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Periodic solitons for the elliptic–elliptic focussing Davey–Stewartson equations

Solitons périodiques du système de Davey–Stewartson elliptique–elliptique focalisant

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ABSTRACT

We consider the elliptic–elliptic, focussing Davey–Stewartson equations, which have an explicit bright line soliton solution. The existence of a family of periodic solitons, which have the profile of the line soliton in the longitudinal spatial direction and are periodic in the transverse spatial direction, is established using dynamical systems arguments. We also show that the line soliton is linearly unstable with respect to perturbations in the transverse direction.

RÉSUMÉ

Nous considérons les équations de Davey–Stewartson focalisantes dans le cas elliptique–elliptique, lorsqu'elles possèdent une solution unidimensionnelle de type soliton. En utilisant des méthodes de la théorie des systèmes dynamiques, nous montrons l'existence d'une famille de solutions bidimensionnelles qui ont le profil d'un soliton dans la direction spatiale longitudinale et sont périodiques dans la direction spatiale transverse. Nous montrons également que le soliton unidimensionnel est linéairement instable vis-à-vis des perturbations transverses.

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1. Introduction

The Davey–Stewartson equations

$$\mathrm{i}A_t + \varepsilon A_{xx} + A_{yy} + (\gamma_1|A|^2 + \gamma_2\phi_x)A = 0, \tag{1}$$

$$\gamma_3\phi_{xx} + \phi_{yy} - \gamma_3|A|^2_x = 0, \tag{2}$$

where $\varepsilon = \pm 1$, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R} \setminus \{0\}$ with $\gamma_2 + \gamma_3 = \pm 2$, arise in the modelling of wave packets on the surface of a three-dimensional body of water; the variables $A = A(x, y, t)$ and $\phi = \phi(x, y, t)$ are the complex wave amplitude and real mean flow and the signs of the parameters depend upon the particular physical regime under consideration (see Ablowitz & Segur [2, §2.2]). In the literature the cases $\gamma_1 + \gamma_2 = 2$ and $\gamma_1 + \gamma_2 = -2$ are termed respectively *focussing* and *defocussing*, and the system is classified as hyperbolic–hyperbolic, hyperbolic–elliptic, elliptic–hyperbolic or elliptic–elliptic according to the signs of ε and γ_3. Certain special cases of the mixed-type systems are often referred to as DS-I and DS-II and are known to be completely integrable (see Ablowitz & Clarkson [1, p. 60]). Note that solutions of (1), (2) which are spatially homogeneous in the y-direction satisfy the cubic nonlinear Schrödinger equation

$$\mathrm{i}A_t + A_{xx} + (\gamma_1 + \gamma_2)|A|^2 A = 0 \tag{3}$$

(where ϕ is recovered from (2)).

Solutions of (3) which converge to an equilibrium as $x \to \pm\infty$ and are 2π-periodic in t are referred to as *line solitons*. In the defocussing case the equation admits a 'dark' line soliton which decays to a nontrivial equilibrium, while in the focussing case it has the 'bright' line soliton

$$A^\star(x, t) = \mathrm{e}^{\mathrm{i}t}\,\mathrm{sech}(x) \tag{4}$$

which satisfies $A^\star(x, t) \to 0$ as $x \to \pm\infty$. In this note we examine *periodic solitons* which decay as $x \to \pm\infty$ and are periodic in y and t, and in particular consider how they emerge from line solitons in a *dimension-breaking bifurcation*. Explicit formulae for dark periodic solitons have been obtained for the integrable versions of the equations by Watanabe & Tajiri [9] and Arai, Takeuchi & Tajiri [3]; here we establish the existence of bright periodic solitons to the elliptic–elliptic, focussing equations ($\varepsilon = 1$, $\gamma_1 + \gamma_2 = 2$, $\gamma_3 > 0$) under the additional condition $\gamma_2 > 0$ by dynamical-systems methods.

Theorem 1.1. *Suppose that $\varepsilon = 1$, $\gamma_1 + \gamma_2 = 2$ and $\gamma_2, \gamma_3 > 0$. There exist an open neighbourhood $\mathcal{N}$ of the origin in $\mathbb{R}$, a positive real number ω_0 and a family of periodic solitons $\{\mathrm{e}^{\mathrm{i}t}u_s(x, y), \phi_s(x, y)\}_{s\in\mathcal{N}}$ to (1), (2) which emerges from the bright line soliton in a dimension-breaking bifurcation. Here*

$$u_s(x, y) = \mathrm{sech}(x) + u_s'(x, y), \qquad \phi_s(x, y) = \tanh(x) + \phi_s'(x, y),$$

in which $u_s'(\cdot,\cdot)$, $\phi_s'(\cdot,\cdot)$ are real, have amplitude $O(|s|)$ and are even in both arguments and periodic in their second with frequency $\omega_0 + O(|s|^2)$.

