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Complex analysis

On estimates for the coefficients of a polynomial

Sur les bornes pour les coefficients d'un polynôme

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A B S T R A C T

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, then it was proved by Rahman and Schmeisser [4] that for every $p \in [0, +\infty]$,

$$|a_n| + |a_0| \leq 2\frac{\|P\|_p}{\|1+z\|_p}.$$

In this paper, various estimates for the coefficients of a polynomial P are obtained which among other things include the above inequality as a special case.

R É S U M É

Si $P(z) = \sum_{j=0}^{n} a_j z^j$ est un polynôme de degré n, Rahman et Schmeisser [4] ont montré que, pour tout $p \in [0, +\infty]$, on a

$$|a_n| + |a_0| \leq 2\frac{\|P\|_p}{\|1+z\|_p}.$$

Nous obtenons ici diverses majorations pour les coefficients d'un polynôme qui, entre autres, incluent l'inégalité ci-dessus comme cas particulier.

1. Introduction and statement of results

Let $\mathcal{P}_n$ be the set of all polynomials of degree n. For $P \in \mathcal{P}_n$ define

$$\|P\|_p := \left(\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right)^{1/p} \qquad (0 < p < \infty),$$

$$\|P\|_\infty := \max_{|z|=1} |P(z)|$$

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and

$$\|P\|_0 := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P(\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta\right).$$

If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$, then by Rouché's theorem, the polynomial

$$P(z) + \lambda\|P\|_\infty = a_n z^n + a_{n-1}z^{n-1} + \ldots + (a_0 + \lambda\|P\|_\infty)$$

does not vanish in the unit disk $|z| < 1$ for any choice of $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. It follows that $|a_0 + \lambda\|P\|_\infty| \geq |a_n|$ for each $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. By choosing the argument of λ suitably, we get

$$|a_n| + |a_0| \leq \|P\|_\infty. \tag{1.1}$$

This is called Visser's inequality [5]. It is well known that equality holds in (1.1) only when $a_j = 0$ for $j = 1, 2, \ldots, n-1$. Different variants of this inequality can be found in [3]. Rahman and Schmeisser [4] obtained an L_p version of inequality (1.1) and proved that if $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$, then for every $p \in [0, +\infty]$,

$$|a_n| + |a_0| \leq 2\frac{\|P\|_p}{\|1+z\|_p}. \tag{1.2}$$

In this paper, we first prove the following result, which among other things includes inequalities (1.1) and (1.2) as special cases. More precisely, we prove the following theorem.

Theorem 1.1. *If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$, then for each $0 \leq p < \infty$,*

$$|a_n| + \frac{|a_k|}{\binom{n}{k}} \leq 2\frac{\|P\|_p}{\|1+z\|_p} \tag{1.3}$$

where $k = 0, 1, \ldots, n-1$.

For $k = 0$, inequality (1.3) reduces to (1.2).

If we let $p \to \infty$ in (1.3), we obtain the following result, from which the Visser's inequality follows when $k = 0$.

Corollary 1.2. *If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$, then*

$$|a_n| + \frac{|a_k|}{\binom{n}{k}} \leq \|P\|_\infty$$

where $k = 0, 1, \ldots, n-1$.

Theorem 1.1 can be improved if we restrict ourselves to the class of polynomials having no zero in $|z| < 1$. In this direction, we prove:

Theorem 1.3. *If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$ does not vanish in $|z| < 1$, then for each $0 \leq p < \infty$,*

$$|a_n| + \frac{|a_k|}{\binom{n}{k}} \leq 2c_p\frac{\|P\|_p}{\|1+z\|_p}$$

where $k = 0, 1, \ldots, n-1$ and $c_p = \begin{cases} 1 & \text{if } k = 0 \\ \dfrac{1}{\|1+z\|_p} & \text{if } k > 0. \end{cases}$

Note that $0 < c_p < 1$ for $k > 0$ and $p > 0$.

2. Lemmas

We first describe a result of Arestov [1].

For $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in \mathbb{C}^{n+1}$ and $P(z) = \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$, we define

$$C_{\boldsymbol{\gamma}} P(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator $C_{\boldsymbol{\gamma}}$ is said to be *admissible* if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $|z| \le 1$,
(ii) $P(z)$ has all its zeros in $|z| \ge 1$.

