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Distance formulas in group algebras

Formules de distance dans les groupes algébriques

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ABSTRACT

Let G be a locally compact amenable group, $A(G)$ and $B(G)$ be the Fourier and the Fourier–Stieltjes algebra of G, respectively. For a given $u \in B(G)$, let $\mathcal{E}_u := \{g \in G : |u(g)| = 1\}$. The main result of this paper particularly states that if $\|u\|_{B(G)} \leq 1$ and $\overline{u(\mathcal{E}_u)}$ is countable (in particular, if $\mathcal{E}_u$ is compact and scattered), then

$$\lim_{n\to\infty} \|u^n v\|_{A(G)} = \operatorname{dist}(v, I_{\mathcal{E}_u}), \ \forall v \in A(G),$$

where $I_{\mathcal{E}_u} = \{v \in A(G) : v(g) = 0, \ \forall g \in \mathcal{E}_u\}$.

RÉSUMÉ

Soit G un groupe compact moyennable et soient $A(G)$ et $B(G)$ l'algèbre de Fourier et l'algèbre de Fourier–Stieltjes de G, respectivement. Pour un $u \in B(G)$ donné, posons $\mathcal{E}_u := \{g \in G : |u(g)| = 1\}$. Le résultat principal de cet article établit que, si $\|u\|_{B(G)} \leq 1$ et si $\overline{u(\mathcal{E}_u)}$ est dénombrable (en particulier si $\mathcal{E}_u$ est compacte et éparpillé), alors

$$\lim_{n\to\infty} \|u^n v\|_{A(G)} = \operatorname{dist}(v, I_{\mathcal{E}_u}), \ \forall v \in A(G),$$

où $I_{\mathcal{E}_u} = \{v \in A(G) : v(g) = 0, \ \forall g \in \mathcal{E}_u\}$.

1. Introduction

Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. As usual, by $\sigma(T)$ we denote the spectrum of $T \in B(X)$. Throughout this paper, we always assume that A is a complex, commutative, and semisimple Banach algebra. By Σ_A we will denote the Gelfand space of A equipped with the w^*-topology and by $\widehat{a}$, where $\widehat{a}(\gamma) = \gamma(a)$, $\gamma \in \Sigma_A$, the Gelfand transform of $a \in A$. A linear mapping $T : A \to A$ is called a *multiplier* of A if $(Ta)b = aT(b)$ holds for all $a, b \in A$. The set $M(A)$ of all multipliers of A is a commutative, unital, closed, and full subalgebra of $B(A)$. The

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Gelfand space of $M(A)$ may be represented as the disjoint union of Σ_A and hull (A), where Σ_A is canonically embedded in $\Sigma_{M(A)}$ and hull (A) denotes the hull of A in $\Sigma_{M(A)}$.

For each $T \in M(A)$, there is a uniquely determined bounded continuous function $\widehat{T}$ $\left(\left\|\widehat{T}\right\|_\infty \le \|T\|\right)$ on Σ_A such that

$$\widehat{(Ta)}(\gamma) = \widehat{T}(\gamma)\widehat{a}(\gamma), \ \forall a \in A, \ \gamma \in \Sigma_A.$$

In fact, $\widehat{T}$ is the restriction to Σ_A of the Gelfand transform of T on $\Sigma_{M(A)}$. The function $\widehat{T}$ is often called the *Helgason–Wang representation* of T [10,12]. It follows from the preceding formula that if $\widehat{T}(\gamma) = 0$ for all $\gamma \in \Sigma_A$, then $T = 0$. If $T \in M(A)$, by Gelfand theory,

$$\sigma(T) = \sigma_{M(A)}(T) = \left\{\widehat{T}(\phi) : \phi \in \Sigma_{M(A)}\right\}.$$

Since Σ_A is a subset of $\Sigma_{M(A)}$, we have $\overline{\widehat{T}(\Sigma_A)} \subseteq \sigma(T)$ for all $T \in M(A)$.

