Partial differential equations/Functional analysis

**Bourgain–Brézis–Mironescu formula for magnetic operators**

**Formule de Brézis–Bourgain–Mironescu pour des opérateurs magnétiques**

Marco Squassina\textsuperscript{a}, Bruno Volzone\textsuperscript{b}

\textsuperscript{a}Dipartimento di Informatica, Universit\'a degli Studi di Verona, Strada Le Grazie 15, 37134 Verona, Italy
\textsuperscript{b}Dipartimento di Ingegneria, Universit\'a di Napoli Parthenope, Centro Direzionale Isola C/4, 80143 Napoli, Italy

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**Abstract**

We prove a Bourgain–Brézis–Mironescu-type formula for a class of nonlocal magnetic spaces, which builds a bridge between a fractional magnetic operator recently introduced and the classical theory.

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**Résumé**

On démontre une formule du type Bourgain–Brézis–Mironescu pour une classe d’espaces magnétiques non locaux, qui jette un pont entre un opérateur magnétique fractionnaire récemment introduit et la théorie classique.

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1. Introduction

Let \( s \in (0, 1) \) and \( N > 2s \). If \( A : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a smooth function, the nonlocal operator

\[
(-\Delta)^s_A u(x) = c(N, s) \lim_{\epsilon \rightarrow 0} \int_{B_\epsilon(x)} \frac{u(x) - e^{i(x-y) \cdot A(x)} (\frac{|x-y|}{2})^{\alpha s}}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,
\]

has been recently introduced in [6], where the ground-state solutions to \((-\Delta)^s_A u + u = |u|^{p-2} u\) in the three-dimensional setting have been obtained via concentration compactness arguments. If \( A = 0 \), then the above operator is consistent with the usual notion of fractional Laplacian. The motivations that led to its introduction are carefully described in [6] and rely essentially on the Lévy–Khinchine formula for the generator of a general Lévy process. We point out that the normalization constant \( c(N, s) \) satisfies

\[
\lim_{s \rightarrow 1} \frac{c(N, s)}{1-s} = \frac{4N \Gamma(N/2)}{2 \pi^{N/2}}.
\]

E-mail addresses: marco.squassina@univr.it (M. Squassina), bruno.volzone@uniparthenope.it (B. Volzone).

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where \( \Gamma \) denotes the Gamma function. For the sake of completeness, we recall that different definitions of nonlocal magnetic operator are viable, see, e.g., [8,9]. All these notions aim to extend the well-known definition of the magnetic Schrödinger operator

\[
-\left( \nabla - iA(x) \right)^2 u = -\Delta u + 2iA(x) \cdot \nabla u + |A(x)|^2 u + iu \text{div} A(x),
\]

namely the differential of the energy functional

\[
E_A(u) = \int_{\mathbb{R}^N} |\nabla u - iA(x)u|^2 dx,
\]

for which we refer the reader to [1,2,11] and the included references. In order to corroborate the justification for the introduction of \((-\Delta)^s_A\), in this note, we prove that a well-known formula due to Bourgain, Brézis and Mironescu (see [3,4,10]) for the limit of the Gagliardo semi-norm of \(H^s(\Omega)\) as \(s \nearrow 1\) extends to the magnetic setting. As a consequence, in a suitable sense, from the nonlocal to the local regime, it holds

\[
(-\Delta)^s_A u \rightharpoonup (\nabla - iA(x))^2 u, \quad \text{for } s \nearrow 1.
\]

We consider

\[
[u]_{H^1_s(\Omega)} := \sqrt{\int_{\Omega} |\nabla u - iA(x)u|^2 dx},
\]

and define \(H^1_s(\Omega)\) as the space of functions \(u \in L^2(\Omega, \mathbb{C})\) such that \([u]_{H^1_s(\Omega)} < \infty\) endowed with the norm

\[
\|u\|_{H^1_s(\Omega)} := \sqrt{\|u\|_{L^2(\Omega)}^2 + [u]_{H^1_s(\Omega)}^2}.
\]

Our main results are the following.

