Mathematical analysis/Functional analysis

Distance formulas in group algebras

Formules de distance dans les groupes algébriques

Heybetkulu Mustafayev

Yuzuncu Yil University, Faculty of Sciences, Department of Mathematics, 65080, Van, Turkey

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A B S T R A C T

Let $G$ be a locally compact amenable group, $A(G)$ and $B(G)$ be the Fourier and the Fourier–Stieltjes algebra of $G$, respectively. For a given $u \in B(G)$, let $\mathcal{E}_u := \{ g \in G : |u(g)| = 1 \}$. The main result of this paper particularly states that if $\|u\|_{B(G)} \leq 1$ and $\bar{u}(\mathcal{E}_u)$ is countable (in particular, if $\mathcal{E}_u$ is compact and scattered), then

$$
\lim_{n \to \infty} \|u^nv\|_{A(G)} = \text{dist}(v, I_{\mathcal{E}_u}), \quad \forall v \in A(G),
$$

where $I_{\mathcal{E}_u} = \{ v \in A(G) : v(g) = 0, \forall g \in \mathcal{E}_u \}$.

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RÉSUMÉ

Soit $G$ un groupe compact moyennable et soient $A(G)$ et $B(G)$ l’algèbre de Fourier et l’algèbre de Fourier–Stieltjes de $G$, respectivement. Pour un $u \in B(G)$ donné, posons $\mathcal{E}_u := \{ g \in G : |u(g)| = 1 \}$. Le résultat principal de cet article établit que, si $\|u\|_{B(G)} \leq 1$ et si $\bar{u}(\mathcal{E}_u)$ est dénombrable (en particulier si $\mathcal{E}_u$ est compact et éparsé), alors

$$
\lim_{n \to \infty} \|u^nv\|_{A(G)} = \text{dist}(v, I_{\mathcal{E}_u}), \quad \forall v \in A(G),
$$

où $I_{\mathcal{E}_u} = \{ v \in A(G) : v(g) = 0, \forall g \in \mathcal{E}_u \}$.

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1. Introduction

Let $X$ be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on $X$. As usual, by $\sigma(T)$ we denote the spectrum of $T \in B(X)$. Throughout this paper, we always assume that $A$ is a complex, commutative, and semisimple Banach algebra. By $\Sigma_A$ we will denote the Gelfand space of $A$ equipped with the $w^*$-topology and by $\hat{a}$, where $\hat{a}(\gamma) = \gamma(a)$, $\gamma \in \Sigma_A$, the Gelfand transform of $a \in A$. A linear mapping $T : A \to A$ is called a multiplier of $A$ if $(Ta)b = aT(b)$ holds for all $a, b \in A$. The set $M(A)$ of all multipliers of $A$ is a commutative, unital, closed, and full subalgebra of $B(A)$. The

E-mail address: hsmustafayev@yahoo.com.

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Gelfand space of $M(A)$ may be represented as the disjoint union of $\Sigma_A$ and hull $(A)$, where $\Sigma_A$ is canonically embedded in $\Sigma_{M(A)}$ and hull $(A)$ denotes the hull of $A$ in $\Sigma_{M(A)}$.

For each $T \in M(A)$, there is a uniquely determined bounded continuous function $\hat{T}$ ($\|\hat{T}\|_\infty \leq \|T\|$) on $\Sigma_A$ such that

$$\hat{(Ta)} (\gamma) = \hat{T} (\gamma) \hat{a} (\gamma), \ \forall a \in A, \ \gamma \in \Sigma_A.$$  

In fact, $\hat{T}$ is the restriction to $\Sigma_A$ of the Gelfand transform of $T$ on $\Sigma_{M(A)}$. The function $\hat{T}$ is often called the Helgason–Wang representation of $T$ [10,12]. It follows from the preceding formula that if $\hat{T} (\gamma) = 0$ for all $\gamma \in \Sigma_A$, then $T = 0$. If $T \in M(A)$, by Gelfand theory,

$$\sigma(T) = \sigma_{M(A)}(T) = \{ \hat{T}(\phi) : \phi \in \Sigma_{M(A)} \}.$$  

Since $\Sigma_A$ is a subset of $\Sigma_{M(A)}$, we have $\overline{T(\Sigma_A)} \subseteq \sigma(T)$ for all $T \in M(A)$.