2. Distance formulas

Recall that an operator T on a Banach space that satisfies

$$C_T := \sup_{n \ge 0} \left\|T^n\right\| < \infty$$

is called *power bounded* (if T is power bounded, then by passing to an equivalent norm T can be made contractive). If $T \in B(X)$ is power bounded, then

$$E_T := \left\{x \in X : \text{l.i.m.}_n \left\|T^n x\right\| = 0\right\}$$

is a closed T-invariant subspace, where l.i.m. is a fixed Banach limit (it can be seen that $\text{l.i.m.}_n \left\|T^n x\right\| = 0$ implies $\lim_{n\to\infty} \left\|T^n x\right\| = 0$). If $x_0 \in E_T$, then from the relations

$$\left\|T^n x\right\| \le \left\|T^n x - T^n x_0\right\| + \left\|T^n x_0\right\| \le C_T \left\|x - x_0\right\| + \left\|T^n x_0\right\|,$$

we have

$$\text{l.i.m.}_n \left\|T^n x\right\| \le C_T \operatorname{dist}(x, E_T). \tag{2.1}$$

We have written $D := \{z \in \mathbb{C} : |z| < 1\}$ and $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$. If $T \in B(X)$ is power bounded, then clearly, $\sigma(T) \subseteq \overline{D}$. A discrete version of [14, Theorem 5.5.10] states that if $T \in B(X)$ is a contraction and the unitary spectrum $\sigma(T) \cap \Gamma$ of T is countable, then

$$\lim_{n\to\infty} \left\|T^n x\right\| = \operatorname{dist}(x, E_T), \ \forall x \in X.$$

Now, let A be a commutative semisimple Banach algebra and let T be a power-bounded multiplier of A. Then

$$\mathcal{I}_T := \left\{a \in A : \text{l.i.m.}_n \left\|T^n a\right\| = 0\right\}$$

is a closed ideal in A. Notice that $\left|\widehat{T}(\gamma)\right| \le 1$ for all $\gamma \in \Sigma_A$. We put

$$\mathcal{E}_T := \left\{\gamma \in \Sigma_A : \left|\widehat{T}(\gamma)\right| = 1\right\}.$$

Recall that a commutative Banach algebra A is said to be *regular* if, given a closed subset S of Σ_A and $\gamma \in \Sigma_A \diagdown S$, there exists an $a \in A$ such that $\widehat{a}(S) = \{0\}$ and $\widehat{a}(\gamma) \neq 0$. Let A be a regular semisimple Banach algebra and $A_{00} := \{a \in A : \operatorname{supp}\widehat{a} \text{ is compact}\}$. For a closed subset S of Σ_A, there are two distinguished closed ideals in A with hull equal to S, namely

$$I_S := \left\{a \in A : \widehat{a}(\gamma) = 0, \ \forall \gamma \in S\right\}$$

is the largest closed ideal whose hull is S and $J_S := \overline{J_S^0}$ is the smallest closed ideal whose hull is S, where

$$J_S^0 := \{a \in A_{00} : \operatorname{supp}\widehat{a} \cap S = \emptyset\}.$$

The set S is said to be a *set of synthesis* for A if $I_S = J_S$ [11, Section 8.3].

Proposition 2.1. *Let A be a commutative, semisimple, and regular Banach algebra and let T be a power-bounded multiplier of A. Then hull $(\mathcal{I}_T) = \mathcal{E}_T$.*

Proof. If $\gamma \in \mathcal{E}_T$ and $a \in \mathcal{I}_T$, then as

$$\left\|T^n a\right\| \geq \left|\widehat{T}(\gamma)\right|^n \left|\widehat{a}(\gamma)\right| = \left|\widehat{a}(\gamma)\right|, \ \forall n \in \mathbb{N},$$

we have $\left|\widehat{a}(\gamma)\right| \leq \text{l.i.m.}_n \left\|T^n a\right\| = 0$. This shows that $\mathcal{E}_T \subseteq \text{hull}(\mathcal{I}_T)$. For the opposite inclusion, assume that $\left|\widehat{T}(\gamma_0)\right| < 1$ for some $\gamma_0 \in \Sigma_A$. Then there is a compact neighborhood U of γ_0 such that $\left|\widehat{T}(\gamma)\right| < 1$ for all $\gamma \in \overline{U}$. Let K be a compact subset of Σ_A such that $\gamma_0 \in K \subset U$. Then there exists an $a \in A$ such that $\widehat{a}(K) = \{1\}$ and $\widehat{a}\left(\Sigma_A \diagdown U\right) = \{0\}$. As $\text{supp}\widehat{a} \subseteq \overline{U}$, we have $\left|\widehat{T}(\gamma)\right| < 1$ for all $\gamma \in \text{supp}\widehat{a}$. Since $\text{supp}\widehat{a}$ is compact, using the formula

$$\overline{\lim_{n\to\infty}} \left\|T^n a\right\|^{\frac{1}{n}} = \max\left\{\left|\widehat{T}(\gamma)\right| : \gamma \in \text{supp}\widehat{a}\right\}$$

