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A maximum principle for the system $\Delta u - \nabla W(u) = 0$

Un principe du maximum pour le système $\Delta u - \nabla W(u) = 0$

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A R T I C L E   I N F O

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A B S T R A C T

A maximum principle is established for minimal solutions to the system $\Delta u - \nabla W(u) = 0$, with a potential W vanishing at the boundary of a closed convex set $C_0 \subset \mathbb{R}^m$, which is either C^2 smooth or coincides with a point $\{a\}$.

R É S U M É

Nous établissons un principe du maximum pour les solutions minimales du système $\Delta u - \nabla W(u) = 0$, dont le potentiel W s'annule à la frontière d'un ensemble fermé convexe $C_0 \subset \mathbb{R}^m$, de classe C^2 ou réduit à un point $\{a\}$.

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1. Introduction and statement of the maximum principle

We will consider the system

$$\Delta u - \nabla W(u) = 0, \; u : A \to \mathbb{R}^m, \; A \subset \mathbb{R}^n, \tag{1}$$

which is the Euler–Lagrange equation of the free energy functional $J(u; A) = \int_A \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dx$. Assuming that the potential W satisfies:

(a) $W \in C^1(\mathbb{R}^m; \mathbb{R})$, $W(a) = 0$ for some $a \in \mathbb{R}^m$, $W \geq 0$,
(b) there exists $r_0 > 0$ such that for every $\xi \in \mathbb{R}^m$ with $|\xi| = 1$: $(0, r_0) \ni r \to W(a + r\xi)$ is strictly increasing,

the following maximum principle was proved in [1].

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Theorem 1.1. *Let $A \subset \mathbb{R}^n$ be an open, connected, bounded set, with ∂A Lipschitz, and suppose that $v(\cdot) \in W^{1,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$ minimizes $J(u;A)$ subject to its boundary conditions on ∂A:*

$$J(v;A) = \min\{J(u;A) : \ u = v \text{ on } \partial A\}.$$

Then, if there holds $|v(x) - a| \le r$ on ∂A, for some $r > 0$ with $0 < 2r \le r_0$, there also holds $|v(x) - a| \le r$ on A.

The scope of this paper is to establish a generalization of Theorem 1.1, by considering potentials W that vanish at the boundary of a closed convex set $C_0 \subset \mathbb{R}^m$, which is either C^2 smooth or coincides with a point $\{a\}$. We mention that the convexity assumption is essential for similar problems involving systems of PDEs (cf. [6,12,13]). Assuming that

(i) $W \in C^1(\mathbb{R}^m;\mathbb{R})$, with $W\big|_{\partial C_0} = 0$, $W \ge 0$ on $\mathbb{R}^m \setminus C_0$,
(ii) there exists $r_0 > 0$ such that for every outer unit normal vector ξ at the point $p \in \partial C_0$ ($|\xi| = 1$, if $C_0 = \{a\}$): $(0, r_0) \ni r \to W(p + r\xi)$ is increasing, and moreover $W(p + r_0\xi) > 0$,

we can state the following maximum principle.

Theorem 1.2. *Let $A \subset \mathbb{R}^n$ be an open, connected, bounded set, with ∂A Lipschitz, and $v \in W^{1,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$. Suppose that v minimizes $J(u;A)$ subject to its boundary conditions on ∂A,*

$$J(v;A) = \min\{J(u;A) : \ u = v \text{ on } \partial A\}.$$

Then, if there holds

$$d(x) := d(v(x), C_0) \le r \text{ on } \partial A, \ \text{for some } r > 0 \text{ with } 0 < 2r \le r_0, \tag{2}$$

where d is the Euclidean distance, there also holds

$$d(x) \le r \text{ on } A. \tag{3}$$

Moreover, the attainment of equality in (3) at an interior point of A,[1]

$$d(\hat{x}) = r, \ \text{for some } \hat{x} \in A, \tag{4}$$

implies that $d(x) = r$, $\forall x \in A$, and in addition if C_0 is strictly convex, $v(x) \equiv \text{constant}$ in A.