The result of Arestov [1, Theorem 2] may now be stated as follows.

Lemma 2.1. *Let $\varphi(x) = \psi(\log x)$, where ψ is a convex non-decreasing function on $\mathbb{R}$. Then for all $P \in \mathcal{P}_n$ and each admissible operator $C_{\boldsymbol{\gamma}}$,*

$$\int_0^{2\pi} \varphi\left(|C_{\boldsymbol{\gamma}} P(e^{i\theta})|\right) d\theta \le \int_0^{2\pi} \varphi\left(c(\boldsymbol{\gamma})|P(e^{i\theta})|\right) d\theta$$

where $c(\boldsymbol{\gamma}) = \max(|\gamma_0|, |\gamma_n|)$.

In particular Lemma 2.1 applies with $\varphi : x \mapsto x^p$ for every $p \in (0, \infty)$ and with $\varphi : x \mapsto \log x$ as well. Therefore, we have for $0 \le p < \infty$,

$$\left\{\int_0^{2\pi} \left|C_{\boldsymbol{\gamma}} P(e^{i\theta})\right|^p d\theta\right\}^{1/p} \le c(\boldsymbol{\gamma}) \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}. \tag{2.1}$$

Lemma 2.2. *If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$ does not vanish for $|z| < 1$, then for $k = 0, 1, \ldots, n-1$, ϕ real and each $p > 0$,*

$$\int_0^{2\pi} \left| \left(a_n e^{in\theta} + \frac{a_k}{\binom{n}{k}} e^{ik\theta}\right) e^{i\phi} + \left(\frac{a_{n-k}}{\binom{n}{k}} e^{i(n-k)\theta} + a_0\right) \right| d\theta \le \Lambda^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta$$

where $k = 0, 1, \ldots, n-1$ and $\Lambda = \begin{cases} |1 + e^{i\phi}| & \text{if } k = 0 \\ 1 & \text{if } 0 < k < n. \end{cases}$

Proof. Since $P(z)$ has all its zeros in $|z| \ge 1$, then all the zeros of $P^\star(z) = z^n \overline{P(1/\bar{z})}$ lie in $|z| \le 1$ and $|P(z)| = |P^\star(z)|$ for $|z| = 1$. Therefore $P^\star(z)/P(z)$ is analytic in $|z| \le 1$. By the maximum modulus principle $|P^\star(z)| \le |P(z)|$ for $|z| \le 1$ or equivalently $|P(z)| \le |P^\star(z)|$ for $|z| \ge 1$. By Rouché's theorem, all the zeros of the polynomial

$$P(z) - \lambda P^\star(z) = \sum_{j=0}^{n} (a_j - \lambda \bar{a}_{n-j}) z^j$$

lie in $|z| \le 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. If $z_1, z_2, \ldots, z_n$ are roots of $P(z) - \lambda P^\star(z)$, then $|z_j| \le 1$, $j = 1, 2, \ldots, n$ and we have by Viète's formula for $k = 0, 1, \ldots, n-1$,

$$(-1)^{n-k} \left(\frac{a_k + \lambda \bar{a}_{n-k}}{a_n + \lambda \bar{a}_0}\right) = \sum_{1 \le i_1 < i_2 \ldots < i_{n-k} \le n} z_{i_1} z_{i_2} \ldots z_{i_{n-k}}$$

which gives

$$\left|\frac{a_k + \lambda \bar{a}_{n-k}}{a_n + \lambda \bar{a}_0}\right| \le \sum_{1 \le i_1 < i_2 \ldots < i_{n-k} \le n} \left|z_{i_1} z_{i_2} \ldots z_{i_{n-k}}\right| \le \binom{n}{n-k} = \binom{n}{k}. \tag{2.2}$$