**Theorem 1.1** (Magnetic Bourgain–Brézis–Mironescu). Let \(\Omega \subset \mathbb{R}^N\) be an open bounded set with Lipschitz boundary and \(A \in C^2(\overline{\Omega})\). Then, for every \(u \in H^1_s(\Omega)\), we have

\[
\lim_{s \nearrow 1} (1 - s) \int_{\Omega} \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^{N+2s}} dx \, dy = K_N \int_{\Omega} |\nabla u - iA(x)u|^2 dx,
\]

where

\[
K_N = \frac{1}{2} \int_{\mathbb{S}^{N-1}} |\omega \cdot e|^2 d\mathcal{H}^{N-1}(\omega),
\]

(1.1)

being \(\mathbb{S}^{N-1}\) the unit sphere and \(e\) any unit vector in \(\mathbb{R}^N\).

As a variant of Theorem 1.1, if \(H^1_{0,A}(\Omega)\) denotes the closure of \(C^\infty_c(\Omega)\) in \(H^1_A(\Omega)\), we get the following theorem.

**Theorem 1.2.** Let \(\Omega \subset \mathbb{R}^N\) be an open bounded set with Lipschitz boundary. Assume that \(A : \mathbb{R}^N \to \mathbb{R}^N\) is locally bounded and \(A \in C^2(\overline{\Omega})\). Then, for every \(u \in H^1_{0,A}(\Omega)\), we have:

\[
\lim_{s \nearrow 1} (1 - s) \int_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y) \cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^{N+2s}} dx \, dy = K_N \int_{\Omega} |\nabla u - iA(x)u|^2 dx.
\]

**Notations.** Let \(\Omega \subset \mathbb{R}^N\) be an open set. We denote by \(L^2(\Omega, \mathbb{C})\) the Lebesgue space of complex valued functions with summable square. For \(s \in (0, 1)\), the magnetic Gagliardo semi-norm is

\[
[u]_{H^s_A(\Omega)} := \sqrt{\int_{\Omega} \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^{N+2s}} dx \, dy}.
\]

We denote by \(H^s_A(\Omega)\) the space of functions \(u \in L^2(\Omega, \mathbb{C})\) such that \([u]_{H^s_A(\Omega)} < \infty\) endowed with
\[ \|u\|_{H^1(\Omega)} := \sqrt{\|u\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2}. \]

We denote by \( B(x_0, R) \) the ball in \( \mathbb{R}^N \) of center \( x_0 \) and radius \( R > 0 \). For any set \( E \subset \mathbb{R}^N \), we will denote by \( E^c \) the complement of \( E \). For \( A, B \subset \mathbb{R}^N \) open and bounded, \( A \cap B \) means \( \overline{A} \subset B \).

2. Preliminary results

We start with the following Lemma.

**Lemma 2.1.** Assume that \( A : \mathbb{R}^N \to \mathbb{R}^N \) is locally bounded. Then, for any compact \( V \subset \mathbb{R}^N \) with \( \Omega \subset V \), there exists \( C = C(A, V) > 0 \) such that

\[
\int_{\mathbb{R}^N} |u(y + h) - e^{ih \cdot A(y + \frac{h}{2})} u(y)|^2 \, dy \leq C|h|^2 \|u\|_{H^1(\mathbb{R}^N)}^2,
\]

for all \( u \in H^1(\mathbb{R}^N) \) such that \( u = 0 \) on \( V^c \) and any \( h \in \mathbb{R}^N \) with \( |h| \leq 1 \).

**Proof.** Assume first that \( u \in C_0^\infty(\mathbb{R}^N) \) with \( u = 0 \) on \( V^c \). Fix \( y, h \in \mathbb{R}^N \) and define

\[
\varphi(t) := e^{i(1-t)h \cdot A(y + \frac{h}{2})} u(y + th), \quad t \in [0, 1].
\]

Then we have

\[
u(y + h) - e^{ih \cdot A(y + \frac{h}{2})} u(y) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t) \, dt,
\]

and since

\[
\varphi'(t) = e^{i(1-t)h \cdot A(y + \frac{h}{2})} h \cdot \left( \nabla_y u(y + th) - iA\left(y + \frac{h}{2}\right) u(y + th) \right),
\]

by Hölder inequality we get

\[
|u(y + h) - e^{ih \cdot A(y + \frac{h}{2})} u(y)|^2 \leq |h|^2 \int_0^1 \left| \nabla_y u(y + th) - iA\left(y + \frac{h}{2}\right) u(y + th) \right|^2 \, dt.
\]