The following extension is also true:

Theorem 1.3. *Let $A \subset \mathbb{R}^n$ be an open, connected, bounded set, with ∂A Lipschitz, and let $\partial_D A \neq \varnothing$ be a Lipschitz portion of ∂A. Assume that $v(\cdot) \in W^{1,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$ minimizes $J(u;A)$ subject to its boundary conditions on $\partial_D A$: $J(v;A) = \min\{J(u;A) : u = v \text{ on } \partial_D A\}$. Then, the condition $d(v(x), C_0) \le r$ on $\partial_D A$, $0 < 2r \le r_0$, implies the same conclusions as in Theorem 1.2 above.*

Note that the strict monotonicity assumption (b) in Theorem 1.1 has been weakened. Theorem 1.2 is a corollary of the replacement result established in Lemma 3.2 below. For $u(\cdot) \in W^{1,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$, satisfying the boundary condition (2), it is shown that if $d(u(x), C_0) > r$ on a non-negligible subset, then there exists a map $\tilde{u}$ coinciding with u on ∂A, and having less energy. To construct the competitor $\tilde{u}$, we utilize the following decomposition of u:

$$u(x) = p(x) + (u(x) - p(x)), \tag{5}$$

where $p(x) := p(u(x))$ is the projection of $u(x)$ onto C_0. Next, we define the distance function $d(x) := d(u(x), C_0) = |u(x) - p(x)|$, and consider a deformation of u of the form:

$$\tilde{u}(x) = p(x) + g(d(x))(u(x) - p(x)), \tag{6}$$

where $g : \mathbb{R} \to [0,\infty)$ is a suitable locally Lipschitz function. We point out that the above expression of $\tilde{u}$ by-passes the difficulty encountered in [1], which utilizes the normal vector: $n(x) := \frac{u(x)-a}{|u(x)-a|}$ if $u(x) \neq a$, and $n(x) := 0$ if $u(x) = a$, and the polar representations of u (respectively $\tilde{u}$): $u(x) = a + |u(x) - a| n(x)$ (resp. $\tilde{u}(x) = a + f(|u(x) - a|) n(x)$), with $f : \mathbb{R} \to [0,\infty)$, locally Lipschitz. Indeed, from (6), the computation of $|\nabla \tilde{u}|$ is elementary. In addition, the decomposition (5) allows us to deal in one shot with the point case (where $p(x) \equiv a$, cf. Theorem 1.1), as well as with the case of smooth convex sets.

[1] We are indebted to A. Savas-Halilaj for suggesting the use of the usual maximum principle at this point. The strong maximum principle part of the theorem follows from the usual maximum principle as developed in [12] and [13].

2. Comparison with the usual maximum principle

[2] The theorems above are different from the usual maximum principle in the following respects:

(a) W is not convex, hence the usual maximum principle is not valid even for the scalar case $m = 1$;
(b) the monotonicity condition (ii) on W is extremely mild, and allows applicability in situations where degeneracy is natural (cf. [3]);
(c) the usual maximum principle is a calculus fact and applies to *all* solutions. This is not true for the result above.

Consider the O.D.E.

$$u'' - W'(u) = 0, \; u : \mathbb{R} \to \mathbb{R}, \text{ with } W : \mathbb{R} \to \mathbb{R}, \; W(u) = \frac{1}{4}(u^2 - 1)^2. \tag{7}$$

Notice that (7) has periodic solutions for $r > 0$ as small as desired such that $\min_{\mathbb{R}} u = -1 + r$ and $\max_{\mathbb{R}} u = 1 - r$. By choosing $A = (0, T)$ such that $u(0) = u(T) = 1 - r$, and $C_0 = \{1\}$, we see that Theorem 1.1 does not apply to these solutions. Similarly, for the potential $H(u) = 1 + \cos(\pi u)$, and for $C_0 = [1, 3]$, the previous periodic solution u does not satisfy the maximum principle of Theorem 1.2.

The following example shows that Theorem 1.1 does not apply even to local minimizers (stable solutions to (1), defined in terms of the definiteness of sign of the second variation). Consider the scalar P.D.E.

$$\epsilon^2 \Delta u - W'(u) = 0 \text{ in } \Omega, \; \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0, \tag{8}$$

where W is as in (7) above and Ω is a dumbbell domain. It is well known (cf. [5,9,10]) that for $\epsilon > 0$ sufficiently small, (8) has a stable solution which on the left and right of the neck is as close to -1 and $+1$, respectively, as desired (by taking $\epsilon > 0$ sufficiently small). By choosing the set $A = \Omega \setminus B$, with $B \subset \Omega$ a closed ball located on the right of the neck, and $\partial_D A = \partial B$, we can secure that $|u(x) - 1| \le r$ on $\partial_D A$, and therefore we see that Theorem 1.3 does not apply.