We also present a corollary to this result which asserts that the bright line soliton is *transversely linearly unstable* and thus confirms the prediction made by Ablowitz & Segur [2, §3.2].

Theorem 1.2. *Suppose that $\varepsilon = 1$, $\gamma_1 + \gamma_2 = 2$ and $\gamma_2, \gamma_3 > 0$. For each sufficiently small positive value of λ the linearisation of (1), (2) at $A^\star(x, t) = \mathrm{e}^{\mathrm{i}t}\,\mathrm{sech}(x)$, $\phi^\star(x, y) = \tanh(x)$ has a solution of the form $\mathrm{e}^{\lambda t + \mathrm{i}t}(A(x, y), \phi(x, y))$, where $(A(x, y), \phi(x, y))$ is periodic in y and satisfies $(A(x, y), \phi(x, y)) \to (0, 0)$ as $x \to \pm\infty$.*

In the remainder of this article we suppose that $\varepsilon = 1$, $\gamma_1 + \gamma_2 = 2$ and $\gamma_2, \gamma_3 > 0$. Equations (1), (2) with these coefficients arise when modelling water waves with weak surface tension. The existence and transverse linear instability of periodic solitons for the water-wave problem in this physical regime has recently been established by Groves, Sun & Wahlén [6].

2. Spatial dynamics

The equations for solutions of (1), (2) for which $A(x, y, t) = \mathrm{e}^{\mathrm{i}t}\big(u_1(x, y, t) + \mathrm{i}u_2(x, y, t)\big)$ (and u_1, u_2 are real-valued) can be formulated as the evolutionary system

$$u_{1y} = v_1, \tag{5}$$

$$v_{1y} = u_{2t} - u_{1xx} + u_1 - (\gamma_1 u_1^2 + \gamma_1 u_2^2 + \gamma_2\phi_x)u_1, \tag{6}$$

$$u_{2y} = v_2, \tag{7}$$

$$v_{2y} = -u_{1t} - u_{2xx} + u_2 - (\gamma_1 u_1^2 + \gamma_1 u_2^2 + \gamma_2\phi_x)u_2, \tag{8}$$

$$\phi_y = \psi, \tag{9}$$

$$\psi_y = -\gamma_3\phi_{xx} + \gamma_3(u_1^2 + u_2^2)_x, \tag{10}$$

where the spatial direction y plays the role of time. To identify an appropriate functional-analytic setting for these equations, let us first specialise to stationary solutions, so that

$$u_{1y} = v_1, \tag{11}$$
$$v_{1y} = -u_{1xx} + u_1 - (\gamma_1 u_1^2 + \gamma_1 u_2^2 + \gamma_2 \phi_x) u_1, \tag{12}$$
$$u_{2y} = v_2, \tag{13}$$
$$v_{2y} = -u_{2xx} + u_2 - (\gamma_1 u_1^2 + \gamma_1 u_2^2 + \gamma_2 \phi_x) u_2, \tag{14}$$
$$\phi_y = \psi, \tag{15}$$
$$\psi_y = -\gamma_3 \phi_{xx} + \gamma_3 (u_1^2 + u_2^2)_x. \tag{16}$$

Equations (11)–(16) constitute a semilinear evolutionary system in the phase space $X = H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R})$; the domain of the linear part of the vector field defined by their right-hand side is $D = H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^2(\mathbb{R}) \times H^1(\mathbb{R})$. This evolutionary system is reversible, that is invariant under $y \mapsto -y$, $(u_1, v_1, u_2, v_2, \phi, \psi) \mapsto S(u_1, v_1, u_2, v_2, \phi, \psi)$, where the reverser $S : X \to X$ is defined by $S(u_1, v_1, u_2, v_2, \phi, \psi) = (u_1, -v_1, u_2, -v_2, \phi, -\psi)$. It is also invariant under the reflection $R : X \to X$ given by $R(u_1(x), v_1(x), u_2(x), v_2(x), \phi(x), \psi(x)) = (u_1(-x), v_1(-x), u_2(-x), v_2(-x), -\phi(-x), -\psi(-x))$, and one may seek solutions which are invariant under this symmetry by replacing X and D by respectively

