Harmonic analysis

$L^p$ harmonic analysis for differential-reflection operators

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Abstract

We introduce and study differential-reflection operators $\Lambda_{A,\varepsilon}$ acting on smooth functions defined on $\mathbb{R}$. Here $A$ is a Sturm–Liouville function with additional hypotheses and $\varepsilon \in \mathbb{R}$. For special pairs $(A, \varepsilon)$, we recover Dunkl’s, Heckman’s and Cherednik’s operators (in one dimension).

As, by construction, the operators $\Lambda_{A,\varepsilon}$ are mixture of $d/dx$ and reflection operators, we prove the existence of an operator $V_{A,\varepsilon}$ so that $\Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} = V_{A,\varepsilon} \circ d/dx$. The positivity of the intertwining operator $V_{A,\varepsilon}$ is also established.

Via the eigenfunctions of $\Lambda_{A,\varepsilon}$, we introduce a generalized Fourier transform $\mathcal{F}_{A,\varepsilon}$. For $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$, we develop an $L^p$-Fourier analysis for $\mathcal{F}_{A,\varepsilon}$, and then we prove an $L^p$-Schwartz space isomorphism theorem for $\mathcal{F}_{A,\varepsilon}$.

Details of this paper will be given in other articles [3] and [4].

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Résumé

Nous introduisons et étudions des opérateurs différentiels aux différences $\Lambda_{A,\varepsilon}$ agissant sur les fonctions régulières définies sur $\mathbb{R}$. Ici $A$ est une fonction de Sturm–Liouville avec des hypothèses supplémentaires et $\varepsilon \in \mathbb{R}$. Pour des cas particuliers de paires $(A, \varepsilon)$, nous obtenons les opérateurs de Dunkl, de Heckman et de Cherednik (unidimensionnels).

Comme, par construction, les opérateurs $\Lambda_{A,\varepsilon}$ entremêlent $d/dx$ et des opérateurs de réflexion, nous prouvons qu’il existe un opérateur $V_{A,\varepsilon}$ tel que $\Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} = V_{A,\varepsilon} \circ d/dx$. La positivité de l’opérateur $V_{A,\varepsilon}$ a été établie.

À l’aide des fonctions propres de $\Lambda_{A,\varepsilon}$, nous introduisons une transformée de Fourier généralisée $\mathcal{F}_{A,\varepsilon}$. Nous développons de l’analyse de Fourier de type $L^p$ pour $\mathcal{F}_{A,\varepsilon}$ quand $-1 \leq \varepsilon \leq 1$ et $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$, et nous caractérisons l’image des $p$-espaces de Schwartz par $\mathcal{F}_{A,\varepsilon}$.


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1. A family of differential-reflection operators

It became apparent long ago that radial Fourier analysis on real-rank-one symmetric spaces is closely connected to certain classes of special functions in one variable:

- Bessel functions in connection with radial Fourier analysis on Euclidean spaces,
- Jacobi functions in connection with radial Fourier analysis on hyperbolic spaces.

We refer to [12] for a detailed exposition.

In the late 80's/early 90's Dunkl [10] found a remarkable family of commuting operators that now bear his name. In one dimension, this reads

\[
D_\alpha f(x) = f'(x) + \frac{2\alpha + 1}{x} \left( \frac{f(x) - f(-x)}{2} \right), \quad \alpha \geq -1/2. \tag{1.1}
\]

The eigenfunctions of Dunkl's operators, known as the Dunkl kernel, are the nonsymmetric version of Bessel functions.

Some years after [10], in [8] Cherednik wrote down a trigonometric variant of the Dunkl operator. In one dimension, this reads

\[
T_{\alpha,\beta} f(x) = f'(x) + \left\{ (2\alpha + 1) \coth x + (2\beta + 1) \tanh x \right\} \left( \frac{f(x) - f(-x)}{2} \right) - \varrho f(-x), \tag{1.2}
\]

where \( \alpha \geq 1 \geq -1/2, \alpha \neq -1/2, \) and \( \varrho = \alpha + \beta + 1 \). The eigenfunctions of Cherednik's operators, known as the Opdam functions [14], are the nonsymmetric version of the Jacobi functions. We mention that the trigonometric Dunkl operators were originally introduced by Hecke in [11] in a different form. In one dimension, his operator reads:

\[
S_{\alpha,\beta} f(x) = f'(x) + \left\{ (2\alpha + 1) \coth x + (2\beta + 1) \tanh x \right\} \left( \frac{f(x) - f(-x)}{2} \right).
\]