The solutions to (1) for which Theorems 1.2 and 1.3 apply are usually called *minimal*. They have the property that they minimize the free energy with respect to their Dirichlet values on the boundary of any open bounded subset of their domain of definition. This is reminiscent of a familiar property of minimal surfaces. That it is shared by the minimal solutions is not surprising, since the functional J is linked to the perimeter functional if scaled appropriately. More precisely, it is well known that the Gamma limit under a blow-down of J is the perimeter functional (cf. [11] for the scalar case, and [2] for the vector), and that the level sets of the rescaled minimizers converge to minimal surfaces.

3. Proof of the maximum principle

In what follows $C_0 \subset \mathbb{R}^m$ is a closed convex set, which is either C^2 smooth or coincides with a point $\{a\}$. We first compute $|\nabla \tilde{u}|^2$ for the map $\tilde{u}$ defined in (6), utilizing the properties of the projection $p(x)$.

Proposition 3.1. *Let $A \subset \mathbb{R}^n$ open and bounded, with Lipschitz boundary, let $u(\cdot) \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$, and let $\tilde{u}(x) = p(x) + g(d(x))(u(x) - p(x))$, where $p(x) := p(u(x))$ is the projection of $u(x)$ onto C_0, $d(x) := d(u(x), C_0) = |u(x) - p(x)|$, and $g : \mathbb{R} \to [0, \infty)$ is a locally Lipschitz function. Then, $\tilde{u}(\cdot) \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$, and*

$$|\nabla \tilde{u}|^2 = |\nabla p|^2 + \big(f'(d)\big)^2 |\nabla d|^2 + \big(g(d)\big)^2 \big(|\nabla(u - p)|^2 - |\nabla d|^2\big) + 2g(d) \sum_{i=1}^{n} \langle p_{x_i}, u_{x_i} - p_{x_i} \rangle,$$

where we have set $f(s) := sg(s)$. In addition, if $|f'| \le 1$ and $0 \le g \le 1$, then $|\nabla \tilde{u}|^2 \le |\nabla u|^2$.

The proof of the maximum principle is based on the following cut-off lemma:

Lemma 3.2. *Let $A \subset \mathbb{R}^n$, open, bounded, connected, with ∂A Lipschitz, and let W satisfy Hypotheses (i), (ii) above. Suppose that $u(\cdot) \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$. If the following two conditions hold,*

(I) *$d(x) := d(u(x), C_0) \le r$ on ∂A, $0 < 2r \le r_0$,*
(II) *$\mathcal{L}^n(A \cap \{d(x) > r\}) > 0$ ($\mathcal{L}^n(E)$, the n-dimensional Lebesgue measure),*

then, there is $\tilde{u}(\cdot) \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$ such that

[2] We want to thank Prof. N.D. Alikakos for providing the examples of this section.

$$\tilde{u} = u \text{ on } \partial A, \tag{9a}$$

$$\tilde{d}(x) := d(\tilde{u}(x), C_0) \leq r, \text{ on } A, \tag{9b}$$

$$J(\tilde{u}; A) < J(u; A). \tag{9c}$$

Proof. Case 1. We will first establish the lemma under the additional hypothesis:

$$d(x) \leq r_0. \tag{10}$$

The argument here is easy since u stays in the monotonicity region of W about C_0. Let $f(s) = \min\{s, r\} = g(s)s$. Clearly, $|f'| \leq 1$ and $0 \leq g \leq 1$. By Proposition 3.1, $\tilde{u}(x) = p(x) + g(d(x))(u(x) - p(x))$ satisfies

$$\int_A |\nabla \tilde{u}(x)|^2 \mathrm{d}x \leq \int_A |\nabla u(x)|^2 \mathrm{d}x. \tag{11}$$

Having a closer look we see that in case of equality in (11), then

$$0 = \int_A |\nabla \tilde{u}|^2 \mathrm{d}x - \int_A |\nabla u|^2 \mathrm{d}x \leq \int_A |\nabla d|^2((f'(d))^2 - 1)\mathrm{d}x \leq -\int_{A\cap\{d>r\}} |\nabla d|^2 \mathrm{d}x,$$

from which it follows that $\nabla d = 0$ a.e. on $A \cap \{d > r\}$, and therefore $\nabla(\tilde{d} - d) = 0$ a.e. on A, where $\tilde{d}(x) = f(d(x))$ via (6). Since $\tilde{d} - d \in W^{1,2}(A)$, we have by connectedness $\tilde{d}(x) - d(x) = \text{constant}$ a.e. in A, and from $\tilde{d} - d = 0$ on ∂A in the sense of trace, we obtain $\tilde{d}(x) - d(x) = 0$ a.e. in A, in contradiction to assumption (II) in Lemma 3.2. Therefore we have strict inequality in (11).