$$X_{\mathrm{r}} := X \cap \operatorname{Fix} R = H^1_{\mathrm{e}}(\mathbb{R}) \times L^2_{\mathrm{e}}(\mathbb{R}) \times H^1_{\mathrm{e}}(\mathbb{R}) \times L^2_{\mathrm{e}}(\mathbb{R}) \times H^1_{\mathrm{o}}(\mathbb{R}) \times L^2_{\mathrm{o}}(\mathbb{R})$$

and

$$D_{\mathrm{r}} := D \cap \operatorname{Fix} R = H^2_{\mathrm{e}}(\mathbb{R}) \times H^1_{\mathrm{e}}(\mathbb{R}) \times H^2_{\mathrm{e}}(\mathbb{R}) \times H^1_{\mathrm{e}}(\mathbb{R}) \times H^2_{\mathrm{o}}(\mathbb{R}) \times H^1_{\mathrm{o}}(\mathbb{R}),$$

where

$$H^n_{\mathrm{e}}(\mathbb{R}) = \{w \in H^n(\mathbb{R}) : w(x) = w(-x) \text{ for all } x \in \mathbb{R}\},$$
$$H^n_{\mathrm{o}}(\mathbb{R}) = \{w \in H^n(\mathbb{R}) : w(x) = -w(-x) \text{ for all } x \in \mathbb{R}\}.$$

It is also possible to replace D_{r} by the extended function space

$$D_\star := H^2_{\mathrm{e}}(\mathbb{R}) \times H^1_{\mathrm{e}}(\mathbb{R}) \times H^2_{\mathrm{e}}(\mathbb{R}) \times H^1_{\mathrm{e}}(\mathbb{R}) \times H^2_{\star,\mathrm{o}}(\mathbb{R}) \times H^1_{\mathrm{o}}(\mathbb{R}),$$

where

$$H^2_{\star,\mathrm{o}}(\mathbb{R}) = \{w \in L^2_{\mathrm{loc}}(\mathbb{R}) : w_x \in H^1(\mathbb{R}), w(x) = -w(-x) \text{ for all } x \in \mathbb{R}\}$$

(a Banach space with norm $\|w\|_{\star,2} := \|w_x\|_1$). This feature allows one to consider solutions to (11)–(16) whose ϕ-component is not evanescent; in particular solutions corresponding to line solitons fall into this category (see below).

Each point in phase space corresponds to a function on the real line which decays as $x \to \infty$, and the dynamics of equations (11)–(16) in y describes the behaviour of their solutions in the y-direction. In particular, equilibria correspond to line solitons (the equilibrium

$$(u_1^\star(x), v_1^\star(x), u_2^\star(x), v_2^\star(x), \phi^\star(x), \psi^\star(x)) = (\operatorname{sech}(x), 0, 0, 0, \tanh(x), 0)$$

corresponds to the line soliton (4)), while periodic orbits correspond to periodic solitons (see Fig. 1). In Section 4 we construct dimension-breaking bifurcations by writing

$$(u_1, v_1, u_2, v_2, \phi, \psi) = (u_1^\star, v_1^\star, \phi^\star, u_2^\star, v_2^\star, \phi^\star, \psi^\star) + (u_1', v_1', u_2', v_2', \phi', \psi') \tag{17}$$

and seeking small-amplitude periodic solutions of the resulting evolutionary system

$$w_y = Lw + N(w) \tag{18}$$

for $w = (u_1', v_1', u_2', v_2', \phi', \psi')$, where

$$L \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \phi \\ \psi \end{pmatrix} = \begin{pmatrix} v_1 \\ -u_{1xx} + u_1 - (3\gamma_1 + \gamma_2)\operatorname{sech}^2(x) u_1 - \gamma_2 \operatorname{sech}(x)\phi_x \\ v_2 \\ -u_{2xx} + u_2 - 2\operatorname{sech}^2(x) u_2 \\ \psi \\ -\gamma_3 \phi_{xx} + 2\gamma_3 (\operatorname{sech}(x) u_1)_x \end{pmatrix},$$

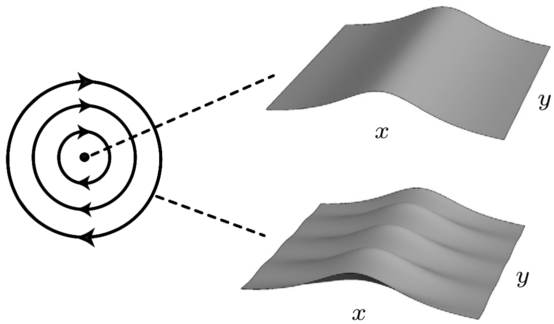

Fig. 1. A family of periodic solutions surrounding a nontrivial equilibrium solution to (11)–(16) in its phase space (left) corresponds to a dimension-breaking bifurcation of a branch of periodic solitons from a line soliton (right, plot of $u(x, y)$).