Therefore, integrating with respect to \( y \) over \( \mathbb{R}^N \) and using Fubini's Theorem, we get

\[
\int_{\mathbb{R}^N} |u(y + h) - e^{ih \cdot A(y + \frac{h}{2})} u(y)|^2 \, dy \leq |h|^2 \int_0^1 \int_{\mathbb{R}^N} \left| \nabla_z u(z + \frac{1-2t}{2}h) - iA\left(z + \frac{1-2t}{2}h\right) u(z) \right|^2 \, dz dy
\]

\[
= |h|^2 \int_0^1 \int_{\mathbb{R}^N} \left| \nabla_z u(z) - iA(z) u(z) \right|^2 \, dz
\]

\[
\leq 2|h|^2 \int_{\mathbb{R}^N} \left| \nabla_z u(z) - iA(z) u(z) \right|^2 \, dz + 2|h|^2 \int_V \left| A\left(z + \frac{1-2t}{2}h\right) - A(z) \right|^2 |u(z)|^2 \, dz.
\]

Then, since \( A \) is bounded on the set \( V \), we have for some constant \( C > 0 \)

\[
\int_{\mathbb{R}^N} |u(y + h) - e^{ih \cdot A(y + \frac{h}{2})} u(y)|^2 \, dy \leq C|h|^2 \left( \int_{\mathbb{R}^N} \left| \nabla_z u(z) - iA(z) u(z) \right|^2 \, dz + \int_{\mathbb{R}^N} |u(z)|^2 \, dz \right)
\]

\[
= C|h|^2 \|u\|_{H^1(\mathbb{R}^N)}^2.
\]

When dealing with a general \( u \) we can argue by a density argument. \( \square \)
Lemma 2.2. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, $V \subset \mathbb{R}^N$ a compact set with $\Omega \Subset V$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ locally bounded. Then there exists $C(\Omega, V, A) > 0$ such that for any $u \in H^1_A(\Omega)$ there exists $E_u \in H^1(\mathbb{R}^N)$ such that $E_u = u$ in $\Omega$, $E_u = 0$ in $V^c$ and

$$\|E_u\|_{H^1(\mathbb{R}^N)} \leq C(\Omega, V, A)\|u\|_{H^1_A(\Omega)}.$$

Proof. Observe that, for any bounded set $W \subset \mathbb{R}^N$ there exist $C_1(A, W), C_2(A, W) > 0$ with

$$C_1(A, W)\|u\|_{H^1(W)} \leq \|u\|_{H^1(W)} \leq C_2(A, W)\|u\|_{H^1(W)}, \text{ for any } u \in H^1(W).$$

This follows easily, via simple computations, by the definition of the norm of $H^1(W)$ and in view of the local boundedness assumption on the potential $A$. Now, by the standard extension property for $H^1(\Omega)$ (see, e.g., [7, Theorem 1, p. 254]), there exists $C(\Omega, V) > 0$ such that, for any $u \in H^1(\Omega)$, there exists a function $E_u \in H^1(\mathbb{R}^N)$ such that $E_u = u$ in $\Omega$, $E_u = 0$ in $V^c$ and $\|E_u\|_{H^1(\mathbb{R}^N)} \leq C(\Omega, V)\|u\|_{H^1(\Omega)}$. Then, for any $u \in H^1_A(\Omega)$, we get

$$\|E_u\|_{H^1(\mathbb{R}^N)} = \|E_u\|_{H^1(\Omega)} \leq C(\Omega, V)\|E_u\|_{H^1(\Omega)} = C(\Omega, V)\|u\|_{H^1_A(\Omega)},$$

which concludes the proof. $\square$

We can now prove the following result:

Lemma 2.3. Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Let $u \in H^1_A(\Omega)$ and $\rho \in L^1(\mathbb{R}^N)$ with $\rho \geq 0$. Then

$$\int_{\Omega} \int_{\Omega} |u(x) - e^{i(x-y) \cdot A(\frac{y+y}{2})} u(y)|^2 \rho(x-y) \, dx \, dy \leq C \rho \|u\|_{H^1_A(\Omega)}^2,$$

where $C$ depends only on $\Omega$ and $A$.