[12, Proposition 4.7.8], we have $\lim_{n\to\infty} \left\|T^n a\right\| = 0$ and therefore, $a \in \mathcal{I}_T$. As $\widehat{a}(\gamma_0) = 1$, we obtain that $\gamma_0 \notin \text{hull}(\mathcal{I}_T)$. □

The same result was obtained in [9, Theorem 2.6]. Our proof is shorter and different.

If $T \in M(A)$, then clearly,

$$\overline{\widehat{T}(\mathcal{E}_T)} \subseteq \sigma(T) \cap \Gamma.$$

We now give an example of a multiplier $T \in M(A)$ such that $\Gamma \subseteq \sigma(T)$, but $\widehat{T}(\mathcal{E}_T)$ is finite.

Let G be a locally compact Abelian group with dual group $\widehat{G}$. As usual, by $L^1(G)$ and $M(G)$ respectively, we denote the group algebra and the convolution measure algebra of G. For every $\mu \in M(G)$, the convolution operator $T_\mu : L^1(G) \to L^1(G)$, defined by $T_\mu f = \mu * f$, $f \in L^1(G)$, is a multiplier of $L^1(G)$. By Wendel–Helson's theorem [10, Theorem 0.1.1], every multiplier of $L^1(G)$ is obtained in this way and the map $\mu \mapsto T_\mu$ is an isometric isomorphism. In other words, $M\left(L^1(G)\right) = M(G)$. By $\widehat{f}$ and $\widehat{\mu}$ respectively, we will denote the Fourier and the Fourier–Stieltjes transform of $f \in L^1(G)$ and $\mu \in M(G)$. Clearly, $\widehat{T_\mu}(\gamma) = \widehat{\mu}(\gamma)$, $\gamma \in \widehat{G}$.

For $n \in \mathbb{N}$, by μ^n we denote the n-th convolution power of $\mu \in M(G)$. A measure $\mu \in M(G)$ is said to be *power bounded* if $\sup_{n\geq 0} \left\|\mu^n\right\|_1 < \infty$. If $\mu \in M(G)$ is power bounded, then

$$\mathcal{I}_\mu := \left\{f \in L^1(G) : \text{l.i.m.}_n \left\|\mu^n * f\right\|_1 = 0\right\}$$

is a closed ideal in $L^1(G)$. Clearly, $\mathcal{I}_{T_\mu} = \mathcal{I}_\mu$. For a power-bounded measure $\mu \in M(G)$, we have $|\widehat{\mu}(\chi)| \leq 1$ for all $\chi \in \widehat{G}$. If

$$\mathcal{E}_\mu := \left\{\chi \in \widehat{G} : |\widehat{\mu}(\chi)| = 1\right\},$$

then as $\widehat{T_\mu} = \widehat{\mu}$, we have $\mathcal{E}_{T_\mu} = \mathcal{E}_\mu$. By Proposition 2.1 (or [9, Theorem 2.6]), $\text{hull}\left(\mathcal{I}_\mu\right) = \mathcal{E}_\mu$.

Recall that the measure $\mu \in M(G)$ has *independent powers* if $\mu^n \perp \mu^m$, whenever $0 \leq m < n < \infty$. Recall also that a measure $\mu \in M(G)$ is said to be *Hermitian* if $\mu(-\Delta) := \overline{\mu(\Delta)}$ for each Borel subset Δ of G. It was proved in [5, Theorem 6.8.1] that if $\mu \in M(G)$ is a Hermitian probability measure with independent powers, then $\sigma_{M(G)}(\mu) = \overline{D}$. As

$$\sigma\left(T_\mu\right) = \sigma_{M(L^1(G))}\left(T_\mu\right) = \sigma_{M(G)}(\mu) = \overline{D},$$

we have that $\Gamma \subseteq \sigma\left(T_\mu\right)$. On the other hand, since $\widehat{\mu}$ is real-valued, $\widehat{T_\mu}\left(\mathcal{E}_{T_\mu}\right) = \widehat{\mu}\left(\mathcal{E}_\mu\right) \subseteq \{-1, 1\}$.

