Group theory/Combinatorics

# Harmonic functions with finite $p$-energy on lamplighter graphs are constant 

# Sur les graphes d'allumeur de réverbères, les fonctions harmoniques de p-énergie finie sont constantes 

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## A R T I CLE IN F O

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#### Abstract

The aim of this note is to show that lamplighter graphs where the space graph is infinite and at most two-ended and the lamp graph is at most two-ended do not admit harmonic functions with gradients in $\ell^{p}$ (i.e. finite $p$-energy) for any $p \in[1, \infty[$ except constants (and, equivalently, that their reduced $\ell^{p}$ cohomology is trivial in degree one). Similar arguments are then applied to many direct products of graphs to conclude the same (including all direct products of Cayley graphs). The proof relies on a theorem of Thomassen [16] on spanning lines in squares of graphs.


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## Ré S U M É

Le but de cette note est de montrer que plusieurs graphes d'allumeurs de réverbères, où le graphe d'espace est infini avec au plus deux bouts et le graphe des lampes a au plus deux bouts, ne possèdent pas de fonction harmonique non constante à gradient $\ell^{p}$ (i.e. une $p$-energie finie) qu'importe le $p \in[1, \infty[$ (et, de manière équivalente, que leur cohomologie $\ell^{p}$ réduite est triviale en degré un). Des arguments similaires permettent aussi de conclure pour plusieurs produits directs de graphes (y compris tous les graphes de Cayley). Les démonstrations reposent sur un théorème de Thomassen [16] sur les lignes couvrantes dans le carré des graphes.
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## 1. Introduction

Given two graphs $H=(X, E)$ (henceforth the "space" graph) and $L=(Y, F)$ (henceforth the "lamp" graph), the lamplighter graph $G:=L_{2} H$ is the graph constructed as follows. Fix some root vertex $o \in Y$ and let $\left(\oplus_{X} Y\right)$ be the set of "finitely

[^0]supported" functions from $X \rightarrow Y$ (i.e. only finitely many elements of $X$ are not sent to $o \in Y$ ). Its vertices are elements of $X \times\left(\oplus_{X} Y\right)$. Two vertices $(x, f)$ and $\left(x^{\prime}, f^{\prime}\right)$ are adjacent if

- either $x \sim x^{\prime}$ in $H$ and $f=f^{\prime}$,
- or $x=x^{\prime}, f(y)=f^{\prime}(y)$ for all $y \neq x$ and $f(x) \sim f^{\prime}(x)$ in $L$.

It is easy to see that $L 2 H$ is connected exactly when both $H$ and $L$ are. In fact, in this note, all graphs will be assumed to be connected (this is not important) and the graphs are locally finite.

The number of ends of a [connected] graph is the limit of the number of infinite components in the complement of $B_{n}$ (where $B_{n}$ is a sequence of balls of radius at some fixed vertex). More precisely, an end $\xi$ is a function from finite sets to infinite connected components of their complement so that $\xi(F) \cap \xi\left(F^{\prime}\right) \neq \varnothing$ (for any $F$ and $F^{\prime}$ ).

Given a graph $G$, a real-valued function $f$ on its vertices $V$ is said to be harmonic if it satisfies the mean value property

$$
\forall v \in V, f(v)=\frac{1}{\operatorname{deg}(v)} \sum_{w \sim v} f(w)
$$

where $v$ is the degree (or valency) of $v$. The gradient of $f$ is the function on the edges ( $v, w$ ) defined by $\nabla f(v, w)=$ $f(w)-f(v)$. The square of the $\ell^{2}$-norm of the gradient is often referred to as the energy of the function.

The main result here is:

Proposition 1. Assume $H$ is infinite and has at most two ends, $L$ has at least one edge, $L$ has two ends or less and that both $L$ and $H$ are locally finite, then there are no non-constant harmonic functions with gradient in $\ell^{p}$ in $L$ 子 $H$ for any $p \in[1, \infty[$.

This result is in contrast with the fact that lamplighter graphs have bounded harmonic functions as soon as $H$ is not recurrent. Indeed, a bounded function has necessarily its gradient in $\ell^{\infty}$.

In fact, Proposition 1 uses (and, when the graphs have bounded valency, is equivalent to) the vanishing of the reduced $\ell^{p}$ cohomology in degree one, see [4] for definitions. The proof of Proposition 1 is essentially a particular case of [4, Question 1.6]. This answers partially questions which may be found (in different guises) in Georgakopoulos [2, Problem 3.1] and Gromov [7, §8. $A_{1} .\left(A_{2}\right)$, p. 226]. Regarding [2], Proposition 1 seems hard to adapt to cases with infinitely many ends, but covers all $p$ (instead of $p=2$ ).