2. Distance formulas

Recall that an operator $T$ on a Banach space that satisfies

$$C_T := \sup_{n \geq 0} \| T^n \| < \infty$$

is called power bounded (if $T$ is power bounded, then by passing to an equivalent norm $T$ can be made contractive). If $T \in B(X)$ is power bounded, then

$$E_T := \{ x \in X : \text{l.i.m.}_{n \rightarrow \infty} \| T^n x \| = 0 \}$$

is a closed $T$-invariant subspace, where l.i.m. is a fixed Banach limit (it can be seen that l.i.m. $\| T^n x \| = 0$ implies $\lim_{n \rightarrow \infty} \| T^n x \| = 0$). If $x_0 \in E_T$, then from the relations

$$\| T^n x \| \leq \| T^n x - T^n x_0 \| + \| T^n x_0 \| \leq C_T \| x - x_0 \| + \| T^n x_0 \|,$$

we have

$$\text{l.i.m.}_{n \rightarrow \infty} \| T^n x \| \leq C_T \text{dist} (x, E_T). \quad (2.1)$$

We have written $D := \{ z \in \mathbb{C} : |z| < 1 \}$ and $\Gamma := \{ z \in \mathbb{C} : |z| = 1 \}$. If $T \in B(X)$ is power bounded, then clearly, $\sigma(T) \subseteq \overline{D}$. A discrete version of [14, Theorem 5.5.10] states that if $T \in B(X)$ is a contraction and the unitary spectrum $\sigma(T) \cap \Gamma$ of $T$ is countable, then

$$\lim_{n \rightarrow \infty} \| T^n x \| = \text{dist} (x, E_T), \ \forall x \in X.$$  

Now, let $A$ be a commutative semisimple Banach algebra and let $T$ be a power-bounded multiplier of $A$. Then

$$I_T := \{ a \in A : \text{l.i.m.}_{n \rightarrow \infty} \| T^n a \| = 0 \}$$

is a closed ideal in $A$. Notice that $|\hat{T}(\gamma)| \leq 1$ for all $\gamma \in \Sigma_A$. We put

$$E_T := \{ \gamma \in \Sigma_A : |\hat{T}(\gamma)| = 1 \}.$$  

Recall that a commutative Banach algebra $A$ is said to be regular if, given a closed subset $S$ of $\Sigma_A$ and $\gamma \in \Sigma_A \setminus S$, there exists an $a \in A$ such that $\hat{a}(S) = \{ 0 \}$ and $\hat{a}(\gamma) \neq 0$. Let $A$ be a regular semisimple Banach algebra and $A_{00} := \{ a \in A : \text{supp} \hat{a} \text{ is compact} \}$. For a closed subset $S$ of $\Sigma_A$, there are two distinguished closed ideals in $A$ with hull equal to $S$, namely

$$I_S := \{ a \in A : \hat{a}(\gamma) = 0, \ \forall \gamma \in S \}$$

is the largest closed ideal whose hull is $S$ and $J_S := \overline{0}$ is the smallest closed ideal whose hull is $S$, where

$$J^0_S := \{ a \in A_{00} : \text{supp} \hat{a} \cap S = \emptyset \}.$$  

The set $S$ is said to be a set of synthesis for $A$ if $I_S = J_S$ [11, Section 8.3].

**Proposition 2.1.** Let $A$ be a commutative, semisimple, and regular Banach algebra and let $T$ be a power-bounded multiplier of $A$. Then hull $(I_T) = E_T$. 

Proof. If $\gamma \in \mathcal{E}_T$ and $a \in \mathcal{I}_T$, then as

$$\|T^n a\| \geq |\hat{T}(\gamma)| \|\hat{a}(\gamma)\| = |\hat{a}(\gamma)|, \quad \forall n \in \mathbb{N},$$

we have $|\hat{a}(\gamma)| \leq \lim_{n \to \infty} \|T^n a\| = 0$. This shows that $\mathcal{E}_T \subseteq \text{hull}(\mathcal{I}_T)$. For the opposite inclusion, assume that $|\hat{T}(\gamma_0)| < 1$ for some $\gamma_0 \in \Sigma_A$. Then there is a compact neighborhood $U$ of $\gamma_0$ such that $|\hat{T}(\gamma)| < 1$ for all $\gamma \in U$. Let $K$ be a compact subset of $\Sigma_A$ such that $\gamma_0 \in K \subset \Sigma_A$. Then there exists an $a \in A$ such that $\hat{a}(K) = \{1\}$ and $\hat{a}(\Sigma_A \setminus U) = \{0\}$. As $\text{supp} \hat{a} \subseteq U$, we have $|\hat{T}(\gamma)| < 1$ for all $\gamma \in \text{supp} \hat{a}$. Since $\text{supp} \hat{a}$ is compact, using the formula

$$\lim_{n \to \infty} \|T^n a\|^{1/2} = \max \{ |\hat{T}(\gamma)| : \gamma \in \text{supp} \hat{a} \}$$