This paper gives some aspects of harmonic analysis associated with the following family of one dimensional \((A, \varepsilon)\)-operators

\[
A_{\varepsilon} f(x) = f'(x) + \frac{A'(x)}{A(x)} \left( \frac{f(x) - f(-x)}{2} \right) - \varepsilon f(-x),
\]

where \( \varepsilon \in \mathbb{R} \) and \( A : \mathbb{R} \to \mathbb{R}^+ \) satisfies the following conditions (cf. [5,6,16]):

(C1) \( A(x) = |x|^{2\alpha+1} B(x), \) where \( \alpha > -\frac{1}{2} \) and \( B \in \mathcal{C}^\infty(\mathbb{R}) \) is even, positive, and \( B(0) = 1 \).

(C2) On \( \mathbb{R}^+ \setminus \{0\} \), \( A \) is increasing, whereas \( A'/A \) is decreasing. This condition implies that the limit \( \varrho := \lim_{x \to +\infty} A'(x)/2A(x) \geq 0 \) exists.

(C3) There exists a constant \( \delta > 0 \) such that for \( x \gg 0 \),

\[
\frac{A'(x)}{A(x)} = \begin{cases} \frac{2\varrho + e^{-\delta x} D(x)}{x} & \text{if } \varrho > 0, \\
\frac{2\alpha + 1}{x} + e^{-\delta x} D(x) & \text{if } \varrho = 0,
\end{cases}
\]

with \( |D^{(k)}(x)| \leq c_k \) for all \( x \gg 0 \) and \( k \in \mathbb{N} \).

The function \( A \) and the real number \( \varepsilon \) are the deformation parameters giving back the above three operators (as special examples) when:

1. \( A(x) = A_\alpha(x) = |x|^{2\alpha+1} \) and \( \varepsilon \) arbitrary (Dunkl's operators \( D_\alpha \)),
2. \( A(x) = A_{\alpha,\beta}(x) = |\sinh x|^{2\alpha+1}(\cosh x)^{2\beta+1} \) and \( \varepsilon = 0 \) (Heckman's operators \( S_{\alpha,\beta} \)),
3. \( A(x) = A_{\alpha,\beta}(x) = |\sinh x|^{2\alpha+1}(\cosh x)^{2\beta+1} \) and \( \varepsilon = 1 \) (Cherednik's operators \( T_{\alpha,\beta} \)).

Let \( \lambda \in \mathbb{C} \) and consider the initial data problem

\[
A_{\varepsilon} f(x) = i\lambda f(x), \quad f(0) = 1, \tag{1.4}
\]

where \( f : \mathbb{R} \to \mathbb{C} \). We prove that:

Theorem 1.1.

1) For \( \lambda \in \mathbb{C} \), there exists a unique solution \( \Psi_{A,\varepsilon}(\lambda, \cdot) \) to the problem (1.4). Further, for every \( x \in \mathbb{R} \), the function \( \lambda \mapsto \Psi_{A,\varepsilon}(\lambda, x) \) is analytic on \( \mathbb{C} \).
II) Under the restriction $-1 \leq \varepsilon \leq 1$, for all $x \in \mathbb{R}$ we have:
1) for $\lambda \in \mathbb{R}$, we have $|\Psi_{A,\varepsilon}(\lambda, x)| \leq \sqrt{2}$.
2) for $\lambda \in i\mathbb{R}$, we have $\Psi_{A,\varepsilon}(\lambda, x) > 0$.
3) Assume that $\lambda \in \mathbb{C}$ and $|x| \geq x_0$ with $x_0 > 0$. Then
$$\left|\partial^N_x \Psi_{A,\varepsilon}(\lambda, x)\right| \leq c(|\lambda| + 1)^N(|x| + 1)e^{(|\text{Im}\lambda| - \varepsilon(1 - \sqrt{1 - \varepsilon^2}))|x|}.$$  
4) Assume that $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$. Then
$$\left|\partial^M_x \Psi_{A,\varepsilon}(\lambda, x)\right| \leq c|x|^M(|x| + 1)e^{(|\text{Im}\lambda| - \varepsilon(1 - \sqrt{1 - \varepsilon^2}))|x|}.$$  

**Sketch of Proof.** 1) The proof is based on the following facts:

Fact 1) Under the conditions (C1) and (C2), the Cauchy problem
$$\begin{cases}
\ddot{h}(x) + \frac{A'(x)}{A(x)}h'(x) = -(\mu^2 + \varepsilon^2)h(x), \\
h(0) = 1, \quad h'(0) = 0,
\end{cases} \quad (1.5)$$

with $\mu \in \mathbb{C}$, admits a unique solution, which we denote by $\varphi_{\mu}$ (see [6,7]).