$$N\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ -3\gamma_1 \operatorname{sech}(x)u_1^2 - \gamma_1 \operatorname{sech}(x)^2 u_2^2 - \gamma_2 u_1 \phi_x - \gamma_1 u_1^3 - \gamma_1 u_1 u_2^2 \\ 0 \\ -2\gamma_1 \operatorname{sech}(x) u_1 u_2 - \gamma_2 u_2 \phi_x - \gamma_1 u_2^3 - \gamma_1 u_1^2 u_2 \\ 0 \\ \gamma_3 (u_1^2 + u_2^2)_x \end{pmatrix}$$

and we have dropped the primes for notational simplicity. Note that (18) has the invariant subspace $\tilde{X} = \{(u_2, v_2) = (0, 0)\}$, and we define $\tilde{X}_\mathrm{r} = X_\mathrm{r} \cap \tilde{X}$, $\tilde{D}_\mathrm{r} = D_\mathrm{r} \cap \tilde{X}$ and $\tilde{D}_\star = D_\star \cap \tilde{X}$.

Returning to (5)–(10), observe that these equations constitute a reversible evolutionary equation with phase space $H^1((-t_0, t_0), X)$; the domain of its vector field is $H^2((-t_0, t_0), X) \cap H^1((-t_0, t_0), D)$ and its reverser is given by the pointwise extension of $S : X \to X$ to $H^1((-t_0, t_0), X)$. Seeking solutions of the form (17), we find that

$$w_y = T w_t + Lw + N(w),$$

where $T(u_1, v_1, u_2, v_2, \phi, \psi) = (0, u_2, 0, -u_1, 0, 0)$ and we have again dropped the primes. In Section 5 we demonstrate that the solution $(u_1^\star, v_1^\star, u_2^\star, v_2^\star, \phi^\star, \psi^\star)$ of (5)–(10) is transversely linearly unstable by constructing a solution of the linear equation

$$w_y = T w_t + Lw \tag{19}$$

of the form $\mathrm{e}^{\lambda t} u_\lambda(y)$, where $u_\lambda \in C_\mathrm{b}^1(\mathbb{R}, X) \cap C_\mathrm{b}(\mathbb{R}, D)$ is periodic, for each sufficiently small positive value of λ.

3. Spectral theory

In this section we determine the purely imaginary spectrum of the linear operator $L : D \subseteq X \to X$. To this end we study the resolvent equations

$$(L - \mathrm{i}kI)w = w^\dagger \tag{20}$$

for L, where $w = (u_1, v_1, u_2, v_2, \phi, \psi)$, $w^\dagger = (u_1^\dagger, v_1^\dagger, u_2^\dagger, v_2^\dagger, \phi^\dagger, \psi^\dagger)$ and $k \in \mathbb{R} \setminus \{0\}$; since L is real and anticommutes with the reverser S it suffices to examine non-negative values of k, real values of u_1, u_2, ϕ, $v_1^\dagger$, $v_2^\dagger$, $\psi^\dagger$ and purely imaginary values of $u_1^\dagger$, $u_2^\dagger$, $\phi^\dagger$, v_1, v_2, ψ. Observe that (20) is equivalent to the decoupled equations

$$(\mathcal{A}_1 + k^2 I)\begin{pmatrix} u_1 \\ \phi \end{pmatrix} = \begin{pmatrix} v_1^\dagger + \mathrm{i}ku_1^\dagger \\ \psi^\dagger + \mathrm{i}k\phi^\dagger \end{pmatrix}, \qquad (\mathcal{A}_2 + k^2 I)u_2 = v_2^\dagger + \mathrm{i}ku_2^\dagger,$$

where $\mathcal{A}_1 : H^2(\mathbb{R}) \times H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and $\mathcal{A}_2 : H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are defined by

$$\mathcal{A}_1 \begin{pmatrix} u_1 \\ \phi \end{pmatrix} = \begin{pmatrix} -u_{1xx} + u_1 - (3\gamma_1 + \gamma_2)\operatorname{sech}^2(x)u_1 - \gamma_2 \operatorname{sech}(x)\phi_x \\ -\gamma_3 \phi_{xx} + 2\gamma_3 (\operatorname{sech}(x)u_1)_x \end{pmatrix},$$

$$\mathcal{A}_2 u_2 = -u_{2xx} + u_2 - 2\operatorname{sech}^2(x)u_2;$$

the values of v_1, v_2 and ψ are recovered from the formulae

$$v_1 = u_1^\dagger + \mathrm{i}ku_1, \qquad v_2 = u_2^\dagger + \mathrm{i}ku_2, \qquad \psi = \phi^\dagger + \mathrm{i}k\phi.$$