A locally compact Hausdorff space Ω is said to be *scattered* if it contains no non-empty compact perfect subset. For example, scattered subsets of the complex plane are precisely countable sets. A locally compact Abelian group is scattered if and only if it is discrete. Recall [12, Lemma 4.8.3] that Ω is scattered if and only if every continuous function on Ω vanishing at $\{\infty\}$ has countable range.

The main result of this paper is the following.

Theorem 2.2. *Let A be a commutative, semisimple, and regular Banach algebra and let T be a contractive multiplier of A. Suppose that $\mathcal{E}_T$ is a set of synthesis for A and $\overline{\widehat{T}(\mathcal{E}_T)}$ is countable. Then*

$$\lim_{n\to\infty} \left\|T^n a\right\| = \text{dist}\left(a, I_{\mathcal{E}_T}\right), \ \forall a \in A,$$

where $I_{\mathcal{E}_T} = \left\{a \in A : \widehat{a}(\gamma) = 0, \ \forall \gamma \in \mathcal{E}_T\right\}$. In particular, if $\mathcal{E}_T$ is a singleton, say $\mathcal{E}_T = \left\{\gamma\right\}$ and $\left\{\gamma\right\}$ is a set of synthesis for A, then

$$\lim_{n\to\infty} \left\|T^n a\right\| = \left|\widehat{a}(\gamma)\right|, \forall a \in A.$$

For the proof of Theorem 2.2, we need some preliminary results.

Recall that the Wiener algebra $\mathcal{A}$ is the space of all continuous functions f on Γ such that

$$\|f\|_1 := \sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right| < \infty,$$

where $\widehat{f}(n)$ is the n-th Fourier coefficient of f. We denote by $\mathcal{A}_+$ the Banach algebra of all functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ analytic on D and satisfying

$$\|f\|_1 := \sum_{n=0}^{\infty} \left|\widehat{f}(n)\right| < \infty.$$

The algebra $\mathcal{A}_+$ can be considered as a subalgebra of $\mathcal{A}$. If $T \in M(A)$ is power bounded, then for arbitrary $f \in \mathcal{A}_+$, we can define $f(T) \in M(A)$, by

$$f(T) = \sum_{n=0}^{\infty} \widehat{f}(n) T^n.$$

Then, the mapping $f \mapsto f(T)$ is a homomorphism and $\|f(T)\| \leq C_T \|f\|_1$. We say that T is a C_1-*multiplier* if $\mathcal{I}_T = \{0\}$. It follows from Proposition 2.1 (or [9, Theorem 2.6]) that if T is a C_1-multiplier on a regular semisimple Banach algebra A, then $\overline{\widehat{T}(\Sigma_A)} \subseteq \sigma(T) \cap \Gamma$.

Lemma 2.3. *Let A be a commutative, semisimple, and regular Banach algebra and let T be a contractive C_1-multiplier on A. If $\overline{\widehat{T}(\Sigma_A)}$ is countable, then T is a surjective isometry.*

Proof. Let $K := \overline{\widehat{T}(\Sigma_A)}$ and let

$$I_K := \{f \in \mathcal{A} : f(K) = \{0\}\}; \; I_K^+ := \{f \in \mathcal{A}_+ : f(K) = \{0\}\}.$$

We know [8, Ch. XI, Section 7] that if K is an arbitrary compact countable subset of Γ, then the map $\alpha : \mathcal{A}_+/I_K^+ \to \mathcal{A}/I_K$, defined by

$$\alpha\left(f + I_K^+\right) = f + I_K, \; f \in \mathcal{A}_+,$$

is an isometric isomorphism. In other words, for every $f \in \mathcal{A}$, there exists an $f_+ \in \mathcal{A}_+$ such that $f = f_+$ on K and

$$\left\|f_+ + I_K^+\right\|_1 = \|f + I_K\|_1.$$

Define a mapping $\beta : \mathcal{A}/I_K \to M(A)$, by

$$\beta(f + I_K) = f_+(T), \; f \in \mathcal{A}.$$

From the identity

$$\widehat{f_+(T)}(\gamma) = f_+\left(\widehat{T}(\gamma)\right) = f\left(\widehat{T}(\gamma)\right), \; \forall \gamma \in \Sigma_A,$$