Fact 2) Define $\mu_\varepsilon$ so that $\mu_\varepsilon^2 = \lambda^2 + (\varepsilon^2 - 1)\varepsilon^2$. For $i\lambda \neq \varepsilon Q$, the function
$$\Psi_{A,\varepsilon}(\lambda, x) := \varphi_{\mu_\varepsilon}(x) + \frac{1}{i\lambda - \varepsilon Q\mu_\varepsilon}(x), \quad (1.6)$$
satisfies the problem (1.4).

Fact 3) We may rewrite (1.6) as
$$\Psi_{A,\varepsilon}(\lambda, x) = \varphi_{\mu_\varepsilon}(x) + (i\lambda + \varepsilon Q)\frac{\text{sg}(x)}{A(x)} \int_0^{|x|} \varphi_{\mu_\varepsilon}(t)A(t) \, dt, \quad (1.7)$$

which implies that $\lambda \mapsto \Psi_{A,\varepsilon}(\lambda, x)$ is analytic, and therefore the restriction on $\lambda$ can be dropped. The uniqueness follows by standard arguments.

II.1) The proof is inspired by Opdam’s proof of Proposition 6.1 in [14]. Using the fact that $\Psi_{A,\varepsilon}$ satisfies
$$\Psi_{A,\varepsilon}(\lambda, x) = -\frac{A'(x)}{2A(x)} \left(\Psi_{A,\varepsilon}(\lambda, x) - \Psi_{A,\varepsilon}(\lambda, -x)\right) + \varepsilon Q\Psi_{A,\varepsilon}(\lambda, -x) + i\lambda \Psi_{A,\varepsilon}(\lambda, x), \quad (1.8)$$
we prove that for all $x \in \mathbb{R}^+$, the derivative $\left\{|\Psi_{A,\varepsilon}(\lambda, -x)|^2 + |\Psi_{A,\varepsilon}(\lambda, x)|^2\right\}' \leq 0$. This implies that for $x \in \mathbb{R}^+$, we have
$$|\Psi_{A,\varepsilon}(\lambda, -x)|^2 + |\Psi_{A,\varepsilon}(\lambda, x)|^2 \leq |\Psi_{A,\varepsilon}(\lambda, 0)|^2 + |\Psi_{A,\varepsilon}(\lambda, 0)|^2 = 2.$$  

II.2) Assume that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is not strictly positive. Since $\Psi_{A,\varepsilon}(\lambda, 0) = 1 > 0$, it follows that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ vanishes. Let $x_0$ be a zero of $\Psi_{A,\varepsilon}(\lambda, \cdot)$ so that $|x_0| = \inf\{|x| : \Psi_{A,\varepsilon}(\lambda, x) = 0\}$. We prove that $\Psi_{A,\varepsilon}(\lambda, \pm x_0) = 0$ and $\Psi'_{A,\varepsilon}(\lambda, \pm x_0) = 0$. Differentiating (1.8), we see that the second derivative of $\Psi_{A,\varepsilon}(\lambda, \cdot)$ vanishes at $\pm x_0$. Repeating the same argument over and over again to get $\Psi^{(k)}_{A,\varepsilon}(\lambda, \pm x_0) = 0$ for all $k \in \mathbb{N}$. Since $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is a real analytic function, we deduce that $\Psi_{A,\varepsilon}(\lambda, x) = 0$ for all $x \in \mathbb{R}$. This contradicts $\Psi_{A,\varepsilon}(\lambda, 0) = 1$.

II.3) If $N = 0$ we show that for $\lambda \in \mathbb{C}$ we have
$$|\Psi_{A,\varepsilon}(\lambda, x)| \leq \Psi_{A,\varepsilon}(0, x) e^{(|\text{Im}\lambda|)|x|}, \quad (1.9)$$
where $\Psi_{A,\varepsilon}(0, x) = 1$ for $\varepsilon = 0$, and $\Psi_{A,\varepsilon}(0, x) \leq c_\varepsilon(|x| + 1)e^{-\varepsilon(1 - \sqrt{1 - \varepsilon^2})|x|}$ for $\varepsilon \neq 0$. So assume $N \geq 1$. The identity (1.8) allows us to express the derivatives of $\Psi_{A,\varepsilon}(\lambda, \cdot)$ in terms of lower-order derivatives. On the other hand, since $A'/2A$ satisfies the condition (C3), it follows that
$$\left|\frac{A'(x)}{2A(x)}\right|^{(N)} \leq C, \quad \forall \, |x| \geq x_0 \text{ with } x_0 > 0.$$  