It follows that $L - \mathrm{i}kI$ is (semi-)Fredholm if $\mathcal{A}_1 + k^2 I$ and $\mathcal{A}_2 + k^2 I$ are (semi-)Fredholm and the dimension of the (generalised) kernel of $L - \mathrm{i}kI$ is the sum of those of $\mathcal{A}_1 + k^2 I$ and $\mathcal{A}_2 + k^2 I$.

Lemmata 3.1 and 3.2 below record the spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$; part (i) of the following proposition (see Drazin [4, Chapter 4.11]) is used in the proof of the former while the latter follows directly from part (ii).

Proposition 3.1.

(i) *The spectrum of the self-adjoint operator* $1 - \partial_x^2 - 6\,\mathrm{sech}^2(x) : H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *consists of essential spectrum* $[1, \infty)$ *and two simple eigenvalues at* -3 *and* 0 *(with corresponding eigenvectors* $\mathrm{sech}^2(x)$ *and* $\mathrm{sech}'(x)$*).*

(ii) *The spectrum of the self-adjoint operator* $1 - \partial_x^2 - 2\,\mathrm{sech}^2(x) : H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *consists of essential spectrum* $[1, \infty)$ *and a simple eigenvalue at* 0 *(with corresponding eigenvector* $\mathrm{sech}(x)$*).*

Lemma 3.1. *The spectrum of the operator* $\mathcal{A}_1$ *consists of essential spectrum* $[0, \infty)$ *and an algebraically simple negative eigenvalue* $-\omega_0^2$ *whose eigenspace lies in* $L^2_{\mathrm{e}}(\mathbb{R}) \times L^2_{\mathrm{o}}(\mathbb{R})$.

Proof. First note that $\mathcal{A}_1$ is a compact perturbation of the constant-coefficient operator $H^2(\mathbb{R}) \times H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$ defined by

$$(u_1, \phi) \mapsto (-u_{1xx} + u_1, -\gamma_3 \phi_{xx}),$$

whose essential spectrum is clearly $[0, \infty)$; it follows that $\sigma_{\mathrm{ess}}(\mathcal{A}_1) = [0, \infty)$ (see Kato [8, Chapter IV, Theorem 5.26]). Because $\mathcal{A}_1$ is self-adjoint with respect to the inner product $\langle (u_1^1, \phi^1), (u_1^2, \phi^2) \rangle = \langle u_1^1, u_1^2 \rangle_0 + \frac{1}{2}\gamma_2\gamma_3^{-1}\langle \phi^1, \phi^2 \rangle_0$ for $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ the remainder of its spectrum consists of negative real eigenvalues with finite multiplicity.

One finds by an explicit calculation that

$$\langle \mathcal{A}_1(u_1, \phi), (u_1, \phi) \rangle = \langle u_1 - u_{1xx} - 6\,\mathrm{sech}^2(x) u_1, u_1 \rangle_0 + \frac{\gamma_2}{2} \int_{\mathbb{R}} (\phi_x - 2\,\mathrm{sech}(x) u_1)^2 \, \mathrm{d}x,$$

which quantity is positive for $(u_1, \phi) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ with

$$\langle (u_1, \phi), (\mathrm{sech}^2(x), 0) \rangle = 0$$

(see Proposition 3.1(ii)). It follows that any subspace of $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ upon which $\mathcal{A}_1$ is strictly negative definite is one-dimensional. The calculation

$$\lim_{R \to \infty} \langle \mathcal{A}_1(\mathrm{sech}(x), 2\phi_R(x)\tanh(x)), (\mathrm{sech}(x), 2\phi_R(x)\tanh(x)) \rangle = -\frac{16}{3},$$

where $\phi(R) = \chi(x/R)$ and $\chi \in C_0^\infty(\mathbb{R})$ is a smooth cut-off function equal to unity in $[-1, 1]$, shows that $\inf \sigma(\mathcal{A}_1) < 0$, so that the spectral subspace of $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ corresponding to the part of the spectrum of $\mathcal{A}_1$ in $(-\infty, -\varepsilon)$ is nontrivial and hence one-dimensional for every sufficiently small value of $\varepsilon > 0$. We conclude that $\mathcal{A}_1$ has precisely one simple negative eigenvalue $-\omega_0^2$.