Therefore, all the zeros of the polynomial

$$M(z) = (a_n + \lambda \bar{a}_0) z^n + \frac{a_k + \lambda \bar{a}_{n-k}}{\binom{n}{k}} z^k = a_n z^n + \frac{a_k}{\binom{n}{k}} z^k + \lambda \left(\bar{a}_0 z^n + \frac{\bar{a}_{n-k}}{\binom{n}{k}} z^k\right)$$

lie in $|z| \le 1$ for $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. So that if $r > 1$, the polynomial

$$M(rz) = a_n (rz)^n + \frac{a_k}{\binom{n}{k}} (rz)^k + \lambda \left(\bar{a}_0 (rz)^n + \frac{\bar{a}_{n-k}}{\binom{n}{k}} (rz)^k\right)$$

has all its zeros in $|z| < 1$. This implies

$$\left| a_n(rz)^n + \frac{a_k}{\binom{n}{k}}(rz)^k \right| \le \left| \bar{a}_0(rz)^n + \frac{\bar{a}_{n-k}}{\binom{n}{k}}(rz)^k \right| \tag{2.3}$$

for $|z| \ge 1$. Indeed, if inequality (2.3) is not true, then there exists a point w with $|w| \ge 1$ such that

$$\left| a_n(rw)^n + \frac{a_k}{\binom{n}{k}}(rw)^k \right| > \left| \bar{a}_0(rw)^n + \frac{\bar{a}_{n-k}}{\binom{n}{k}}(rw)^k \right|.$$

Since all the zeros of $P^\star(z)$ lie in $|z| \le 1$, then by the similar argument as in (2.2), we have $|\bar{a}_0| \ge |\bar{a}_{n-k}|/\binom{n}{k}$, which implies that $\bar{a}_0(rw)^n + \bar{a}_{n-k}/\binom{n}{k}(rw)^k \neq 0$. We take

$$\lambda = -\frac{a_n(rw)^n + \frac{a_k}{\binom{n}{k}}(rw)^k}{\bar{a}_0(rw)^n + \frac{\bar{a}_{n-k}}{\binom{n}{k}}(rw)^k},$$

then λ is a well defined complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $M(rw) = 0$ where $|w| \ge 1$. This contradicts the fact that all the zeros of $M(rz)$ lie in $|z| < 1$. Thus (2.3) holds. Letting $r \to 1$ in (2.3) and using continuity, in particular, we obtain

$$\left| a_n z^n + \frac{a_k}{\binom{n}{k}} z^k \right| \le \left| \bar{a}_0 z^n + \frac{\bar{a}_{n-k}}{\binom{n}{k}} z^k \right| = \left| \frac{a_{n-k}}{\binom{n}{k}} z^{n-k} + a_0 \right| \tag{2.4}$$

for $|z| = 1$. Again, since $|\bar{a}_0| \ge |\bar{a}_{n-k}|/\binom{n}{k}$ then the polynomial $\frac{a_{n-k}}{\binom{n}{k}} z^{n-k} + a_0$ does not vanish in $|z| < 1$. Hence by the maximum modulus principle,

$$\left| a_n z^n + \frac{a_k}{\binom{n}{k}} z^k \right| < \left| \frac{a_{n-k}}{\binom{n}{k}} z^{n-k} + a_0 \right| \quad \text{for} \quad |z| < 1.$$

A direct application of Rouché's theorem shows that with $P(z) = a_n z^n + \cdots + a_0$,

$$C_\gamma P(z) = \left(a_n z^n + \frac{a_k}{\binom{n}{k}} z^k \right) e^{i\phi} + \left(\frac{a_{n-k}}{\binom{n}{k}} z^{n-k} + a_0 \right)$$

has all its zeros in $|z| \ge 1$. Therefore, C_γ is an admissible operator. Applying (2.1) of Lemma 2.1, the desired result follows immediately for each $p > 0$. □

The next two lemmas can be found in [2].

Lemma 2.3. *Let α be a complex number independent of θ, where θ is real. Then for each $p > 0$*

$$\int_0^{2\pi} \left| \alpha + e^{i\theta} \right|^p d\theta = \int_0^{2\pi} \left| 1 + |\alpha| e^{i\theta} \right|^p d\theta.$$

Lemma 2.4. *Let n be a positive integer and $0 \le p \le \infty$,*

$$\left\| 1 + z^n \right\|_p = \| 1 + z \|_p .$$

Lemma 2.5. *Let $\alpha, \beta \in \mathbb{C}$ and n, k are positive integers with $n > k$. Then for each $p > 0$,*

$$\left\| \alpha z^n + \beta z^k \right\|_p \ge \frac{|\alpha| + |\beta|}{2} \| 1 + z \|_p . \tag{2.5}$$