we can see that if $f \in \mathcal{A}$ vanishes on K, then $\widehat{f_+(T)}$ vanishes on Σ_A and therefore, $f_+(T) = 0$. It follows that if $g \in \mathcal{A}_+$ is another function for which $g(\xi) = f(\xi)$ for all $\xi \in K$, then $g_+(T) = f_+(T)$. Consequently, β is an algebra-homomorphism. Further, if $f_+^0 \in I_K^+$, then as $f_+^0(T) = 0$, we can write

$$\|f_+(T)\| = \left\|f_+(T) + f_+^0(T)\right\| \leq \left\|f_+ + f_+^0\right\|_1,$$

which implies

$$\|f_+(T)\| \leq \left\|f_+ + I_K^+\right\|_1 = \|f + I_K\|_1.$$

Hence β is a contractive homomorphism. If $S := \beta\left(e^{-it} + I_K\right)$, then as $T = \beta\left(e^{it} + I_K\right)$ and $I = \beta(1 + I_K)$, we have $TS = I$ and $\|S\| \leq 1$. This shows that T is a surjective isometry. $\square$

Let T be a contraction on a Banach space X and let T/E_T be the quotient operator induced by T on the quotient space X/E_T;

$$T/E_T : x + E_T \mapsto Tx + E_T, x \in X.$$

Lemma 2.4. *If T is a contraction on a Banach space X, then*

$$\lim_{n\to\infty} \left\|(T/E_T)^n (x + E_T)\right\| = \lim_{n\to\infty} \left\|T^n x\right\|, \; \forall x \in X.$$

Proof. Let $x \in X$ and

$$d(x) := \lim_{n\to\infty} \left\| (T/E_T)^n (x + E_T) \right\| = \lim_{n\to\infty} \left\| T^n x + E_T \right\|.$$

Clearly, $d(x) \le \lim_{n\to\infty} \left\| T^n x \right\|$. On the other hand, since

$$d(x) = \inf_{n\ge 0} \left\| T^n x + E_T \right\|,$$

for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and $y \in E_T$ such that

$$\left\| T^{n_0} x - y \right\| \le d(x) + \varepsilon.$$

It follows that

$$\left\| T^{n+n_0} x \right\| \le \left\| T^{n+n_0} x - T^n y \right\| + \left\| T^n y \right\| \le d(x) + \varepsilon + \left\| T^n y \right\|.$$

As $n \to \infty$, we have

$$\lim_{n\to\infty} \left\| T^n x \right\| \le d(x) + \varepsilon,$$

so that $\lim_{n\to\infty} \left\| T^n x \right\| \le d(x)$. □

Now, we are in a position to prove Theorem 2.2.

Proof of Theorem 2.2. By Proposition 2.1 (or [9, Theorem 2.6]), $\mathrm{hull}(\mathcal{I}_T) = \mathcal{E}_T$. Since $\mathcal{E}_T$ is a set of synthesis for A, we have $\mathcal{I}_T = I_{\mathcal{E}_T}$. Therefore, $A/\mathcal{I}_T$ is a regular semisimple Banach algebra whose Gelfand space is $\mathcal{E}_T$. Notice also that $\widehat{T/\mathcal{I}_T}(\mathcal{E}_T) = \widehat{T}(\mathcal{E}_T)$. Since the set $\overline{\widehat{T/\mathcal{I}_T}(\mathcal{E}_T)}$ is countable, by Lemma 2.3, the operator $T/\mathcal{I}_T$ is a surjective isometry on $A/\mathcal{I}_T$. Consequently, by Lemma 2.4 we can write

$$\lim_{n\to\infty} \left\| T^n a \right\| = \| a + \mathcal{I}_T \| = \mathrm{dist}\left(a, I_{\mathcal{E}_T}\right), \ \forall a \in A.$$

If $\mathcal{E}_T = \{\gamma\}$ and $\{\gamma\}$ is a set of synthesis for A, then as $\mathrm{dist}\left(a, I_{\mathcal{E}_T}\right) = \left|\widehat{a}(\gamma)\right|$, we have

$$\lim_{n\to\infty} \left\| T^n a \right\| = \left|\widehat{a}(\gamma)\right|, \ \forall a \in A.$$

The proof is complete. □

Next, we present several corollaries of Theorem 2.2.