Finally, the same argument shows that $\mathcal{A}_1|_{L^2_{\mathrm{e}}(\mathbb{R}) \times L^2_{\mathrm{o}}(\mathbb{R})}$ also has precisely one simple negative eigenvalue. It follows that this eigenvalue is $-\omega_0^2$, whose eigenspace therefore lies in $L^2_{\mathrm{e}}(\mathbb{R}) \times L^2_{\mathrm{o}}(\mathbb{R})$. □

Lemma 3.2. *The spectrum of the operator* $\mathcal{A}_2$ *consists of essential spectrum* $[1, \infty)$ *and an algebraically simple negative eigenvalue at* 0 *whose eigenspace lies in* $L^2_{\mathrm{e}}(\mathbb{R})$.

Corollary 3.3. *The purely imaginary number* $\mathrm{i}k$ *belongs to the resolvent set of* L *for* $k \in \mathbb{R} \setminus \{0, \pm\omega_0\}$ *and* $\pm\mathrm{i}\omega_0$ *are algebraically simple purely imaginary eigenvalues of* L *whose eigenspace lies in* $\tilde{X}_{\mathrm{r}}$.

4. Application of the Lyapunov–Iooss theorem

Our existence theory for periodic solitons is based upon an application of the following version of the Lyapunov centre theorem for reversible systems (see Iooss [7]) which allows for a violation of the classical nonresonance condition at the origin due to the presence of essential spectrum there (a feature typical of spatial dynamics formulations for problems in unbounded domains) provided that the 'Iooss condition at the origin' (hypothesis (viii)) is satisfied.

Theorem 4.1 *(Iooss–Lyapunov). Consider the differential equation*

$$w_\tau = L(w) + N(w), \tag{21}$$

in which $w(\tau)$ *belongs to a real Banach space* $\mathcal{X}$. *Suppose that* $\mathcal{Y}$, $\mathcal{Z}$ *are further real Banach spaces with the properties that*

(i) *$\mathcal{Z}$ is continuously embedded in $\mathcal{Y}$ and continuously and densely embedded in $\mathcal{X}$,*
(ii) *$L : \mathcal{Z} \subseteq \mathcal{X} \to \mathcal{X}$ is a closed linear operator,*
(iii) *there is an open neighbourhood $\mathcal{U}$ of the origin in $\mathcal{Y}$ such that $L \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and $N \in C^3_{\mathrm{b,u}}(\mathcal{U}, \mathcal{X})$ (and hence $N \in C^3_{\mathrm{b,u}}(\mathcal{U} \cap \mathcal{Z}, \mathcal{X})$) with $N(0) = 0$, $\mathrm{d}N[0] = 0$.*

Suppose further that

(iv) *equation (21) is reversible: there exists an involution $S \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Y}) \cap \mathcal{L}(\mathcal{Z})$ with $SLw = -LSw$ and $SN(w) = -N(Sw)$ for all $w \in \mathcal{U}$,*

and that the following spectral hypotheses are satisfied.

(v) *$\pm \mathrm{i}\omega_0$ are nonzero simple eigenvalues of L;*
(vi) *$\mathrm{i}n\omega_0 \in \rho(L)$ for $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$;*
(vii) *$\|(L - \mathrm{i}n\omega_0 I)^{-1}\|_{\mathcal{X}\to\mathcal{X}} = o(1)$ and $\|(L - \mathrm{i}n\omega_0 I)^{-1}\|_{\mathcal{X}\to\mathcal{Z}} = O(1)$ as $n \to \pm\infty$;*
(viii) *For each $w^\dagger \in \mathcal{U}$ the equation*

$$Lw = -N(w^\dagger)$$

has a unique solution $w \in \mathcal{Y}$ and the mapping $w^\dagger \mapsto w$ belongs to $C^3_{\mathrm{b,u}}(\mathcal{U}, \mathcal{Y})$.

Under these hypotheses there exist an open neighbourhood I of the origin in $\mathbb{R}$ and a continuously differentiable branch $\{(v(s), \omega(s))\}_{s\in I}$ of reversible, $2\pi/\omega(s)$-periodic solutions in $C^1_{\mathrm{per}}(\mathbb{R}, \mathcal{Y} \oplus \mathcal{X}) \cap C_{\mathrm{per}}(\mathbb{R}, \mathcal{Y} \oplus \mathcal{Z})$ to (21) with amplitude $O(|s|)$. Here the direct sum refers to the decomposition of a function into its mode 0 and higher-order Fourier components, the subscript 'per' indicates a $2\pi/\omega(s)$-periodic function and $\omega(s) = \omega_0 + O(|s|^2)$.