[12, Proposition 4.7.8], we have $\lim_{n \to \infty} \|T^n a\| = 0$ and therefore, $a \in \mathcal{I}_T$. As $\hat{a}(\gamma_0) = 1$, we obtain that $\gamma_0 \notin \text{hull}(\mathcal{I}_T)$. □

The same result was obtained in [9, Theorem 2.6]. Our proof is shorter and different. If $T \in M(A)$, then clearly,

$$\mathcal{T}(\mathcal{E}_T) \subseteq \sigma(T) \cap \Gamma.$$ 

We now give an example of a multiplier $T \in M(A)$ such that $\Gamma \subseteq \sigma(T)$, but $\mathcal{T}(\mathcal{E}_T)$ is finite.

Let $G$ be a locally compact Abelian group with dual group $\hat{G}$. As usual, by $L^1(G)$ and $M(G)$ respectively, we denote the group algebra and the convolution measure algebra of $G$. For every $\mu \in M(G)$, the convolution operator $T_\mu : L^1(G) \to L^1(G)$, defined by $T_\mu f = \mu * f$, $f \in L^1(G)$, is a multiplier of $L^1(G)$. By Wendel–Helson’s theorem [10, Theorem 0.1.1], every multiplier of $L^1(G)$ is obtained in this way and the map $\mu \mapsto T_\mu$ is an isometric isomorphism. In other words, $M(L^1(G)) = M(G)$. By $\hat{f}$ and $\hat{\mu}$ respectively, we will denote the Fourier and the Fourier–Stieltjes transform of $f \in L^1(G)$ and $\mu \in M(G)$. Clearly, $\hat{T_\mu}(\gamma) = \hat{\mu}(\gamma)$, $\gamma \in \hat{G}$.

For $n \in \mathbb{N}$, by $\mu^n$ we denote the $n$-th convolution power of $\mu \in M(G)$. A measure $\mu \in M(G)$ is said to be power bounded if $\sup_{n \geq 0} \|\mu^n\|_1 < \infty$. If $\mu \in M(G)$ is power bounded, then

$$\mathcal{I}_\mu := \left\{ f \in L^1(G) : \lim_{n \to \infty} \|\mu^n * f\|_1 = 0 \right\}$$

is a closed ideal in $L^1(G)$. Clearly, $\mathcal{I}_{T_\mu} = \mathcal{I}_\mu$. For a power-bounded measure $\mu \in M(G)$, we have $|\hat{\mu}(\chi)| \leq 1$ for all $\chi \in \hat{G}$. If

$$\mathcal{E}_\mu := \left\{ \chi \in \hat{G} : |\hat{\mu}(\chi)| = 1 \right\},$$

then as $\hat{T_\mu} = \hat{\mu}$, we have $\mathcal{E}_{T_\mu} = \mathcal{E}_\mu$. By Proposition 2.1 (or [9, Theorem 2.6]), $\text{hull}(\mathcal{I}_\mu) = \mathcal{E}_\mu$.

Recall that the measure $\mu \in M(G)$ has independent powers if $\mu^m \perp \mu^n$, whenever $0 \leq m < n < \infty$. Recall also that a measure $\mu \in M(G)$ is said to be Hermitian if $\mu(-\Delta) := \overline{\mu(\Delta)}$ for each Borel subset $\Delta$ of $G$. It was proved in [5, Theorem 6.8.1] that if $\mu \in M(G)$ is a Hermitian probability measure with independent powers, then $\sigma_{M(G)}(\mu) = \overline{D}$. As

$$\sigma(T_\mu) = \sigma_{M(L^1(G))}(T_\mu) = \sigma_{M(G)}(\mu) = \overline{D},$$

we have that $\Gamma \subseteq \sigma(T_\mu)$. On the other hand, since $\hat{\mu}$ is real-valued, $\hat{T_\mu}(\mathcal{E}_\mu) = \hat{\mu}(\mathcal{E}_\mu) \subseteq \{\pm 1\}$.