Let G be a locally compact Abelian group and let $\mu \in M(G)$ be a power-bounded measure. The classical Foguel theorem [2] states that $\lim_{n\to\infty} \left\| \mu^n * f \right\|_1 = 0$ for all $f \in L^1(G)$ such that $\widehat{f}(0) = 0$ if and only if $\mathcal{E}_\mu = \{0\}$. In [6, Theorem 2], Granirer has proved that if $f \in L^1(G)$, then $\lim_{n\to\infty} \left\| \mu^n * f \right\|_1 = 0$ if and only if $\widehat{f}$ vanishes on $\mathcal{E}_\mu$.

Recall that the *coset ring* of a locally compact group G (not necessarily Abelian), denoted by $\mathcal{R}(G)$, is the smallest Boolean algebra of subsets of G containing left cosets of all subgroups of G. As in [3], define the *closed coset ring* $\mathcal{R}_c(G)$ of G, by

$$\mathcal{R}_c(G) = \{E \in \mathcal{R}(G_d) : E \text{ is closed in } G\},$$

where G_d is the algebraic group G with the discrete topology. As proved in [13, Lemma 6.1], if $\mu \in M(G)$ is power bounded, where G is Abelian, then $\mathcal{E}_\mu \in \mathcal{R}_c\left(\widehat{G}\right)$. On the other hand, each subset of $\mathcal{R}_c\left(\widehat{G}\right)$ is a set of synthesis for $L^1(G)$ [4, Theorem 3.9]. Consequently, $\mathcal{E}_\mu$ is a set of synthesis for $L^1(G)$.

Corollary 2.5. *Let G be a locally compact Abelian group and let $\mu \in M(G)$ such that $\|\mu\|_1 \le 1$. If $\overline{\widehat{\mu}\left(\mathcal{E}_\mu\right)}$ is countable (in particular, if $\mathcal{E}_\mu$ is compact and scattered), then*

$$\lim_{n\to\infty} \left\| \mu^n * f \right\|_1 = \mathrm{dist}\left(f, I_{\mathcal{E}_\mu}\right), \ \forall f \in L^1(G),$$

where $I_{\mathcal{E}_\mu} = \left\{ f \in L^1(G) : \widehat{f}(\chi) = 0, \ \forall \chi \in \mathcal{E}_\mu \right\}$. In particular, if $\mathcal{E}_\mu = \{\chi\}$, then

$$\lim_{n\to\infty} \left\| \mu^n * f \right\|_1 = \left|\widehat{f}(\chi)\right|, \ \forall f \in L^1(G).$$

If G is a compact Abelian group, then $L^p(G)$ $(1 \le p < \infty)$ with the convolution as multiplication and the usual norm is a commutative, semisimple, and regular Banach algebra. The Gelfand space of $L^p(G)$ is $\widehat{G}$ and the Gelfand transform of $f \in L^p(G)$ is just the Fourier transform of f. As $\widehat{G}$ is discrete, every subset of $\widehat{G}$ is a set of synthesis for $L^p(G)$. Further, for every $\mu \in M(G)$, the convolution operator $T_\mu : L^p(G) \to L^p(G)$, defined by $T_\mu f = \mu * f$, $f \in L^p(G)$, is a multiplier of $L^p(G)$ and $\left\| T_\mu \right\|_p \le \|\mu\|_1$. Moreover, we have $\widehat{T_\mu} = \widehat{\mu}$ and therefore, $\mathcal{E}_{T_\mu} = \mathcal{E}_\mu$.

Corollary 2.6. *Let G be a compact Abelian group and let $\mu \in M(G)$ such that $\|\mu\|_1 \leq 1$. If $\overline{\widehat{\mu}(\mathcal{E}_\mu)}$ is countable (in particular, if $\mathcal{E}_\mu$ is finite), then*

$$\lim_{n\to\infty} \left\|\mu^n * f\right\|_p = \operatorname{dist}\left(f, I_{\mathcal{E}_\mu}\right), \ \forall f \in L^p(G) \ \ (1 < p < \infty),$$

where $I_{\mathcal{E}_\mu} = \left\{f \in L^p(G) : \widehat{f}(\chi) = 0, \ \forall \chi \in \mathcal{E}_\mu\right\}$. In particular, if $\mathcal{E}_\mu = \{\chi\}$, then

$$\lim_{n\to\infty} \left\|\mu^n * f\right\|_p = \left|\widehat{f}(\chi)\right|, \ \forall f \in L^p(G).$$