Theorem 1.1 is proved by applying the Iooss–Lyapunov theorem to (18), taking $\mathcal{X} = \tilde{X}_{\mathrm{r}}$, $\mathcal{Y} = \tilde{D}_\star$, $\mathcal{Z} = \tilde{D}_{\mathrm{r}}$ and $\mathcal{U} = \tilde{D}_\star$ (and of course $\tau = y$ and $S(u_1, v_1, \phi, \psi) = (u_1, -v_1, \phi, -\psi)$). The spectral hypotheses (v) and (vi) follow from Corollary 3.3, while (vi) and (vii) are verified in respectively Lemma 4.2 and 4.3 below.

Lemma 4.2. *The operator L satisfies the resolvent estimates $\|(L - \mathrm{i}kI)^{-1}\|_{\tilde{X}_{\mathrm{r}}\to\tilde{X}_{\mathrm{r}}} = O(|k|^{-1})$ and $\|(L - \mathrm{i}kI)^{-1}\|_{\tilde{X}_{\mathrm{r}}\to\tilde{D}_{\mathrm{r}}} = O(1)$ as $|k| \to \infty$.*

Proof. Notice that

$$L(u_1, v_1, \phi, \psi) = (\mathcal{B}_1(u_1, v_1), \mathcal{B}_2(\phi, \psi)) + \mathcal{C}(u_1, v_1, \phi, \psi),$$

where $\mathcal{B}_1(u_1, v_1) = (v_1, -u_{1xx} + u_1)$, $\mathcal{B}_2(\phi, \psi) = (\psi, -\gamma_3\phi_{xx} + \gamma_3\phi)$ and

$$\mathcal{C}(u_1, v_1, \phi, \psi) = (0, -(3\gamma_1 + \gamma_2)\,\mathrm{sech}^2(x)u_1 - \gamma_2\,\mathrm{sech}(x)\phi_x, 0, -\gamma_3\phi + 2\gamma_3(\mathrm{sech}(x)u_1)_x).$$

Writing $\tilde{X}_{\mathrm{r}} = \tilde{X}_1 \times \tilde{X}_2$, $\tilde{D}_{\mathrm{r}} = \tilde{Y}_1 \times \tilde{Y}_2$ and equipping $\tilde{X}_1$ with the usual inner product and $\tilde{X}_2$ with the inner product $\langle(\phi^1, \psi^1), (\phi^2, \psi^2)\rangle = \langle\phi^1, \phi^2\rangle_1 + \gamma_3^{-1}\langle\psi^1, \psi^2\rangle_0$, observe that $\mathcal{B}_j : \tilde{Y}_j \subset \tilde{X}_j \to \tilde{X}_j$ is self-adjoint, so that

$$\|(\mathcal{B}_j - \mathrm{i}kI)^{-1}\|_{\tilde{X}_j\to\tilde{X}_j} \le |k|^{-1}$$

for $k \neq 0$. Furthermore $\|\mathcal{B}_j(\cdot)\|_{\tilde{X}_j} = \|\cdot\|_{\tilde{Y}_j}$, so that

$$\|(\mathcal{B}_j - \mathrm{i}kI)^{-1}\|_{\tilde{X}_j\to\tilde{Y}_j} = \|\mathcal{B}_j(\mathcal{B}_j - \mathrm{i}kI)^{-1}\|_{\tilde{X}_j\to\tilde{X}_j} = \|I + \mathrm{i}kI(\mathcal{B}_j - \mathrm{i}kI)^{-1}\|_{\tilde{X}_j\to\tilde{X}_j} \le 2$$

for $k \neq 0$. It follows that $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 : \tilde{D}_{\mathrm{r}} \subseteq \tilde{X}_{\mathrm{r}} \to \tilde{X}_{\mathrm{r}}$ satisfies the estimates

$$\|(\mathcal{B} - \mathrm{i}kI)^{-1}\|_{\tilde{X}_{\mathrm{r}}\to\tilde{X}_{\mathrm{r}}} \le |k|^{-1}, \qquad \|(\mathcal{B} - \mathrm{i}kI)^{-1}\|_{\tilde{X}_{\mathrm{r}}\to\tilde{D}_{\mathrm{r}}} \le 2 \tag{22}$$

for $k \neq 0$.