II.4) If $M = 0$, this is just (1.9). So assume $M \geq 1$. If $x = 0$, the statement follows from Liouville’s theorem. If $x \neq 0$, apply Cauchy’s integral formula for $\Psi_{A,\varepsilon}(\lambda, x)$ over a circle with radius proportional to $\frac{1}{|x|}$ centered at $\lambda$ in the complex plane. □
2. The existence and the positivity of an intertwining operator

Recall from the (sketch of) proof of Theorem 1.1 the function \( \varphi_{\mu} \) which is the unique solution to the Cauchy problem (1.5). By [6] we have the following Laplace type representation

\[
\varphi_{\mu}(x) = \int \frac{1}{|x|} K(|x|, \cdot) \cos(\mu \cdot \cdot) \, dy \quad x \in \mathbb{R}^*,
\]  

(2.1)

where \( K(|x|, \cdot) \) is a non-negative even continuous function supported in \([-|x|, |x|]\). Using a Delsarte type operator introduced in \([15, \text{Proposition 2.1}] \) (see also Theorem 5.1 in \([13]\)), we prove that the integral representation (2.1) can be rewritten as

\[
\varphi_{\mu_\epsilon}(x) = \int \frac{1}{|x|} K_\epsilon(|x|, y) \cos(\lambda \cdot y) \, dy \quad x \in \mathbb{R}^*,
\]  

(2.2)

where the relationship between \( \mu_\epsilon \) and \( \lambda \) is given by \( \mu_\epsilon^2 = \lambda^2 + (\epsilon^2 - 1)q^2 \). Here \( K_\epsilon(|x|, \cdot) \) is even, continuous and supported in \([-|x|, |x|]\). Now, in view of the expression (1.7) of the eigenfunction \( \Psi_{\mu_\epsilon}(\lambda, x) \), we deduce that

\[
\Psi_{\mu_\epsilon}(\lambda, x) = \int_{|y| < |x|} K_\epsilon(x, y) e^{i\lambda y} \, dy \quad x \in \mathbb{R}^*,
\]  

(2.3)

where \( K_\epsilon(x, \cdot) \) is a continuous function supported in \([-|x|, |x|]\). This integral representation of \( \Psi_{\mu_\epsilon}(\lambda, x) \) is the starting point for obtaining an intertwining operator between the operator \( \Lambda_{\mu_\epsilon} \) and the ordinary derivative \( d/dx \). More precisely, for \( f \in C^\infty(\mathbb{R}) \), we define \( V_{A,\epsilon} f \) by

\[
V_{A,\epsilon} f(x) = \begin{cases} \int_{|y| < |x|} K_\epsilon(x, y) f(y) \, dy & x \neq 0 \\ f(0) & x = 0 \end{cases}
\]  

(2.4)

where the kernel \( K_\epsilon(x, y) \) is as in (2.3).

**Theorem 2.1.**

1) The operator \( V_{A,\epsilon} \) is the unique automorphism of \( C^\infty(\mathbb{R}) \) such that

\[
\Lambda_{A,\epsilon} \circ V_{A,\epsilon} = V_{A,\epsilon} \circ \frac{d}{dx}.
\]  

(2.5)

2) For all \((x, y) \in \mathbb{R}^+ \times \mathbb{R}\), the kernel \( K_\epsilon(x, y) \) is positive.

The positivity of \( V_{A,\epsilon} \) played a fundamental role in [2] in establishing an analogue of Beurling’s theorem, and its relatives such as theorems of type Gelfand–Shilov, Morgan’s, Hardy’s, and Cowling–Price in the setting of this paper.

For \( \varepsilon = 0 \) and \( 1 \), the positivity of \( K_\epsilon(x, y) \) can be found in [17] and [18].