Proof. Let $V \subset \mathbb{R}^N$ be a fixed compact set with $\Omega \Subset V$. Given $u \in H^1_A(\Omega)$, by Lemma 2.2, there exists a function $\tilde{u} \in H^1_A(\mathbb{R}^N)$ with $\tilde{u} = u$ on $\Omega$ and $\tilde{u} = 0$ on $V^c$. By Lemma 2.1 and 2.2,

$$\int_{\mathbb{R}^N} |\tilde{u}(y+h) - e^{ihA(y+\frac{h}{2})} \tilde{u}(y)|^2 \, dy \leq C |h|^2 \|\tilde{u}\|_{H^1_A(\mathbb{R}^N)}^2 \leq C |h|^2 \|u\|_{H^1_A(\Omega)}^2,$$

for some positive constant $C$ depending on $\Omega$ and $A$. Then, in light of (2.1), we get

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - e^{i(x-y) \cdot A(\frac{y+y}{2})} u(y)|^2 \rho(x-y) \, dx \, dy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho(h) \left| \tilde{u}(y+h) - e^{ihA(y+\frac{h}{2})} \tilde{u}(y) \right|^2 \, dy \, dh$$

$$\leq \int_{\mathbb{R}^N} \rho(h) \left| \tilde{u}(y+h) - e^{ihA(y+\frac{h}{2})} \tilde{u}(y) \right|^2 \, dy \, dh$$

which concludes the proof. $\square$

Lemma 2.4. Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded and let $u \in H^1_{0,A}(\Omega)$. Then, we have

$$(1-S) \int_{\mathbb{R}^{2N}} |u(x) - e^{i(x-y) \cdot A(\frac{y+y}{2})} u(y)|^2 \, dx \, dy \leq C \|u\|_{H^1_A(\Omega)}^2,$$

where $C$ depends only on $\Omega$ and $A$.

Proof. Given $u \in C^\infty_0(\Omega)$, by Lemma 2.1 we have

$$\int_{\mathbb{R}^N} |u(y+h) - e^{ihA(y+\frac{h}{2})} u(y)|^2 \, dy \leq C |h|^2 \|u\|_{H^1_A(\Omega)}^2,$$
for some $C > 0$ depending on $\Omega$ and $A$ and all $h \in \mathbb{R}^N$ with $|h| \leq 1$. Then, we get

$$(1 - s) \int_{\mathbb{R}^{2n}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \leq (1 - s) \int_{\mathbb{R}^{2n}} \frac{|u(y + h) - e^{iA(y + \frac{h}{2})} u(y)|^2}{|h|^{N+2s}} \, dy \, dh$$

$$= (1 - s) \int_{|h| \leq 1} \frac{1}{|h|^{N+2s}} \left( \int_{\mathbb{R}^N} |u(y + h) - e^{iA(y + \frac{h}{2})} u(y)|^2 \, dy \right) \, dh$$

$$+ 4(1 - s) \int_{|h| \geq 1} \frac{1}{|h|^{N+2s}} \, dh \|u\|^2_{L^2(\Omega)}$$

$$\leq (1 - s) \int_{|h| \leq 1} \frac{1}{|h|^{N+2s}} \, dh \|u\|^2_{H^s_\lambda(\Omega)} + C \|u\|^2_{L^2} \leq C \|u\|^2_{H^s_\lambda(\Omega)};$$

The assertion then follows by a density argument. $\square$

If $A|_{\Omega}$ is smooth (and extended if necessary to a locally bounded field on $\Omega^c$), we get the following result.

**Theorem 2.5.** Assume that $A \in C^2(\bar{\Omega})$. Let $u \in H^1_A(\Omega)$ and consider a sequence $\{\rho_n\}_{n \in \mathbb{N}}$ of nonnegative radial functions in $L^1(\mathbb{R}^N)$ with

$$\lim_{n \to \infty} \int_{0}^{\infty} \rho_n(r) r^{N-1} \, dr = 1,$$  \hspace{1cm} (2.2)

and such that, for every $\delta > 0$,

$$\lim_{n \to \infty} \int_{\delta}^{\infty} \rho_n(r) r^{N-1} \, dr = 0.$$  \hspace{1cm} (2.3)

Then, we have

$$\lim_{n \to \infty} \int_{\Omega} \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x - y|^2} \rho_n(x - y) \, dx \, dy = 2K_n \int_{\Omega} |\nabla u - i A(x) u|^2 \, dx$$ \hspace{1cm} (2.4)

being $K_n$ the constant introduced in (1.1).

