb{R}^{3}$. We write $\mathfrak{C}$ for positive constants that may depend on $\tau$ or $\varrho$. Constants additionally depending on parameters $\sigma_{1}, \ldots, \sigma_{n} \in(0, \infty)$ for some $n \in \mathbb{N}$ are denoted by $\mathfrak{C}\left(\sigma_{1}, \ldots, \sigma_{n}\right)$. Next we introduce some fundamental solutions. Let $N(x):=(4 \pi|x|)^{-1}$ for $x \in \mathbb{R}^{3} \backslash\{0\}$ ("Newton potential", the fundamental solution to the Poisson equation in $\mathbb{R}^{3}$ ), $\mathfrak{O}(x):=(4 \pi|x|)^{-1} \mathrm{e}^{-\tau\left(|x|-x_{1}\right) / 2}$ for $x \in \mathbb{R}^{3} \backslash\{0\}$ (fundamental solution to the scalar Oseen equation $-\Delta v+\tau \partial_{1} v=g$ in $\left.\mathbb{R}^{3}\right), K(x, t):=(4 \pi t)^{-3 / 2} \mathrm{e}^{-|x|^{2} /(4 t)}$ for $x \in \mathbb{R}^{3}, t \in(0, \infty)$ (fundamental solution to the heat equation in $\left.\mathbb{R}^{3}\right), \psi(r):=\int_{0}^{r}\left(1-\mathrm{e}^{-t}\right) t^{-1} \mathrm{~d} t(r \in \mathbb{R}), \Phi(x):=(4 \pi \tau)^{-1} \psi\left(\tau\left(|x|-x_{1}\right) / 2\right)\left(x \in \mathbb{R}^{3}\right), E_{j k}(x):=\left(\delta_{j k} \Delta-\partial_{j} \partial_{k}\right) \Phi(x)$, $E_{4 k}(x):=x_{k}\left(4 \pi|x|^{3}\right)^{-1}\left(x \in \mathbb{R}^{3} \backslash\{0\}, 1 \leq j, k \leq 3\right)$ (the fundamental solution to the Oseen system (5), with $\left(E_{j k}\right)_{1 \leq j, k \leq 3}$ the velocity part and $\left(E_{4 k}\right)_{1 \leq k \leq 3}$ the pressure part). Following Solonnikov [11, (40)], we introduce the velocity part $\left(T_{j k}\right)_{1 \leq j, k \leq 3}$ of a fundamental solution to the time-dependent Stokes system, setting $T_{j k}(x, t):=\delta_{j k} K(x, t)+\partial_{j} \partial_{k}\left(\int_{\mathbb{R}^{3}} N(x-y) K(y, t) \mathrm{d} y\right)$ $\left(x \in \mathbb{R}^{3}, t>0,1 \leq j, k \leq 3\right)$.

Let:

$$
\begin{equation*}
\Gamma(x, y, t):=T\left(x-\tau t e_{1}-\mathrm{e}^{-t \Omega} \cdot y, t\right) \cdot \mathrm{e}^{-t \Omega} \quad \text { for } x, y \in \mathbb{R}^{3}, t>0 \tag{10}
\end{equation*}
$$

The matrix-valued function $\left(\Gamma_{j k}\right)_{1 \leq j, k \leq 3}$ is the velocity part of a fundamental solution to the time-dependent variant of the linearized equation of (1), precisely:

$$
\begin{equation*}
-\Delta v+\tau \partial_{1} v-(\omega \times x) \cdot \nabla v+\omega \times v+\nabla \Pi=f, \operatorname{div} v=0 \tag{11}
\end{equation*}
$$

This fundamental solution was constructed by Guenther, Thomann [12] via a procedure involving the Kummer functions, an approach also used in [1-5]. However, Guenther, Thomann [12, (3.9)] showed that $\Gamma$ is given by (10) as well, thus providing an access to this function that is more convenient in many respects.