**Sketch of Proof of Theorem 2.1.**

1) Write \( f \) as the superposition \( f = f_\varepsilon + f_\odot \) of an even function \( f_\varepsilon \) and an odd function \( f_\odot \). We prove that \( V_{A,\epsilon} \) can be expressed as

\[
V_{A,\epsilon} f(x) = \left( \text{id} + \varepsilon \mathfrak{D} \mathfrak{M} \right) \circ A_\varepsilon f_\varepsilon(x) + \mathfrak{M} \circ A_\varepsilon f_\odot(x),
\]  

(2.6)

where

\[
\mathfrak{M} h(x) := \frac{\text{sg}(x)}{A(x)} \int_0^{|x|} h(t) A(t) \, dt
\]

and

\[
A_\varepsilon f(x) := \frac{1}{2} \int_{|y| < |x|} K_\epsilon(|x|, y) f(y) \, dy,
\]
with $K_{\varepsilon}(|x|, y)$ as in (2.2). The transform $\mathcal{M}$ is an isomorphism from $C_0^\infty(\mathbb{R})$ to $C_0^\infty(\mathbb{R})$ and its inverse is given by $\mathcal{M}^{-1} = \frac{\partial}{\partial \varepsilon} + \frac{A'(\lambda)}{A(\lambda)}$, while $A_{\varepsilon}$ is an automorphism of $C_0^\infty(\mathbb{R})$. Further, $(d^2/dx^2 + (A'(\lambda)/A(\lambda))dx) \circ A_{\varepsilon} = A_{\varepsilon} \circ (d^2/dx^2 - \varepsilon^2 \delta^2)$ and $A_{\varepsilon} \circ \mathcal{M} = \text{Id} + \varepsilon \theta \mathcal{M}$. Now, the first statement follows from (2.6). The uniqueness of $V_{A,\varepsilon}$ is due to the fact that the unique solution $V_{A,\varepsilon}$ to the problem (1.4) can be written as $\Psi_{A,\varepsilon}(\lambda, x) = V_{A,\varepsilon}(e^{i \lambda})$ (see (2.3)).

2) For a linear operator $L$ on $\mathcal{D}(\mathbb{R})$ we denote by $\mathcal{L}$ its dual operator in the sense that $\int_{\mathbb{R}} Lf(x)g(x)A(x)dx = \int_{\mathbb{R}} f(y)^*g(y)dy$.

It is more convenient to deal with the dual operator $\mathcal{L}_{A,\varepsilon}$ than with $V_{A,\varepsilon}$. For $g \in \mathcal{D}(\mathbb{R})$, we have $\mathcal{L}_{A,\varepsilon}g(y) = \int_{|y|>\varepsilon} K_{\varepsilon}(x, y)g(x)A(x)dx$. We shall prove that if $g \geq 0$ then $\mathcal{L}_{A,\varepsilon}g \geq 0$. For $s > 0$ and $u, v \in \mathbb{R}$, let $p_s(u, v) := \frac{e^{i (u-v)^2/2\sqrt{s}}}{2\sqrt{s}}$ be the Euclidean heat kernel. The key observation is that

$$\int_{\mathbb{R}} g(x)V_{A,\varepsilon}(p_s(u, \cdot))(x)A(x)dx = \int_{\mathbb{R}} \mathcal{L}_{A,\varepsilon}g(x)p_s(x, u)dx = \int_{\mathbb{R}} \mathcal{L}_{A,\varepsilon}g \ast q_s(u)(u) \rightarrow \mathcal{L}_{A,\varepsilon}g(u)$$

as $s \to 0$, where $q_s(r) := p_s(r, 0)$ and $\ast$ is the Euclidean convolution product. Thus, the positivity of $\mathcal{L}_{A,\varepsilon}g$ reduces to the positivity of $V_{A,\varepsilon}(p_s(u, \cdot))$. Now, by (2.4) and (2.3) we prove that for every $s > 0$ and $u, x \in \mathbb{R}$, we have

$$V_{A,\varepsilon}(p_s(u, \cdot))(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \Psi_{A,\varepsilon}(-\lambda, x)e^{-\varepsilon^2\lambda^2}e^{i\lambda u}d\lambda,$$

which allowed us to show that $V_{A,\varepsilon}(p_s(u, \cdot))(x) \geq 0$. □

3. $L^p$-Fourier analysis

For $f \in L^1(\mathbb{R}, A(x)dx)$ put

$$\mathcal{F}_{A,\varepsilon}f(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{A,\varepsilon}(\lambda, -x)A(x)dx,$$

which is well defined by Theorem 1.1.II.1.

For $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}}$, set $\vartheta_p,\varepsilon := \frac{2}{p - 1} - \sqrt{1 - \varepsilon^2}$. Observe that $1 \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}} < 2$. We introduce the tube domain

$$\mathcal{C}_{p,\varepsilon} := \{\lambda \in \mathbb{C} | |\text{Im}\lambda| \leq \vartheta_p,\varepsilon\}.$$

**Theorem 3.1.** Let $f \in L^p(\mathbb{R}, A(x)dx)$ with $1 \leq p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}}$. Then the following properties hold.