**Proof.** Let us first observe that by (2.2) and (2.3) we easily obtain that, for every $\delta > 0$,

$$\lim_{n \to \infty} \int_{0}^{\delta} \rho_n(r) r^{N-1} \, dr = \lim_{n \to \infty} \int_{0}^{\delta} \rho_n(r) r^{N+1} \, dr = 0.$$ \hspace{1cm} (2.5)

In fact, taken any $0 < \tau < \delta$, we have

$$\int_{0}^{\delta} \rho_n(r) r^{N} \, dr = \int_{0}^{\tau} \rho_n(r) r^{N} \, dr + \int_{\tau}^{\delta} \rho_n(r) r^{N} \, dr \leq \tau \int_{0}^{\infty} \rho_n(r) r^{N-1} \, dr + \delta \int_{\tau}^{\infty} \rho_n(r) r^{N-1} \, dr,$$

from which formula (2.5) follows using (2.2), (2.3) and letting $\tau \searrow 0$. We follow the main lines of the proof in [3]. Setting

$$F_n^u(x, y) := \frac{u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)}{|x - y|} \rho_n^{1/2}(x - y), \quad x, y \in \Omega, \quad n \in \mathbb{N},$$

by virtue of Lemma 2.3, for all $u, v \in H^1_A(\Omega)$, recalling (2.2) we have

$$\|F_n^u\|_{L^2(\Omega \times \Omega)} - \|F_n^v\|_{L^2(\Omega \times \Omega)} \leq \|F_n^u - F_n^v\|_{L^2(\Omega \times \Omega)} \leq C \|u - v\|_{H^1_A(\Omega)},$$

for some $C > 0$ depending on $\Omega$ and $A$. This allows to reduce the proof of (2.4) to $u \in C^2(\bar{\Omega})$. If we set
\[
\varphi(y) := e^{i(x-y) \cdot A(y/2)} u(y),
\]

since
\[
\nabla_y \varphi(y) = e^{i(x-y) \cdot A(y/2)} \left( \nabla_y u(y) - iA(y/2) u(y) + \frac{i}{2} u(y)(x-y) \cdot \nabla_y A(y/2) \right).
\]

if \( x \in \Omega \), a second-order Taylor expansion gives (since \( u, A \in C^2 \), then \( \nabla^2 \varphi \) is bounded on \( \bar{\Omega} \))
\[
u(x) - e^{i(x-y) \cdot A(y/2)} u(y) = \varphi(x) - \varphi(y) = (\nabla u(x) - iA(x) u(x)) \cdot (x - y) + O(|x - y|^2).
\]

Hence, for any fixed \( x \in \Omega \),
\[
\frac{|u(x) - e^{i(x-y) \cdot A(y/2)} u(y)|}{|x - y|} = \left| (\nabla u(x) - iA(x) u(x)) \cdot \frac{x - y}{|x - y|} \right| + O(|x - y|).
\]

(2.6)

Fix \( x \in \Omega \). If we set \( R_x := \text{dist}(x, \partial \Omega) \), integrating with respect to \( y \), we have
\[
\int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A(y/2)} u(y)|^2}{|x - y|^2} \rho_n(x - y) \, dy = \int_{B(x, R_x)} \frac{|u(x) - e^{i(x-y) \cdot A(y/2)} u(y)|^2}{|x - y|^2} \rho_n(x - y) \, dy
\]
\[
+ \int_{\Omega \setminus B(x, R_x)} \frac{|u(x) - e^{i(x-y) \cdot A(y/2)} u(y)|^2}{|x - y|^2} \rho_n(x - y) \, dy.
\]

(2.7)

The second integral goes to zero by conditions (2.3), since
\[
\lim_{n \to \infty} \int_{\Omega \setminus B(x, R_x)} \frac{|u(x) - e^{i(x-y) \cdot A(y/2)} u(y)|^2}{|x - y|^2} \rho_n(x - y) \, dy \leq C \lim_{n \to \infty} \int_{B^c(0, R_x)} \rho_n(z) \, dz = 0.
\]