Proof. If $\alpha = 0$ then (2.5) follows by the fact $\|1 + z\|_p < 2$. Henceforth, we assume that $\alpha \neq 0$, therefore by Lemma 2.3 and Lemma 2.4, we obtain

$$\begin{aligned} \left\| \alpha z^n + \beta z^k \right\|_p &= |\alpha| \left\| z + \frac{\beta}{\alpha} \right\|_p \\ &= |\alpha| \left\| 1 + \left| \frac{\beta}{\alpha} \right| z \right\|_p . \end{aligned} \tag{2.6}$$

From the inequality

$$\left|\frac{1+r\mathrm{e}^{\mathrm{i}\theta}}{1+\mathrm{e}^{\mathrm{i}\theta}}\right| \geq \frac{1+r}{2} \quad \text{with} \quad r=\left|\frac{\beta}{\alpha}\right| \text{ and } 0\leq\theta<2\pi$$

we deduce

$$|\alpha|\left|1+\left|\frac{\beta}{\alpha}\right|\mathrm{e}^{\mathrm{i}\theta}\right| \geq \frac{|\alpha|+|\beta|}{2}\left|1+\mathrm{e}^{\mathrm{i}\theta}\right|.$$

This implies

$$|\alpha|\left\|1+\left|\frac{\beta}{\alpha}\right|z\right\|_p \geq \frac{|\alpha|+|\beta|}{2}\left\|1+\mathrm{e}^{\mathrm{i}\theta}\right\|_p.$$

Using this in conjunction with (2.6), the desired result follows. □

3. Proof of theorems

Proof of Theorem 1.1. By hypothesis $P\in\mathcal{P}_n$, we can write

$$P(z)=P_1(z)P_2(z),$$

where all the zeros of $P_1(z)$ lie in $|z|\leq 1$ and all the zeros of $P_2(z)$ lie in $|z|>1$. First, we suppose that $P_1(z)$ has no zero on $|z|=1$. Let the degree of polynomial $P_2(z)$ be k then the polynomial $P_2^\star(z)=z^k\overline{P_2(1/\overline{z})}$ has all its zeros in $|z|<1$ and $|P_2^\star(z)|=|P_2(z)|$ for $|z|=1$. Now consider the polynomial

$$F(z)=P_1(z)P_2^\star(z),$$

then all the zeros of $F(z)$ lie in $|z|<1$ and for $|z|=1$,

$$|F(z)|=|P_1(z)|\left|P_2^\star(z)\right|=|P_1(z)|\,|P_2(z)|=|P(z)|.$$

By the maximum modulus principle, it follows that

$$|P(z)|\leq|F(z)| \quad \text{for} \quad |z|\geq 1.$$

Since $F(z)\neq 0$ for $|z|\geq 1$, a direct application of Rouché's theorem shows that the polynomial $H(z)=P(z)+\lambda F(z)$ has all its zeros in $|z|<1$, for every $\lambda\in\mathbb{C}$ with $|\lambda|>1$. Let $F(z)=\sum_{j=0}^n b_j z^j$, then the polynomial

$$H(z)=\sum_{j=0}^{n}(a_j+\lambda b_j)z^j$$

has all its zeros in $|z|<1$. If $w_1,w_2,\dots,w_n$ be roots of $H(z)$, then $|w_j|<1$, $j=1,2,\dots,n$ and we have by Viète's formula for $k=0,1,\dots,n-1$,

$$(-1)^{n-k}\left(\frac{a_k+\lambda b_k}{a_n+\lambda b_n}\right)=\sum_{1\leq i_1<i_2\dots<i_{n-k}\leq n} w_{i_1}w_{i_2}\dots w_{i_{n-k}}.$$

Now, proceeding similarly as in Lemma 2.2, we obtain

$$\left|a_n z^n+\frac{a_k}{\binom{n}{k}}z^k\right| \leq \left|b_n z^n+\frac{b_k}{\binom{n}{k}}z^k\right|$$

for $|z|\geq 1$. This implies for each $p>0$ and $0\leq\theta<2\pi$,

$$\int_0^{2\pi}\left|a_n\mathrm{e}^{\mathrm{i}n\theta}+\frac{a_k}{\binom{n}{k}}\mathrm{e}^{\mathrm{i}k\theta}\right|^p\mathrm{d}\theta \leq \int_0^{2\pi}\left|b_n\mathrm{e}^{\mathrm{i}n\theta}+\frac{b_k}{\binom{n}{k}}\mathrm{e}^{\mathrm{i}k\theta}\right|^p\mathrm{d}\theta. \tag{3.1}$$