1) For $p > 1$, the Fourier transform $\mathcal{F}_{A,\varepsilon}(f)(\lambda)$ is well defined for all $\lambda$ in $\mathcal{C}_{p,\varepsilon}$, the interior of $\mathcal{C}_{p,\varepsilon}$. Moreover, for all $\lambda \in \mathcal{C}_{p,\varepsilon}$, we have $\|\mathcal{F}_{A,\varepsilon}(f)(\lambda)\| \leq c\|f\|p$. For $p = 1$, we may replace above the open domain $\mathcal{C}_{p,\varepsilon}$ by $\mathcal{C}_{p,\varepsilon}$.

2) The function $\mathcal{F}_{A,\varepsilon}(f)$ is holomorphic on $\mathcal{C}_{p,\varepsilon}$.

3) (Riemann–Lebesgue lemma) We have $\lim_{\lambda \in \mathcal{C}_{p,\varepsilon}, \lambda \to \infty} |\mathcal{F}_{A,\varepsilon}(f)(\lambda)| = 0$.

4) The Fourier transform $\mathcal{F}_{A,\varepsilon}$ is injective on $L^p(\mathbb{R}, A(x)dx)$ for $1 \leq p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}}$.

**Sketch of Proof.** The first two statements follow from the estimate of $\Psi_{A,\varepsilon}(\lambda, x)$ given in Theorem 1.1.II.4 (with $N = 0$), the fact that $A(x) \leq c|x|^p e^{\varepsilon^2|x|^2}$ (a consequence of the hypothesis (C) on the function $A$), the fact that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is holomorphic in $\lambda$, and Morera’s theorem. To extend the first statement from $\mathcal{C}_{p,\varepsilon}$ to $\mathcal{C}_{p,\varepsilon}$ when $p = 1$, in addition, we show that $|\Psi_{A,\varepsilon}(\lambda, x)| \leq 2$ for all $\lambda \in \mathcal{C}_{1,\varepsilon}$ and for all $x \in \mathbb{R}$. The proof uses the maximum modulus principle and the fact that $|\Psi_{A,\varepsilon}(\lambda, \lambda)| \leq \Psi_{A,\varepsilon}(1, \text{Im}\lambda, x)$. For the Riemann–Lebesgue lemma, a classical proof for the Euclidean Fourier transform carries over. The forth statement is based on the following steps:

Step 1) For $f \in L^p(\mathbb{R}, A(x)dx)$ get $g \in \mathcal{D}(\mathbb{R})$, we show, by means of Hölder’s inequality and the first statement, that the mappings $f \mapsto (f, g)_A := \int_{\mathbb{R}} f(x)g(-x)A(x)dx$ and $f \mapsto (\mathcal{F}_{A,\varepsilon}(f), \mathcal{F}_{A,\varepsilon}(g))_{\pi_\varepsilon} := \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon}(f)(\lambda)\mathcal{F}_{A,\varepsilon}(g)(\lambda)(\frac{1}{\sqrt{1 - \varepsilon^2\lambda^2}} - \sqrt{1 - \varepsilon^2\lambda^2})d\lambda$ are continuous functionals on $L^p(\mathbb{R}, A(x)dx)$. Here $\pi_\varepsilon$ is a positive measure with support $\mathbb{R}

Step 2) We show that $(f, g)_A = (\mathcal{F}_{A,\varepsilon}(f), \mathcal{F}_{A,\varepsilon}(g))_{\pi_\varepsilon}$ for all $f, g \in \mathcal{D}(\mathbb{R})$. Thus, by Step 1), $(f, g)_A = (\mathcal{F}_{A,\varepsilon}(f), \mathcal{F}_{A,\varepsilon}(g))_{\pi_\varepsilon}$ for all $f \in L^p(\mathbb{R}, A(x)dx)$. 
Hence, if we assume that \( f \in L^p(\mathbb{R}, A(x) \, dx) \) and that \( \mathcal{F}_{A,\varepsilon}(f) = 0 \), then for all \( g \in \mathcal{D}(\mathbb{R}) \), we have \((f, g)_A = 0 \) and therefore \( f = 0. \) \( \square \)

For \(-1 \leq \varepsilon \leq 1 \) and \( 0 < p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}} \), denote by \( \mathcal{S}_p(\mathbb{R}) \) the space consisting of all functions \( f \in C^\infty(\mathbb{R}) \) such that

\[
\sigma^{(p)}_{s,k}(f) := \sup_{x \in \mathbb{R}} (|x| + 1)^s e^{\frac{2}{\varepsilon^2} |x|} |f^{(k)}(x)| < \infty
\]

for any \( s,k \in \mathbb{N} \). The topology of \( \mathcal{S}_p(\mathbb{R}) \) is defined by the seminorms \( \sigma^{(p)}_{s,k} \). The space \( \mathcal{D}(\mathbb{R}) \) of smooth functions with compact support on \( \mathbb{R} \) is a dense subspace of \( \mathcal{S}_p(\mathbb{R}) \); see for instance [9, Appendix A].