As for [7], the question there concerns other types of graphs; for lamplighter graphs of Cayley graphs, the answer to this question is essentially complete. Indeed, a wreath product (i.e. lamplighter group) is amenable exactly when the lamp and space groups are amenable. Since amenable groups have at most 2 ends, Proposition 1 shows the reduced $\ell^{p}$-cohomology of any amenable wreath product is trivial. Note that Martin \& Valette [11, Theorem-(iv)] show this is still true when $L$ is not amenable and has infinitely many ends (and $H$ is infinite).

Proposition 1 extends probably to graphs with finitely many ends. To do this, one would need to answer the following question. Assume $\mathcal{G}$ is the set of graphs obtained by taking a cycle and attaching to it finitely many (half-infinite) rays. Is the lamplighter graph $L$ z $H$ with $L, H \in \mathcal{G}$ Liouville? This seems to follow from classical consideration of Furstenberg (coupling), since both $H$ and $L$ are recurrent.

Our other application concerns the direct product. Given two graph $H_{1}=\left(X_{1}, E_{1}\right)$ and $H_{2}=\left(X_{2}, E_{2}\right)$, the direct product $H_{1} \times H_{2}$ is defined as follows. Its vertices are elements of $X_{1} \times X_{2}$. Two vertices ( $x_{1}, x_{2}$ ) and ( $x_{1}^{\prime}, x_{2}^{\prime}$ ) are adjacent if either $x_{1} \sim x_{1}^{\prime}$ or $x_{2} \sim x_{2}^{\prime}$, but not both.

Proposition 2. Assume $G$ is a direct product of graphs $H_{1} \times H_{2}$, so that $H_{1}$ has 1 or 2 ends and $H_{2}$ is a Cayley graph with volume growth at least polynomial of degree $d$, then there are no non-constant harmonic functions with gradient in $\ell^{p}$ for all $p<\frac{d+1}{2}$.
$H_{1}$ is only locally finite, but $H_{2}$ will be of bounded valency. This generalises a result of Martin \& Valette [11, Theorem-(v)] (on products of groups and which requires that one group in the direct product be non-amenable):

Corollary 3. Let $\Gamma$ be a direct product of infinite [finitely generated] groups. Then there are no non-constant harmonic functions with gradient in $\ell^{p}$ in any Cayley graph of $\Gamma$ (and the reduced $\ell^{p}$ cohomology in degree 1 is trivial for all $p \in[1, \infty[$ ).

Proposition 1 and Corollary 3 also have consequences on the cohomology of Hilbertian representations with $\ell^{p}$-coefficients, see [6, Corollary 2.6]. The same can be said for some representations given by $G \curvearrowright L^{q}$ (with coefficients in $\ell^{p}$ ) modulo the following remark:

Remark 4. There is a non-linear analogue of harmonic equations called $p$-harmonic equation (with $p \in] 1, \infty[$ ). The proofs of the Propositions 1 and 2 also apply to $q$-harmonic functions with gradient in $\ell^{p}$. Indeed, $q$ is irrelevant, since only the fact that harmonic functions satisfy the maximum principle is required to conclude (and $q$-harmonic functions also satisfy the maximum principle).

## 2. Preliminaries

Let $D^{p}(G)$ be the space of functions on the vertices of the graph $G$ with gradient in $\ell^{p}$ and $\mathcal{H} D^{p}(G)$ be the subset of $D^{p}(G)$ consisting of functions that are furthermore harmonic. Lastly, $B \mathcal{H} \mathrm{D}^{p}(G)$ are the bounded functions in $\mathcal{H} \mathrm{D}^{p}(G)$. The notation $\mathcal{H} D^{p}(G) \simeq \mathbb{R}$ means that the only functions in $\mathcal{H D}^{p}(G)$ are constants.

For $F \subset X$ a subset of the vertices, let $\partial F$ be the edges between $F$ and $F^{c}$. Let $d \in \mathbb{R}_{\geq 1}$. Then, a graph $G=(X, E)$ has
$\mathrm{IS}_{d}$ if there is a $\kappa>0$ such that for all finite $F \subset X,|F|^{(d-1) / d} \leq \kappa|\partial F|$.
Quasi-homogeneous graphs with a certain (uniformly bounded below) volume growth in $n^{d}$ will satisfy these isoperimetric profiles, see Woess' book [17, (4.18) Theorem]. For example, the Cayley graph of a group $G$ satisfies $\mathrm{IS}_{d}$ for all $d$ if and only if $G$ is not virtually nilpotent.