A locally compact Hausdorff space $\Omega$ is said to be scattered if it contains no non-empty compact perfect subset. For example, scattered subsets of the complex plane are precisely countable sets. A locally compact Abelian group is scattered if and only if it is discrete. Recall [12, Lemma 4.8.3] that $\Omega$ is scattered if and only if every continuous function on $\Omega$ vanishing at $\{\infty\}$ has countable range.

The main result of this paper is the following.

Theorem 2.2. Let $A$ be a commutative, semisimple, and regular Banach algebra and let $T$ be a contractive multiplier of $A$. Suppose that $\mathcal{E}_T$ is a set of synthesis for $A$ and $\overline{T}(\mathcal{E}_T)$ is countable. Then

$$\lim_{n \to \infty} \|T^n a\| = \text{dist}(a, I_{\mathcal{E}_T}), \quad \forall a \in A,$$

where $I_{\mathcal{E}_T} = \{ a \in A : \hat{a}(\gamma) = 0, \quad \forall \gamma \in \mathcal{E}_T \}$. In particular, if $\mathcal{E}_T$ is a singleton, say $\mathcal{E}_T = \{\gamma\}$ and $\{\gamma\}$ is a set of synthesis for $A$, then

$$\lim_{n \to \infty} \|T^n a\| = |\hat{a}(\gamma)|, \quad \forall a \in A.$$

For the proof of Theorem 2.2, we need some preliminary results. Recall that the Wiener algebra $\mathcal{A}$ is the space of all continuous functions $f$ on $\Gamma$ such that
Lemma 2.3. Let $A$ be a commutative, semisimple, and regular Banach algebra and let $T$ be a contractive $C_1$-multiplier on $A$. If $\hat{T}(\Sigma_A)$ is countable, then $T$ is a surjective isometry.

Proof. Let $K := \hat{T}(\Sigma_A)$ and let

$$I_k := \{f \in \mathcal{A} : f(K) = \{0\}\}; ~ I_k^\perp := \{f \in \mathcal{A}_+ : f(K) = \{0\}\}.$$

We know [8, Ch. XI, Section 7] that if $K$ is an arbitrary compact countable subset of $\Gamma$, then the map $\alpha : \mathcal{A} / I_k^\perp \to \mathcal{A} / I_k$, defined by

$$\alpha(f + I_k^\perp) = f + I_k, \quad f \in \mathcal{A}_+,$$

is an isometric isomorphism. In other words, for every $f \in \mathcal{A}$, there exists an $f_+ \in \mathcal{A}_+$ such that $f = f_+$ on $K$ and

$$\|f_+ + I_k^\perp\|_1 = \|f + I_k\|_1.$$

Define a mapping $\beta : \mathcal{A} / I_k \to M(A)$, by

$$\beta(f + I_k) = f_+(T), \quad f \in \mathcal{A}.$$

From the identity

$$\hat{f}(T)(\gamma) = f_+(\hat{T}(\gamma)) = f(\hat{T}(\gamma)), \quad \forall \gamma \in \Sigma_A,$$

we can see that if $f \in \mathcal{A}$ vanishes on $K$, then $\hat{f}(T)$ vanishes on $\Sigma_A$ and therefore, $f_+(T) = 0$. It follows that if $g \in \mathcal{A}_+$ is another function for which $g(\xi) = f(\xi)$ for all $\xi \in K$, then $g_+(T) = f_+(T)$. Consequently, $\beta$ is an algebra-homomorphism. Further, if $f_+^0 \in I_k^\perp$, then as $f_+^0(T) = 0$, we can write

$$\|f_+(T)\| = \|f_+(T) + f_+^0(T)\| \leq \|f_+ + f_+^0\|_1,$$

which implies

$$\|f_+(T)\| \leq \|f_+ + I_k^\perp\|_1 = \|f + I_k\|_1.$$

Hence $\beta$ is a contractive homomorphism. If $S := \beta(e^{i\pi} + I_k)$, then as $T = \beta(e^{i\pi} + I_k)$ and $I = \beta(1 + I_k)$, we have $TS = I$ and $\|S\| \leq 1$. This shows that $T$ is a surjective isometry. $\square$