Let \( \mathcal{C}_{p,\varepsilon} \) be the Schwartz space consisting of all complex valued functions \( h \) that are analytic in the interior of \( C_{p,\varepsilon} \), and such that \( h \) together with all its derivatives extend continuously to \( C_{p,\varepsilon} \) and satisfy

\[
\tau_{t,\ell}^{(\theta, p, \varepsilon)}(h) := \sup_{\lambda \in C_{p,\varepsilon}} (|\lambda| + 1)^t |h^{(\ell)}(\lambda)| < \infty
\]

for any \( t, \ell \in \mathbb{N} \). The topology of \( \mathcal{C}_{p,\varepsilon} \) is defined by the seminorms \( \tau_{t,\ell}^{(\theta, p, \varepsilon)} \).

Using Anker’s approach [1], we prove the following result:

**Theorem 3.2.** Let \(-1 \leq \varepsilon \leq 1 \) and \( 0 < p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}} \). Then the Fourier transform \( \mathcal{F}_{A,\varepsilon} \) is a topological isomorphism between \( \mathcal{S}_p(\mathbb{R}) \) and \( \mathcal{C}_{p,\varepsilon} \).

**Sketch of Proof.** The proof is based on the following steps:

Step 1) The transform \( \mathcal{F}_{A,\varepsilon} \) maps \( \mathcal{S}_p(\mathbb{R}) \) continuously into \( \mathcal{C}_{p,\varepsilon} \) and is injective.

Step 2) The inverse Fourier transform \( \mathcal{F}_{A,\varepsilon}^{-1} : \mathcal{PW}(\mathbb{C}) \rightarrow \mathcal{S}_p(\mathbb{R}) \) given by

\[
\mathcal{F}_{A,\varepsilon}^{-1} h(x) = c \int \mathbb{R} h(\lambda) \Psi_{A,\varepsilon}(\lambda, x) \left( 1 - \frac{\varepsilon \lambda}{ix} \right) \pi_\varepsilon(d\lambda)
\]

is continuous for the topologies induced by \( \mathcal{C}_{p,\varepsilon} \) and \( \mathcal{S}_p(\mathbb{R}) \). Here \( \mathcal{PW}(\mathbb{C}) \) is the space of entire functions on \( \mathbb{C} \) which are of exponential type and rapidly decreasing, and \( \pi_\varepsilon \) is a positive measure with support \( \mathbb{R} \setminus ]-\sqrt{1 - \varepsilon^2}, \sqrt{1 - \varepsilon^2}[^G \right) \). We pin down that \( \mathcal{PW}(\mathbb{C}) \) is dense in \( \mathcal{C}_{p,\varepsilon} \).

For Step 1), we prove that \( \mathcal{F}_{A,\varepsilon}(f) \) is well defined for all \( f \in \mathcal{S}_p(\mathbb{R}) \). This is due to the growth estimates for \( \Psi_{A,\varepsilon}(\lambda, x) \) stated in Theorem 1.1.II.4. Moreover, since the map \( \lambda \mapsto \Psi_{A,\varepsilon}(\lambda, x) \) is holomorphic on \( \mathbb{C} \), it follows that for all \( f \in \mathcal{S}_p(\mathbb{R}) \), the function \( \mathcal{F}_{A,\varepsilon}(f) \) is analytic in the interior of \( C_{p,\varepsilon} \), and continuous on \( C_{p,\varepsilon} \). Finally, we prove that given a continuous seminorm \( \tau \) on \( \mathcal{C}_{p,\varepsilon} \), there exists a continuous seminorm \( \sigma \) on \( \mathcal{S}_p(\mathbb{R}) \) such that \( \tau(\mathcal{F}_{A,\varepsilon}(f)) \leq c \sigma(f) \) for all \( f \in \mathcal{S}_p(\mathbb{R}) \). Indeed, by means of the growth estimates for \( A_{A,\varepsilon}(\lambda, x) \) stated in Theorem 1.1.II.4, we show that