Now, in light of (2.6), following [3], we compute
\[
\int_{B(x, R_x)} \frac{|u(x) - e^{i(x-y) \cdot A(y/2)} u(y)|^2}{|x - y|^2} \rho_n(x - y) \, dy = Q_N |\nabla u(x) - iA(x) u(x)|^2 \int_0^{R_x} r^{N-1} \rho_n(r) \, dr
\]
\[
+ O \left( \int_0^{R_x} r^N \rho_n(r) \, dr \right) + O \left( \int_0^{R_x} r^{N+1} \rho_n(r) \, dr \right),
\]

where we have set
\[
Q_N = \int_{S^{N-1}} |\omega \cdot e|^2 d\mathcal{H}^{N-1}(\omega),
\]

being \( e \in \mathbb{R}^N \) a unit vector. Letting \( n \to \infty \) in (2.7), the result follows by dominated convergence, taking into account formulas (2.5).

\[\Box\]

3. Proofs of Theorem 1.1 and 1.2

3.1. Proof of Theorem 1.1

If \( r_\Omega := \text{diam}(\Omega) \), we consider a radial cut-off \( \psi \in C_c^\infty(\mathbb{R}^N) \), \( \psi(x) = \psi_0(|x|) \) with \( \psi_0(t) = 1 \) for \( t < r_\Omega \) and \( \psi_0(t) = 0 \) for \( t > 2r_\Omega \). Then, by construction, \( \psi_0(|x - y|) = 1 \), for every \( x, y \in \Omega \). Furthermore, let \( \{s_n\}_{n \in \mathbb{N}} \subset (0, 1) \) be a sequence with \( s_n \nearrow 1 \) as \( n \to \infty \) and consider the sequence of radial functions in \( L^1(\mathbb{R}^N) \)
\[
\rho_n(|x|) = \frac{2(1 - s_n)}{|x|^{N+2s_n-2}} \psi_0(|x|), \quad x \in \mathbb{R}^N, \ n \in \mathbb{N}.
\]

(3.1)

Notice that (2.2) holds, since
\[
\lim_{n \to \infty} \int_{0}^{r_{\Omega}} \rho_{n}(r)r^{N-1}dr = \lim_{n \to \infty} 2(1-s_{n}) \int_{0}^{r_{\Omega}} \frac{1}{r^{2s_{n}-1}}dr = \lim_{n \to \infty} \frac{2}{\Omega s_{n}} = 1,
\]

and
\[
\lim_{n \to \infty} \int_{r_{\Omega}}^{2r_{\Omega}} \rho_{n}(r)r^{N-1}dr = \lim_{n \to \infty} 2(1-s_{n}) \int_{r_{\Omega}}^{2r_{\Omega}} \frac{1}{t^{2s_{n}-1}}dt = 0.
\]

In a similar fashion, for any \( \delta > 0 \), there holds
\[
\lim_{n \to \infty} \int_{\delta}^{\infty} \rho_{n}(r)r^{N-1}dr \leq \lim_{n \to \infty} 2(1-s_{n}) \int_{\delta}^{r_{\Omega}} \frac{1}{t^{2s_{n}-1}}dt = 0.
\]

Then Theorem 1.1 follows directly from Theorem 2.5 using \( \rho_{n} \) as defined in (3.1).

3.2. Proof of Theorem 1.2

In light of Theorem 1.1 and since \( u = 0 \) on \( \Omega^{c} \), we have
\[
\lim_{s \to 1} \int_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y) \cdot A(x)}(\frac{xy}{2})|^{2}}{|x-y|^{N+2s}}dxdy = K_{N} \int_{\Omega} |\nabla u - iA(x)u|^{2}dx + \lim_{s \to 1} R_{s},
\]

where
\[
R_{s} \leq 2(1-s) \int_{\Omega} \int_{\mathbb{R}^{N} \setminus \Omega} \frac{|u(x)|^{2}}{|x-y|^{N+2s}}dxdy.
\]

On the other hand, arguing as in the proof of [5, Proposition 2.8], we get \( R_{s} \to 0 \) as \( s \to 1 \) when \( u \in C^{\infty}_{c}(\Omega) \) and, on account of Lemma 2.4, for general function in \( H^{s}_{0,A}(\Omega) \) by a density argument.

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References