Let G be a locally compact (not necessarily Abelian) group. By $A(G)$ and $B(G)$ respectively, we denote the Fourier and the Fourier–Stieltjes algebra of G. With pointwise multiplication $A(G)$ is a commutative, semisimple, and regular Banach algebra whose Gelfand space is G (via Dirac measures) [7]. For each $u \in B(G)$, the operator $L_u : A(G) \to A(G)$, defined by $L_u v = uv$, $v \in A(G)$, is a multiplier of $A(G)$. If G is amenable, then every multiplier of $A(G)$ is of this form and the map $u \mapsto L_u$ is isometric [1]. For a power-bounded element u of $B(G)$, we put

$$\mathcal{E}_u := \{g \in G : |u(g)| = 1\}.$$

As proved in [9, Theorem 4.1], $\mathcal{E}_u \in \mathcal{R}_c(G)$. On the other hand, if G is amenable, then every subset of $\mathcal{R}_c(G)$ is a set of synthesis for $A(G)$ [3, Lemma 2.2]. Therefore, in the case when G is amenable, $\mathcal{E}_u$ is a set of synthesis for $A(G)$.

Corollary 2.7. *Let G be a locally compact amenable group and let $u \in B(G)$ such that $\|u\|_{B(G)} \leq 1$. If $\overline{u(\mathcal{E}_u)}$ is countable (in particular, if $\mathcal{E}_u$ is compact and scattered), then*

$$\lim_{n\to\infty} \left\|u^n v\right\|_{A(G)} = \operatorname{dist}\left(v, I_{\mathcal{E}_u}\right), \ \forall v \in A(G),$$

where $I_{\mathcal{E}_u} = \{v \in A(G) : v(g) = 0, \ \forall g \in \mathcal{E}_u\}$. In particular, if $\mathcal{E}_u = \{g\}$, then

$$\lim_{n\to\infty} \left\|u^n v\right\|_{A(G)} = |v(g)|, \ \forall v \in A(G).$$

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References

[1] A. Derighetti, Some results on the Fourier–Stieltjes algebra of a locally compact group, Comment. Math. Helv. 45 (1970) 219–228.
[2] S.R. Foguel, On iterates of convolutions, Proc. Amer. Math. Soc. 47 (1975) 368–370.
[3] B. Forrest, E. Kaniuth, A.T. Lau, N. Spronk, Ideals with bounded approximate identities in Fourier algebras, J. Funct. Anal. 203 (2003) 286–304.
[4] J.E. Gilbert, On projections of $L^\infty(G)$ onto translation-invariant subspaces, Proc. Lond. Math. Soc. 19 (1969) 69–88.
[5] C.C. Graham, O.C. McGehee, Essays in Commutative Harmonic Analysis, Springer-Verlag, New York, Heidelberg, Berlin, 1979.
[6] E.E. Granirer, On some properties of the Banach algebras $A_p(G)$ for locally compact groups, Proc. Amer. Math. Soc. 95 (1985) 375–381.
[7] C. Herz, Harmonic synthesis for subgroups, Ann. Inst. Fourier (Grenoble) 23 (1973) 91–123.
[8] J.P. Kahane, Series de Fourier Absolument Convergentes, Mir, Moscow, 1976 (in Russian).
[9] E. Kaniuth, A. Lau, A. Ülger, Multipliers of commutative Banach algebras, power boundedness and Fourier–Stieltjes algebras, J. Lond. Math. Soc. 81 (2010) 255–275.
[10] R. Larsen, An Introduction to the Theory of Multipliers, Springer-Verlag, New York, 1971.
[11] R. Larsen, Banach Algebras, Marcel Dekker, New York, 1973.
[12] K.B. Laursen, M.M. Neumann, An Introduction to the Local Spectral Theory, Clarendon Press, Oxford, 2000.
[13] B.M. Schreiber, Measures with bounded convolution powers, Trans. Amer. Math. Soc. 151 (1970) 405–431.
[14] J. Van Neerven, The Asymptotic Behaviour of Semigroups of Linear Operators, Oper. Theory, Adv. Appl., vol. 88, Birkhäuser, Basel, Boston, Berlin, 1996.