From [1], we have $\int_{0}^{\infty}|\Gamma(x, y, t)| \mathrm{d} t<\infty$ for $x, y \in \mathbb{R}^{3}$ with $x \neq y$, so we may define

$$
\mathcal{Z}_{j k}(x, y):=\int_{0}^{\infty} \Gamma_{j k}(x, y, t) \mathrm{d} t \quad \text { for } x, y \in \mathbb{R}^{3} \text { with } x \neq y, 1 \leq j, k \leq 3
$$

This function $\mathcal{Z}$ was introduced in [12, p. 96] as the velocity part of a fundamental solution to (11).

## 3. Statement of our main result

Theorem 3.1. Let $\mathfrak{D} \subset \mathbb{R}^{3}$ be open, $p \in(1, \infty)$, $f \in L^{p}\left(\mathbb{R}^{3}\right)^{3}$ with $\operatorname{supp}(f)$ compact. Let $S_{1} \in(0, \infty)$ with $\overline{\mathfrak{D}} \cup \operatorname{supp}(f) \subset B_{S_{1}}$. Let $u \in L^{6}\left(\overline{\mathfrak{D}}^{c}\right)^{3} \cap W_{\text {loc }}^{1,1}\left(\overline{\mathfrak{D}}^{c}\right)^{3}, \pi \in L_{\text {loc }}^{2}\left(\overline{\mathfrak{D}}^{c}\right)$ with $\nabla u \in L^{2}\left(\overline{\mathfrak{D}}^{c}\right)^{9}, \operatorname{div} u=0$ and

$$
\begin{align*}
\int_{\overline{\mathfrak{D}}^{c}} & {\left[\nabla u \cdot \nabla \varphi+\left(\tau \partial_{1} u+\tau(u \cdot \nabla) u-(\omega \times z) \cdot \nabla u+\omega \times u\right) \cdot \varphi-\pi \operatorname{div} \varphi\right] \mathrm{d} z }  \tag{12}\\
& =\int_{\overline{\mathfrak{D}}^{c}} f \cdot \varphi \mathrm{~d} z \text { for } \varphi \in C_{0}^{\infty}\left(\overline{\mathfrak{D}}^{c}\right)^{3} .
\end{align*}
$$

(This means that the pair $(u, \pi)$ is a Leray solution to (1), (2).) Suppose in addition that

$$
\begin{equation*}
\mathfrak{D} \text { is } C^{2} \text {-bounded, } \quad u\left|\partial \mathfrak{D} \in W^{2-1 / p, p}(\partial \mathfrak{D})^{3}, \quad \pi\right| B_{S_{1}} \backslash \overline{\mathfrak{D}} \in L^{p}\left(B_{S_{1}} \backslash \overline{\mathfrak{D}}\right) \tag{13}
\end{equation*}
$$

Let $n$ denote the outward unit normal to $\mathfrak{D}$, and define

$$
\begin{align*}
\beta_{k}:= & \int_{\overline{\mathfrak{D}}^{c}} f_{k}(y) \mathrm{d} y  \tag{14}\\
& +\int_{\partial \mathfrak{D}} \sum_{l=1}^{3}\left(-\partial_{l} u_{k}(y)+\delta_{k l} \pi(y)+\left(\tau e_{1}-\omega \times y\right)_{l} u_{k}(y)-\tau\left(u_{l} u_{k}\right)(y)\right) n_{l}(y) \mathrm{d} o_{y} \tag{15}
\end{align*}
$$