Finally, note that $\mathcal{C} : \tilde{X}_{\mathrm{r}} \to \tilde{X}_{\mathrm{r}}$ is bounded, whence

$$\|\mathcal{C}(\mathcal{B} - \mathrm{i}kI)^{-1}\|_{\tilde{X}_{\mathrm{r}}\to\tilde{X}_{\mathrm{r}}} = O(|k|^{-1})$$

as $|k| \to \infty$. Consequently $I - \mathcal{C}(\mathcal{B} - \mathrm{i}kI)^{-1} : \tilde{X}_{\mathrm{r}} \to \tilde{X}_{\mathrm{r}}$ is invertible for sufficiently large values of $|k|$ with

$$\|(I - \mathcal{C}(\mathcal{B} - \mathrm{i}kI)^{-1})^{-1}\|_{\tilde{X}_{\mathrm{r}}\to\tilde{X}_{\mathrm{r}}} = O(1) \tag{23}$$

as $|k| \to \infty$, and the stated result follows from the identity

$$(L - \mathrm{i}kI)^{-1} = (\mathcal{B} - \mathrm{i}kI)^{-1}\big(I - \mathcal{C}(\mathcal{B} - \mathrm{i}kI)^{-1}\big)^{-1}$$

and the estimates (22), (23). □

Lemma 4.3. *The equation*

$$Lw = -N(w^\dagger) \tag{24}$$

has a unique solution $w \in \tilde{D}_\star$ for each $w^\dagger \in \tilde{D}_\star$ and the formula $w^\dagger \mapsto w$ defines a smooth mapping $\tilde{D}_\star \to \tilde{D}_\star$.

Proof. Equation (24) is equivalent to the equations

$$u_1 - u_{1xx} - 6\,\mathrm{sech}^2(x)u_1 = f(w^\dagger),$$

where $f(w^\dagger) = (3\gamma_1 + \gamma_2)\,\mathrm{sech}(x)(u_1^\dagger)^2 + \gamma_2 u_1^\dagger \phi_x^\dagger + \gamma_1 (u_1^\dagger)^3$, and

$$\phi_x = (u_1^\dagger)^2 + 2\,\mathrm{sech}(x)u_1, \qquad v_1 = 0, \qquad \psi = 0.$$

The result thus follows from Proposition 3.1(i) and the fact that f and $(u_1, u_1^\dagger) \mapsto (u_1^\dagger)^2 + 2\,\mathrm{sech}(x)u_1$ define smooth mappings $\tilde{D}_\star \to L^2_{\mathrm{e}}(\mathbb{R})$ and $H^1_{\mathrm{e}}(\mathbb{R}) \times H^1_{\mathrm{e}}(\mathbb{R}) \to H^1_{\mathrm{e}}(\mathbb{R})$. □

5. Transverse linear instability

Finally, we demonstrate the transverse linear instability of the line soliton using the following general result due to Godey [5].

Theorem 5.1 *(Godey). Consider the differential equation*

$$v_\tau = T v_t + L v, \tag{25}$$

in which $v(\tau, t)$ belongs to a real Banach space $\mathcal{X}$. Suppose that $\mathcal{Y}$, $\mathcal{Z}$ are further real Banach spaces with the properties that

(i) *$L : \mathcal{Z} \subseteq \mathcal{X} \to \mathcal{X}$ and $T : \mathcal{Y} \subseteq \mathcal{X} \to \mathcal{X}$ are closed linear operators with $\mathcal{Z} \subseteq \mathcal{Y}$,*
(ii) *the equation is reversible: there exists an involution $S \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Y}) \cap \mathcal{L}(\mathcal{Z})$ with $LSv = -SLv$ and $TSv = -STv$ for all $v \in \mathcal{Z}$,*
(iii) *L has a pair $\pm\mathrm{i}\omega_0$ of isolated purely imaginary eigenvalues with odd algebraic multiplicity.*

Under these hypotheses equation (25) has a solution of the form $\mathrm{e}^{\lambda t} v_\lambda(\tau)$, where $v_\lambda \in C^1(\mathbb{R}, \mathcal{X}) \cap C(\mathbb{R}, \mathcal{Z})$ is periodic, for each sufficiently small positive value of λ; its period tends to $2\pi/\omega_0$ as $\lambda \to 0$.

Theorem 1.2 is proved by applying Godey's theorem to (19), taking $\mathcal{X} = X$, $\mathcal{Y} = X$ and $\mathcal{Z} = D$ (and of course $\tau = y$ and $S(u_1, v_1, u_2, v_2, \phi, \psi) = (u_1, -v_1, u_2, -v_2, \phi, -\psi)$). The spectral hypothesis (iii) follows from Corollary 3.3.

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