\[
\left| (i\lambda)^t \mathcal{F}_{A,\varepsilon}(f)(\lambda) \right|^{(t)} \leq c \int \mathbb{R} |A_{A,\varepsilon}(\lambda, x)| (|x| + 1)^{t+1} e^{(|\lambda| \varepsilon (1 - \sqrt{1 - \varepsilon^2}))} A(x) \, dx,
\]

and then we prove that \( |A_{A,\varepsilon}(\lambda, x)| \) is bounded by finite sums of the derivatives of \( f \). Thus \( \tau(\mathcal{F}_{A,\varepsilon}(f)) \leq c \sum_{\text{finite}} \sigma(f) \) for all \( f \in \mathcal{S}_p(\mathbb{R}) \). The injectivity of \( \mathcal{F}_{A,\varepsilon} \) on \( \mathcal{S}_p(\mathbb{R}) \) follows from Theorem 3.1.4 and the fact that \( \mathcal{S}_p(\mathbb{R}) \subset L^p(\mathbb{R}, A(x) \, dx) \) for all \( q < \infty \) so that \( p \leq q \).

For Step 2), we start by proving a Paley–Wiener theorem for \( \mathcal{F}_{A,\varepsilon} \), i.e. we prove that \( \mathcal{F}_{A,\varepsilon} \) is a linear isomorphism between the space \( \mathcal{S}_p(\mathbb{R}) \) of smooth compactly supported functions with support inside \([R, R] \right) \) and the space \( \mathcal{PW}_R(\mathbb{C}) \) of entire functions that are of R-exponential type and rapidly decreasing. We note that \( \mathcal{PW}(\mathbb{C}) = \bigcup_{R>0} \mathcal{PW}_R(\mathbb{C}) \).

Next, we take \( f \in \mathcal{S}_p(\mathbb{R}) \) and \( h \in \mathcal{PW}(\mathbb{C}) \) so that \( f = \mathcal{F}_{A,\varepsilon}^{-1}(h) \). Denote by \( g \) the image of \( h \) by the inverse Euclidean Fourier transform \( \mathcal{F}_{\text{eu}}^{-1} \). Making use of the Paley–Wiener theorem for \( \mathcal{F}_{A,\varepsilon} \) and the classical Paley–Wiener theorem for \( \mathcal{F}_{\text{eu}} \), we have the following support conservation property: supp(\( f \)) \( \subset \) \([R, R] \right) \) \( \iff \) supp(\( g \)) \( \subset \) \([R, R] \right) .

For \( j \in \mathbb{N}_{\geq 2} \), let \( \omega_j \in \mathcal{C}^\infty(\mathbb{R}) \) with \( \omega_j = 0 \) on \( I_{j-1} \) and \( \omega_j = 1 \) outside of \( I_j \). Assume that \( \omega_j \) and all its derivatives are bounded, uniformly in \( j \). We write \( g_j = \omega_j g \), and define \( h_j := \mathcal{F}_{\text{eu}}(g_j) \) and \( f_j := \mathcal{F}_{A,\varepsilon}(h_j) \). Note that \( g_j = g \) outside \( I_j \).

Hence, by the above support property, \( f_j = f \) outside \( I_j \).

In view of the growth estimate for \( A_{A,\varepsilon}(\lambda, x) \) stated in Theorem 1.1.II.3, we prove that for all \( j \in \mathbb{N}_{\geq 1} \),

\[
\sup_{x \in I_{j+1} \setminus I_j} (|x| + 1)^s e^{\frac{2}{\varepsilon^2} |x|} |f^{(k)}(x)| \leq c \sum_{r=0}^{s+3} \tau_{r,\varepsilon}(h_j),
\]
for some integer \( t > 0 \). For \( I_1 \), we show first that there exists an integer \( m_k \geq 1 \) such that

\[
|d^k_x \psi_{A, \epsilon}(\lambda, x)| \leq c(\lambda|+1)^{m_k}|(x|+1)e^{-\epsilon|x|},
\]

for \( \lambda \in \mathbb{R} \) such that \(|\lambda| \geq \sqrt{1-\epsilon^2}\rho\). Then, using the compactness of \( I_1 \), we prove that

\[
\sup_{x \in I_1} |x|^{s} e^{2\epsilon|x|} |f^{(k)}(x)| \leq c_{s, t, 0}(\epsilon),
\]

for some integer \( t > 0 \).

Details of this paper will be given in other articles \cite{3} and \cite{4}.

References