for $1 \leq k \leq 3$,

$$
\begin{aligned}
\mathfrak{F}_{j}(x):= & \int_{\mathfrak{\mathfrak { D }}^{c}}\left[\sum_{k=1}^{3}\left(\mathcal{Z}_{j k}(x, y)-\mathcal{Z}_{j k}(x, 0)\right) f_{k}(y)-\tau \cdot \sum_{k, l=1}^{3} \mathcal{Z}_{j k}(x, y)\left(u_{l} \partial_{l} u_{k}\right)(y)\right] \mathrm{d} y \\
& +\int_{\partial \mathfrak{D}} \sum_{k=1}^{3}\left[\left(\mathcal{Z}_{j k}(x, y)-\mathcal{Z}_{j k}(x, 0)\right) \sum_{l=1}^{3}\left(-\partial_{l} u_{k}(y)+\delta_{k l} \pi(y)+\left(\tau e_{1}-\omega \times y\right)_{l} u_{k}(y)\right) n_{l}(y)\right. \\
& +\left(E_{4 j}(x-y)-E_{4 j}(x)\right) u_{k}(y) n_{k}(y) \\
& \left.+\sum_{l=1}^{3}\left(\partial y_{l} \mathcal{Z}_{j k}(x, y)\left(u_{k} n_{l}\right)(y)+\tau \mathcal{Z}_{j k}(x, 0)\left(u_{l} u_{k} n_{l}\right)(y)\right)\right] \mathrm{d} o_{y}
\end{aligned}
$$

for $x \in{\overline{B_{S_{1}}}}^{c}, 1 \leq j \leq 3$. The preceding integrals are absolutely convergent. Moreover, $\mathfrak{F} \in C^{1}\left(\overline{B_{S_{1}}}{ }^{c}\right)^{3}$ and equation

$$
\begin{equation*}
u_{j}(x)=\sum_{k=1}^{3} \beta_{k} \mathcal{Z}_{j k}(x, 0)+\left(\int_{\partial \Omega} u \cdot n \mathrm{do}_{\chi}\right) x_{j}\left(4 \pi|x|^{3}\right)^{-1}+\mathfrak{F}_{j}(x) \tag{16}
\end{equation*}
$$

holds. In addition, for any $S \in\left(S_{1}, \infty\right)$, there is a constant $C>0$ that depends on $\tau, \varrho, S_{1}, S, f, u$, and $\pi$, and which is such that

$$
\begin{equation*}
\left|\partial^{\alpha} \mathfrak{F}(x)\right| \leq C(|x| s(x))^{-3 / 2-|\alpha| / 2} \ln (2+|x|) \quad \text { for } x \in{\overline{B_{S}}}^{c}, \alpha \in \mathbb{N}_{0}^{3} \text { with }|\alpha| \leq 1 \tag{17}
\end{equation*}
$$

Proof. We will just give main lines of proof. For complete details, see [4, Theorem 3.1]. The main tool of proof is the representation formula

$$
\begin{equation*}
u_{j}(y)=\mathcal{R}_{j}(f-\tau(u \cdot \nabla) u)(y)+\mathcal{B}_{j}(u, \pi(y)) \tag{18}
\end{equation*}
$$

where

$$
\begin{align*}
\mathcal{B}_{j}(y):= & \mathcal{B}_{j}(u, \pi)(y)  \tag{19}\\
:= & \int_{\partial \mathfrak{D}} \sum_{k=1}^{3}\left[\sum _ { l = 1 } ^ { 3 } \left(\mathcal{Z}_{j k}(y, z) \cdot\left(-\partial_{l} u_{k}(z)+\delta_{k l} \cdot \pi(z)+u_{k}(z) \cdot\left(\tau \cdot e_{1}-\omega \times z\right)_{l}\right)\right.\right. \\
& \left.\left.+\partial z_{l} \mathcal{Z}_{j k}(y, z) \cdot u_{k}(z)\right) \cdot n_{l}^{(\mathfrak{D})}(z)+E_{4 j}(y-z) \cdot u_{k}(z) \cdot n_{k}^{(\mathfrak{D})}(z)\right] \mathrm{d} o_{z}
\end{align*}
$$

for $y \in \overline{\mathfrak{D}}^{c}$, with outer normal $n^{(\mathfrak{D})}$ to $\mathfrak{D}$.