An important ingredient of the proofs is a result from [4]. Let $B_{n}$ be a sequence of balls in the graph with the same centre and $B_{n}^{c}$ its complement. On a connected graph, a function $f: X \rightarrow \mathbb{R}$ takes only one value at infinity if $\exists c \in \mathbb{R}$ so that $\forall \varepsilon>0, \exists n_{\varepsilon}$ satisfying $f\left(B_{n_{\varepsilon}}^{\mathrm{c}}\right) \subset[c-\varepsilon, c+\varepsilon]$. Define for $p \geq 1$ :
(1p) the reduced $\ell^{p}$-cohomology in degree one vanishes (for short, $\underline{\ell}^{p} H^{1}=\{0\}$ );
$\left(2_{p}\right)$ all functions in $D^{p}(G)$ take only one value at infinity;
$\left(3_{p}\right)$ there are no non-constant functions in $\mathcal{H D}^{p}(G)$;
$\left(4_{p}\right)$ there are no non-constant functions in $\mathrm{BH}^{p}(G)$.
For the record, note that $\left(1_{1}\right) \Longleftrightarrow\left(2_{1}\right) \Longleftrightarrow$ the number of ends is $\leq 1$ (see [4, Proposition A.2]). Let us sum up [4, Theorem 1.2] here again:

Theorem 5. Assume a graph $G$ is of bounded valency and has $\mathrm{IS}_{d}$. For $1<p<d / 2,\left(1_{p}\right) \Longleftrightarrow\left(2_{p}\right) \Longrightarrow\left(3_{p}\right) \Longrightarrow\left(4_{p}\right)$ and, for $q \geq \frac{d p}{d-2 p},\left(4_{q}\right) \Longrightarrow\left(1_{p}\right)$.

If $G$ has $\mathrm{IS}_{d}$ for all $d$, then " $\left.\forall p \in\right] 1, \infty\left[,\left(i_{p}\right)\right.$ holds" where $i \in\{1,2,3,4\}$ are four equivalent conditions.
The important corollary of the above theorem (see [4, Corollary 4.2.1]) is that if a graph $G$ has a connected spanning subgraph which is Liouville and has $\mathrm{IS}_{d}$ for some $d$ (resp. for all $d$ ), then $\left(1_{p}\right),\left(2_{p}\right)$ and ( $3_{p}$ ) hold for any $p<d / 2$ (resp. for all $p<\infty$ ). Indeed, Liouville implies that $\left(4_{q}\right)$ holds for all $q$, and the condition ( $2_{p}$ ) passes from a connected spanning subgraph to the whole graph.

## 3. Proof of Proposition 1

The main second ingredient of the proof of Proposition 1 is the following. Let $G_{0}=L_{0}$ 乙 $H_{0}$ the lamplighter graph where $L_{0}$ is either finite or a Cayley graph of $\mathbb{Z}$ and $H_{0}$ is a Cayley graph of $\mathbb{Z}$. For our current purpose, it will suffice to note that $G_{0}$ has $\mathrm{IS}_{d}$ for any $d \geq 1$, see Woess' book [17, (4.16) Corollary]. A second important ingredient is that, using Kaimanovich [9, Theorem 3.3], $G_{0}$ is Liouville, i.e. a bounded harmonic function is constant.

The proof will be split in three steps for convenience.
Step 1 - Assume that $H$ and $L$ have bounded valency. Note that if a spanning subgraph of $G$ has $\mathrm{IS}_{d}$, it implies that $G$ has $I S_{d}$. Summing up, if a graph $G$ admits $G_{0}$ as a subgraph then $\left(1_{q}\right)$ holds in $G$ for any $q<\infty$ and, equivalently, ( $3_{p}$ ) holds in $G$ for any $p<\infty$.

It is also possible to work only up to quasi-isometry: if two graphs of bounded valency $\Gamma$ and $\Gamma^{\prime}$ are quasi-isometric, then they have the same $\ell^{p}$-cohomology (in all degrees, reduced or not), see Élek [1, §3] or Pansu [12].

Recall that the $k$-fuzz of a graph $G$, is the graph $G^{[k]}$ with the same vertices as $G$, but now two vertices are neighbours in $G^{[k]}$ if their distance in $G$ is $\leq k . G^{[2]}$ is often called the square of $G$.

Lastly, using either Thomassen [16] or Seward [14, Theorem 1.6], the graphs $L$ and $H$ in Proposition 1 are bi-Lipschitz equivalent to graphs containing a spanning line (or a cycle if the graph is finite). In fact, this bi-Lipschitz equivalence is given by taking the $k$-fuzz of these graphs. An interested reader could probably show that $k=4$ is sufficient. This means that $L_{2} H$ is bi-Lipschitz equivalent (and so quasi-isometric) to a graph containing $G_{0}$. This finishes the proof of Proposition 1 when both $H$ and $L$ have bounded valency.

Step 2 - Assume from now on that both $H$ and $L$ have connected spanning subgraphs of bounded valency, say $H^{\prime}$ and $L^{\prime}$, respectively. If there is a non-constant $f \in \mathcal{H} D^{p}(G)$ (where $G=L \imath H$ ). Then $f$ is not constant at infinity. Indeed, since $f$ is harmonic, the maximum principle would then imply that $f$ is constant.