On the other hand, since $f(d) = g(d)d \leq d \leq r_0$, we have by (ii):

$$\int_A W\big(p(x) + g(d(x))(u(x) - p(x))\big)\mathrm{d}x \leq \int_A W\big(p(x) + (u(x) - p(x))\big)\mathrm{d}x,$$

$$\int_A W(\tilde{u}(x))\mathrm{d}x \leq \int_A W(u(x))\mathrm{d}x,$$

hence Case 1 is settled. Notice that in this case the strictness in (9c) was obtained via the gradient term.

Case 2. Assume

$$\mathcal{L}^n(A \cap \{d > r_0\}) > 0. \tag{12}$$

Consider the following cut-off functions:

$$\alpha(s) := \begin{cases} 1 & \text{for } s \leq r \\ \frac{2r-s}{r} & \text{for } r \leq s \leq 2r \\ 0 & \text{for } s \geq 2r, \end{cases} \qquad f(s) = \min\{s, r\}\alpha(s) = g(s)s.$$

Again, it is clear that $|f'| \leq 1$ and $0 \leq g(s) \leq 1$, thus $\tilde{u}(x) = p(x) + g(d(x))(u(x) - p(x))$ satisfies thanks to Proposition 3.1:

$$|\nabla \tilde{u}(x)|^2 \leq |\nabla u(x)|^2. \tag{13}$$

We note in passing that $\tilde{u}$ is a reflection of u along $d(u, C_0) = r$, and thus (13) is expected. Unlike in Case 1, the strictness of the inequality in (9c) will follow from the potential term. We will need the following proposition.

Proposition 3.3. *('Continuity' of Sobolev functions, cf. [4].) Let $A \subset \mathbb{R}^n$, open, bounded and connected, with Lipschitz boundary, and assume that $f \in W^{1,2}(A; \mathbb{R})$ satisfies*

$$f \leq \hat{r} \text{ on } \partial A \text{ (in the sense of trace) and } \mathcal{L}^n(A \cap \{\hat{s} < f\}) > 0 \text{ for some } \hat{r} < \hat{s}. \tag{14}$$

Then, $\mathcal{L}^n(A \cap \{\hat{r} < f \leq \hat{s}\}) > 0$.

Proof. Let $\sigma, \tau : A \to \mathbb{R}$ be defined by

$$\sigma(x) = \min\{f(x), \hat{s}\} = \begin{cases} f(x) & \text{for } x \in E_1 := A \cap \{f \leq \hat{r}\} \\ \hat{s} & \text{for } x \in E_3 := A \cap \{\hat{s} < f\}, \end{cases} \qquad \tau(x) = \max\{\sigma(x), \hat{r}\} = \begin{cases} \hat{r} & \text{for } x \in E_1 \\ \hat{s} & \text{for } x \in E_3. \end{cases}$$

Suppose, for the sake of contradiction, that $\mathcal{L}^n(A \cap \{\hat{r} < f \le \hat{s}\}) = 0$. Therefore, τ is a step function. Thus $\nabla\tau = 0$ a.e. in A. On the other hand, σ, τ are in $W^{1,2}(A;\mathbb{R})$ (cf. [8, p. 130]). This and the connectedness of A imply that $\tau \equiv$ constant (cf. [7, p. 307]). Hence $\tau \equiv \hat{s}$ since $\mathcal{L}^n(E_3) > 0$. It follows that $\mathcal{L}^n(E_1) = 0$ and $f > \hat{s}$ a.e. in A. Thus, $f \ge \hat{s}$ on ∂A in the sense of trace, which is contradicting (14). The proof is complete. □

Conclusion: Let $\epsilon > 0$ such that $W(u) > 0$ on $r_0 \le d(u, C_0) \le r_0 + \epsilon$. We define the sets

$$E_1 := A \cap \{d \le r_0\},\ E_2 := A \cap \{r_0 < d \le r_0 + \epsilon\},\ E_3 := A \cap \{d > r_0 + \epsilon\}.$$

From (12), we obtain that in the event that $\mathcal{L}^n(E_2) = 0$, then necessarily $\mathcal{L}^n(E_3) > 0$. But $d \le r < r_0$ on ∂A, hence by Proposition 3.3:

$$\mathcal{L}^n(E_2) > 0. \tag{15}$$

Therefore, (15) holds under any circumstances. On $A \cap \{0 \le d \le 2r\}$ we have:

$$W(\tilde{u}(x)) = W\big(p(x) + g(d(x)(u(x) - p(x))\big) \le W\big(p(x) + (u(x) - p(x))\big) = W(u(x)),$$

since $g(d)d = f(d) \le d \le 2r \le r_0$. On the other hand, on $A \cap \{d > 2r\}$ we have $0 = W(\tilde{u}(x)) \le W(u(x))$, while on E_2: $0 = W(\tilde{u}(x)) < W(u(x))$. Therefore, we have $J(\tilde{u}; A) < J(u; A)$ and the proof of Lemma 3.2 is complete. □

Now the proof of Theorem 1.2 is straightforward.