Let $T$ be a contraction on a Banach space $X$ and let $T / E_T$ be the quotient operator induced by $T$ on the quotient space $X / E_T$:

$$T / E_T : x + E_T \mapsto Tx + E_T, x \in X.$$

Lemma 2.4. If $T$ is a contraction on a Banach space $X$, then

$$\lim_{n \to \infty} \| (T / E_T)^n (x + E_T) \| = \lim_{n \to \infty} \| T^n x \|, \forall x \in X.$$
**Proof.** Let $x \in X$ and 
$$d(x) := \lim_{n \to \infty} \left\| (T/E_T)^n (x + E_T) \right\| = \lim_{n \to \infty} \left\| T^nx + E_T \right\|.$$ 
Clearly, $d(x) \leq \lim_{n \to \infty} \left\| T^nx \right\|$. On the other hand, since 
$$d(x) = \inf_{n \geq 0} \left\| T^nx + E_T \right\|,$$
for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and $y \in E_T$ such that 
$$\left\| T^{n_0}x - y \right\| \leq d(x) + \varepsilon.$$ 
It follows that 
$$\left\| T^{n+n_0}x \right\| \leq \left\| T^{n+n_0}y - T^ny \right\| + \left\| T^ny \right\| \leq d(x) + \varepsilon + \left\| T^ny \right\|.$$ 
As $n \to \infty$, we have 
$$\lim_{n \to \infty} \left\| T^nx \right\| \leq d(x) + \varepsilon,$$ 
so that $\lim_{n \to \infty} \left\| T^nx \right\| \leq d(x)$. \qed

Now, we are in a position to prove **Theorem 2.2**.

**Proof of Theorem 2.2.** By Proposition 2.1 (or [9, Theorem 2.6]), hull $(I_T) = \mathcal{E}_T$. Since $\mathcal{E}_T$ is a set of synthesis for $A$, we have $I_T = I_{\mathcal{E}_T}$. Therefore, $A/I_T$ is a regular semisimple Banach algebra whose Gelfand space is $\mathcal{E}_T$. Notice also that $\overline{T/I_T}(\mathcal{E}_T) = \widehat{T}(\mathcal{E}_T)$. Since the set $\overline{T/I_T}(\mathcal{E}_T)$ is countable, by Lemma 2.3, the operator $T/I_T$ is a surjective isometry on $A/I_T$. Consequently, by Lemma 2.4 we can write 
$$\lim_{n \to \infty} \left\| T^n a \right\| = \left\| a + I_T \right\| = \text{dist}(a, I_{\mathcal{E}_T}), \ \forall a \in A.$$ 
If $\mathcal{E}_T = \{y\}$ and $\{y\}$ is a set of synthesis for $A$, then as dist $(a, I_{\mathcal{E}_T}) = |\widehat{a}(y)|$, we have 
$$\lim_{n \to \infty} \left\| T^n a \right\| = |\widehat{a}(y)|, \ \forall a \in A.$$ 
The proof is complete. \qed

Next, we present several corollaries of **Theorem 2.2**.

Let $G$ be a locally compact Abelian group and let $\mu \in M(G)$ be a power-bounded measure. The classical Foguel theorem [2] states that $\lim_{n \to \infty} \left\| \mu^n * f \right\|_1 = 0$ for all $f \in L^1(G)$ such that $\widehat{f}(0) = 0$ if and only if $\mathcal{E}_\mu = \{0\}$. In [6, Theorem 2.2], Granirer has proved that if $f \in L^1(G)$, then $\lim_{n \to \infty} \left\| \mu^n * f \right\|_1 = 0$ if and only if $\widehat{f}$ vanishes on $\mathcal{E}_\mu$.

Recall that the **coset ring** of a locally compact group $G$ (not necessarily Abelian), denoted by $\mathcal{R}(G)$, is the smallest Boolean algebra of subsets of $G$ containing left cosets of all subgroups of $G$. As in [3], define the **closed coset ring** $\mathcal{R}_c(G)$ of $G$, by 
$$\mathcal{R}_c(G) = \{E \in \mathcal{R}(G) : E \text{ is closed in } G\},$$ 
where $G_d$ is the algebraic group $G$ with the discrete topology. As proved in [13, Lemma 6.1], if $\mu \in M(G)$ is power bounded, where $G$ is Abelian, then $\mathcal{E}_\mu \subseteq \mathcal{R}_c(G)$. On the other hand, each subset of $\mathcal{R}_c(G)$ is a set of synthesis for $L^1(G)$ [4, Theorem 3.9]. Consequently, $\mathcal{E}_\mu$ is a set of synthesis for $L^1(G)$.