But $f$ is also a function on the vertices of $G^{\prime}=L^{\prime} \imath H^{\prime}$ and it is also in $D^{p}\left(G^{\prime}\right)$ (because deleting edges only reduces the $\ell^{p}$ norm of the gradient). So ( $2_{p}$ ) cannot hold on $G^{\prime}$. On the other hand, $G^{\prime}$ contains $G_{0}$ up to quasi-isometry (as in step 1) and hence $\underline{\ell^{p} H^{1}}\left(G^{\prime}\right)=\{0\}$. However, by Theorem 5 above, " $\left(1_{p}\right)$ for all $p$ " implies " $\left(2_{p}\right)$ for all $p$ ".

Step 3 - Now assume $H$ and $L$ are only locally finite. The result of Thomassen [16] still implies that (for some $k$ ) the $k$-fuzz of $H$ and $L$ have a spanning line (or cycle if the graph $L$ is finite). However, given a function $f \in D^{p}(G)$, it may no
longer be in $D^{p}\left(G^{[k]}\right)$ if $k>1$ and $G$ does not have bounded valency. To circumvent this problem, construct a graph $H^{\dagger}$ by adding (when necessary) to $H$ the edges of the spanning line in $H^{[k]}$. Construct $L^{\dagger}$ similarly.

Given $f \in \mathrm{D}^{p}(G)$ where $G=L \imath H$, one has that $f \in \mathrm{D}^{p}\left(G^{\dagger}\right)$ with $G^{\dagger}=L^{\dagger} \imath H^{\dagger}$. Indeed, in passing from $G$ to $G^{\dagger}$, at most four edges are added to each vertex, and the gradient along these edges is expressed as a sum of $k$ values of the gradient of $f$ on $G$. The triangle inequality ensures that the $\ell^{p}$-norm of $\nabla f$ (on $G^{\dagger}$ ) is at most $(4 k+1)$ times the $\ell^{p}$-norm of the gradient of $f$ on $G$.

This last reduction yields the conclusion. Indeed, if there is an $f \in \mathcal{H} D^{p}(G)$ that is not constant, then there is an $f \in \mathrm{D}^{p}\left(G^{\dagger}\right)$ that takes different values at infinity. This is however excluded by step 2 (since $H^{\dagger}$ and $L^{\dagger}$ have connected spanning subgraphs which are of valency $\leq 2$ ).

## 4. Proof of Proposition 2 and Corollary 3

The main second ingredient for the proof of Proposition 2 is that if $G$ is a Cayley graph of a [finitely generated] group and this group has infinitely many finite conjugacy classes (e.g., infinite centre) then $\underline{\ell}^{p} H^{1}(G)=\{0\}$ (there are many possible proofs: see Kappos [10, Theorem 6.4], Martin \& Valette [11, Theorem 4.3] Puls [13, Theorem 5.3], Tessera [15, Proposition 3] or [3, Theorem 3.2]).

Proof of Proposition 2. Let $\Gamma$ be the group whose Cayley graph is $H_{2}$, let $\Gamma_{0}=\mathbb{Z} \times \Gamma$ and let $G_{0}$ be the direct product of the bi-infinite line and $H_{2}$ (a Cayley graph of $\Gamma_{0}$ ). By the result quoted in the previous paragraph, $\underline{\ell^{p}} H^{1}\left(G_{0}\right)=\{0\}$. The growth condition (see Woess' book [17, (4.16) Corollary]) implies that $G_{0}$ has $\mathrm{IS}_{d+1}$. By Theorem 5, one deduces that $G$ has no non-constant harmonic functions with gradient in $\ell^{p}$ for $p<\frac{d+1}{2}$.

To realise $G_{0}$ as a spanning subgraph, the arguments are absolutely identical to those of the proof of Proposition 1 (§ 3 above).

Proof of Corollary 3. The proof requires to distinguish two cases:

- if one of the two groups (say $\Gamma_{2}$ ) is not virtually nilpotent, then its Cayley graphs have $\mathrm{IS}_{d}$ for all $d$. By Theorem 5 , " $\left(3_{p}\right)$ for all $p$ " is equivalent to " $\left(1_{p}\right)$ for all $p$ " (which does not depend on the generating set). Take a generating set so the graph is a direct product and take $H_{2}$ to be a Cayley graph of $\Gamma_{2}$. Apply Proposition 2 to conclude.
- if both groups are virtually nilpotent, so is the direct product, then it is well known that there are no non-constant harmonic functions with gradient in $c_{0}$ (see for example [5, Lemma 5]) and even no non-constant functions with sublinear growth (see Hebisch \& Saloff-Coste [8, Theorem 6.1]). Note that in this second case, one still has that ( $1_{p}$ ) holds $\forall p \in] 1, \infty[$.


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