Proof of Theorem 1.2. We proceed by contradiction. So suppose that (3) does not hold, hence $\mathcal{L}^n(A \cap d(v(x), C_0) > r\}) > 0$. But this contradicts the minimality of v by Lemma 3.2. Thus (3) holds. Next, suppose (4) holds and notice that $d^2 = d^2(v, C_0)$ satisfies

$$\Delta(d^2)(x) = 2d(x)\langle \nabla d(x), \nabla W(v(x))\rangle + \sum_{i=1}^{n} D^2(d^2)(v(x))(v_{x_i}(x), v_{x_i}(x)) \ge 0,$$

for every $x \in A$ such that $0 < d(v(x), C_0) \le r_0$, thanks to Hypothesis (ii) on W, to the convexity of the function $u \to d^2(u, C_0)$ in the complement of C_0, and to the fact that the minimizer v solves system (1). By the strong maximum principle, it follows that $d(v, C_0)$ is constant in A. In addition, if C_0 is strictly convex, then the function $u \to d^2(u, C_0)$ is strictly convex in the complement of C_0. This implies that

$$\Delta(d^2)(x) \ge \epsilon|\nabla v(x)|^2,\ \forall x \in A:\ 0 < d(v(x), C_0) \le r_0,$$

for some $\epsilon > 0$. Thus, $\nabla v \equiv 0$ since $d(v, C_0)$ is constant, and as a consequence v is constant in A. □

Proof of Theorem 1.3. First we note that if in Lemma 3.2, specifically in condition (I), one replaces ∂A with $\partial_D A$, then the same conclusion (9) holds, where ∂A is now replaced with $\partial_D A$. The argument is completely unaltered. Similarly, in Proposition 3.3, ∂A is replaced with $\partial_D A$, without change in the proof. The proof of Theorem 1.3 is complete. □

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References

[1] N.D. Alikakos, G. Fusco, A maximum principle for systems with variational structure and an application to standing waves, J. Eur. Math. Soc. 17 (7) (2015) 1547–1567.
[2] S. Baldo, Minimal interface criterion for phase transitions in mixtures of Cahn–Hilliard fluids, Ann. Inst. Henri Poincaré 7 (2) (1990) 67–90.
[3] J.M. Ball, E.C.M. Crooks, Local minimizers and planar interfaces in a phase-transition model with interfacial energy, Calc. Var. Partial Differ. Equ. 40 (3–4) (2011) 501–538.
[4] H. Brézis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer-Verlag, New York, 2011.
[5] R.G. Casten, C.J. Holland, Instability results for reaction-diffusion equations with Neumann boundary conditions, J. Differ. Equ. 27 (1978) 266–273.
[6] L.C. Evans, A strong maximum principle for parabolic systems in a convex set with arbitrary boundary, Proc. Amer. Math. Soc. 138 (9) (2010) 3179–3185.
[7] L.C. Evans, Partial Differential Equations, second edition, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, 2010.
[8] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton, FL, USA, 1992.
[9] R.V. Kohn, P. Sternberg, Local minimisers and singular perturbations, Proc. R. Soc. Edinb., Sect. A 111 (1–2) (1989) 69–84.
[10] H. Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations, Publ. Res. Inst. Math. Sci. 15 (2) (1979) 401–454.

[11] L. Modica, S. Mortola, Un esempio di Γ-convergenza, Boll. Unione Mat. Ital., B 14 (1) (1977) 285–299.
[12] H. Weinberger, Invariant sets for weakly coupled parabolic and elliptic systems, Rend. Mat. Ser. VI 8 (1975) 295–310.
[13] H. Weinberger, Some remarks on invariant sets for systems, in: P.W. Schaefer (Ed.), Maximum Principles and Eigenvalue Problems in Partial Differential Equations, in: Pitman Research Notes in Mathematics, vol. 175, Longman, 1988, pp. 189–207.