We have to consider the term $\mathcal{R}_{j}((u \cdot \nabla) u)(x)=\int_{\overline{\mathcal{D}}^{c}} \sum_{k=1}^{3} \mathcal{Z}_{j k}(x, y)\left[(u \cdot \nabla) u_{k}\right](y) \mathrm{d} y$. Integrating by parts [1, Lemma 4.2, Theorem 2.19, Theorem 5.5, Theorem 3.3] and [2, Theorem 1.1], we get the corresponding estimate. Concerning the leading term for the velocity gradient, $\int_{\overline{\mathcal{D}}^{c}} \sum_{k=1}^{3} \partial_{m} \mathcal{Z}_{j k}(x, y)\left[(u \cdot \nabla) u_{k}\right](y) \mathrm{d} y$, we have first to divide the integral into two parts $\int_{\overline{\mathcal{D}}^{c} \backslash B_{1}(x)}$ and $\int_{B_{1}(x)}$. Again integrating by parts and using [1, Lemmas 3.3, 5.4], and [2, Theorem 1], we get the leading term for the velocity part.

Theorem 3.2. Let $\mathfrak{D}, p, f, S_{1}, u, \pi$ satisfy the assumptions of Theorem 3.1, including (13). Let $\beta_{1}, \beta_{2}, \beta_{3}$ and $\mathfrak{F}$ be defined as in Theorem 3.1. Define the function $\mathfrak{G}$ as

$$
\begin{equation*}
\mathfrak{G}_{j}(x):=\sum_{k=2}^{3} \beta_{k} \mathcal{Z}_{j k}(x, 0)+\mathfrak{F}_{j}(x) \quad\left(x \in{\overline{B_{S_{1}}}}^{c}, 1 \leq j \leq 3\right) . \tag{20}
\end{equation*}
$$

Then $\mathfrak{G} \in C^{1}\left(\overline{B_{S_{1}}}{ }^{c}\right)^{3}$, equation

$$
\begin{equation*}
u_{j}(x)=\beta_{1} E_{j 1}(x)+\left(\int_{\partial \Omega} u \cdot n \operatorname{do}_{x}\right) x_{j}\left(4 \pi|x|^{3}\right)^{-1}+\mathfrak{G}_{j}(x) \quad\left(x \in{\overline{B_{S_{1}}}}^{c}, 1 \leq j \leq 3\right) \tag{21}
\end{equation*}
$$

holds, and for any $S \in\left(S_{1}, \infty\right)$, there is a constant $C>0$ that depends on $\tau, \varrho, S_{1}, S, f, u$, and $\pi$, and which is such that

$$
\left|\partial^{\alpha} \mathfrak{G}(x)\right| \leq C(|x| S(x))^{-3 / 2-|\alpha| / 2} \ln (2+|x|) \quad \text { for } x \in{\overline{B_{S}}}^{c}, \alpha \in \mathbb{N}_{0}^{3} \text { with }|\alpha| \leq 1 .
$$

Corollary 3.3. Take $\mathfrak{D}, p, f, S_{1}, u, \pi$ as in Theorem 3.1, but without requiring (13). (This means that $(u, \pi)$ is only assumed to be a Leray solution to (1), (2).) Let $\tilde{p}:=\min \{3 / 2, p\}$.

Then $u \in W_{\text {loc }}^{2, \tilde{p}}\left(\overline{\mathfrak{D}}^{c}\right)^{3}$ and $\pi \in W_{\text {loc }}^{1, \tilde{p}}\left(\overline{\mathfrak{D}}^{c}\right)$.
Fix some number $S_{0} \in\left(0, S_{1}\right)$ with $\overline{\mathfrak{D}} \cup \operatorname{supp}(f) \subset B_{S_{0}}$, and define $\beta_{1}, \beta_{2}, \beta_{3}$ and $\mathfrak{F}$ as in Theorem 3.1, but with $\mathfrak{D}$ replaced by $B_{S_{0}}$, and $n(x)$ by $S_{0}^{-1} x$, for $x \in \partial B_{S_{0}}$. Moreover, define $\mathfrak{G}$ as in (20).

Then all the conclusions of Theorem 3.2 are valid.

Proof. The proof is based on the Fourier transformation of the Oseen resolvent to get a Fourier transformation of our fundamental solution and on the application of [12, Lemma 13]. For complete details, see [5].

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