**Corollary 2.5.** Let $G$ be a locally compact Abelian group and let $\mu \in M(G)$ such that $\|\mu\|_1 \leq 1$. If $\widehat{\mathcal{E}_\mu}$ is countable (in particular, if $\mathcal{E}_\mu$ is compact and scattered), then 
$$\lim_{n \to \infty} \left\| \mu^n * f \right\|_1 = \text{dist}(f, I_{\mathcal{E}_\mu}), \ \forall f \in L^1(G),$$ 
where $I_{\mathcal{E}_\mu} = \{f \in L^1(G) : \widehat{f}(\chi) = 0, \ \forall \chi \in \mathcal{E}_\mu\}$. In particular, if $\mathcal{E}_\mu = \{\chi\}$, then 
$$\lim_{n \to \infty} \left\| \mu^n * f \right\|_1 = |\widehat{f}(\chi)|, \ \forall f \in L^1(G).$$ 

If $G$ is a compact Abelian group, then $L^p(G)$ ($1 \leq p < \infty$) with the convolution as multiplication and the usual norm is a commutative, semisimple, and regular Banach algebra. The Gelfand space of $L^p(G)$ is $\hat{G}$ and the Gelfand transform of $f \in L^p(G)$ is just the Fourier transform of $f$. As $\hat{G}$ is discrete, every subset of $\hat{G}$ is a set of synthesis for $L^p(G)$. Further, for every $\mu \in M(G)$, the convolution operator $T_\mu : L^p(G) \to L^p(G)$, defined by $T_\mu f = \mu * f$, $f \in L^p(G)$, is a multiplier of $L^p(G)$ and $\|T_\mu\|_p \leq \|\mu\|_1$. Moreover, we have $T_\mu = \widehat{\mu}$ and therefore, $\mathcal{E}_{T_\mu} = \mathcal{E}_\mu$. 

Corollary 2.6. Let $G$ be a compact Abelian group and let $\mu \in M(G)$ such that $\|\mu\|_1 \leq 1$. If $\overline{\mu(E)}$ is countable (in particular, if $E_\mu$ is finite), then
\[
\lim_{n \to \infty} \|\mu^n \ast f\|_p = \text{dist}(f, I_{E_\mu}), \quad \forall f \in L^p(G) \ (1 < p < \infty),
\]
where $I_{E_\mu} = \{ f \in L^p(G) : \hat{f}(\chi) = 0, \forall \chi \in E_\mu \}$. In particular, if $E_\mu = \{ \chi \}$, then
\[
\lim_{n \to \infty} \|\mu^n \ast f\|_p = |\hat{f}(\chi)|, \quad \forall f \in L^p(G).
\]

Let $G$ be a locally compact (not necessarily Abelian) group. By $A(G)$ and $B(G)$ respectively, we denote the Fourier and the Fourier–Stieltjes algebra of $G$. With pointwise multiplication $A(G)$ is a commutative, semisimple, and regular Banach algebra whose Gelfand space is $G$ (via Dirac measures) [7]. For each $u \in B(G)$, the operator $L_u : A(G) \to A(G)$, defined by $L_uv = uv, \ v \in A(G)$, is a multiplier of $A(G)$. If $G$ is amenable, then every multiplier of $A(G)$ is of this form and the map $u \mapsto L_u$ is isometric [1]. For a power-bounded element $u$ of $B(G)$, we put
\[
E_u := \{ g \in G : |u(g)| = 1 \}.
\]
As proved in [9, Theorem 4.1], $E_u \in \mathcal{R}_\mu(G)$. On the other hand, if $G$ is amenable, then every subset of $\mathcal{R}_\mu(G)$ is a set of synthesis for $A(G)$ [3, Lemma 2.2]. Therefore, in the case when $G$ is amenable, $E_u$ is a set of synthesis for $A(G)$.

Corollary 2.7. Let $G$ be a locally compact amenable group and let $u \in B(G)$ such that $\|u\|_{B(G)} \leq 1$. If $\overline{\mu(E_u)}$ is countable (in particular, if $E_u$ is compact and scattered), then
\[
\lim_{n \to \infty} \|u^n v\|_{A(G)} = \text{dist}(v, I_{E_u}), \quad \forall v \in A(G),
\]
where $I_{E_u} = \{ v \in A(G) : v(g) = 0, \forall g \in E_u \}$. In particular, if $E_u = \{ g \}$, then
\[
\lim_{n \to \infty} \|u^n v\|_{A(G)} = |v(g)|, \quad \forall v \in A(G).
\]

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References