Again, since all the zeros of $F(z)=\sum_{j=0}^n b_j z^j$ lie in $|z|<1$, similarly as shown before, the polynomial $b_n z^n+\frac{b_k}{\binom{n}{k}}z^k$ also has all its zeros in $|z|<1$. Therefore, the operator C_γ defined by

$$C_\gamma F(z)=b_n z^n+\frac{b_k}{\binom{n}{k}}z^k$$

is admissible. Hence by (2.1) of Lemma 2.1, for each $p > 0$, we have

$$\int_0^{2\pi} \left| b_n e^{in\theta} + \frac{b_k}{\binom{n}{k}} e^{ik\theta} \right|^p d\theta \le (c(\boldsymbol{\gamma}))^p \int_0^{2\pi} \left| F(e^{i\theta}) \right|^p d\theta, \tag{3.2}$$

where $c(\boldsymbol{\gamma}) = \max(|\gamma_0|, |\gamma_n|) = 1$. Combining inequalities (3.1), (3.2) and noting that $|M(e^{i\theta})| = |P(e^{i\theta})|$, we obtain for each $p > 0$,

$$\left\{ \int_0^{2\pi} \left| a_n e^{in\theta} + \frac{a_k}{\binom{n}{k}} e^{ik\theta} \right|^p d\theta \right\}^{1/p} \le \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \right\}^{1/p}.$$

Using this in conjunction with Lemma 2.5, we obtain

$$|a_n| + \frac{|a_k|}{\binom{n}{k}} \le 2 \frac{\|P(z)\|_p}{\|1+z\|_p}. \tag{3.3}$$

In case $P_1(z)$ has a zero on $|z| = 1$, inequality (3.3) follows by continuity. This proves Theorem 1.1 for $p > 0$. To obtain this result for $p = 0$, we simply make $p \to 0+$. □

Proof of Theorem 1.3. Since $P(z) = \sum_{j=0}^{n} a_j z^j$ does not vanish in $|z| < 1$, therefore by (2.4) for $k = 0, 1, 2, \ldots, n-1$,

$$\left| a_n e^{in\theta} + \frac{a_k}{\binom{n}{k}} z^{ik\theta} \right| \le \left| \frac{a_{n-k}}{\binom{n}{k}} e^{i(n-k)\theta} + a_0 \right|. \tag{3.4}$$

Also, by Lemma 2.2,

$$\int_0^{2\pi} \left| G(\theta) + e^{i\phi} Q(\theta) \right|^p d\theta \le \Lambda^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta, \tag{3.5}$$

where $G(\theta) = a_n e^{in\theta} + \frac{a_k}{\binom{n}{k}} e^{ik\theta}$ and $Q(\theta) = \frac{a_{n-k}}{\binom{n}{k}} e^{i(n-k)\theta} + a_0$.

Integrating both sides of (3.5) with respect to ϕ from 0 to 2π, we get for each $p > 0$ and ϕ real

$$\int_0^{2\pi} \int_0^{2\pi} \left| G(\theta) + e^{i\phi} Q(\theta) \right|^p d\phi d\theta \le \int_0^{2\pi} \Lambda^p d\phi \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \tag{3.6}$$

Now for every real ϕ and $t \ge 1$ and $p > 0$, we have

$$\int_0^{2\pi} |t + e^{i\phi}|^p d\phi \ge \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi.$$

If $G(\theta) \ne 0$, we take $t = |Q(\theta)|/|G(\theta)|$, then by (3.4) $t \ge 1$ and by using Lemma 2.3 we get

$$\begin{aligned} \int_0^{2\pi} \left| G(\theta) + e^{i\phi} Q(\theta) \right|^p d\phi &= |G(\theta)|^p \int_0^{2\pi} \left| 1 + e^{i\phi} \frac{Q(\theta)}{G(\theta)} \right|^p d\phi \\ &= |G(\theta)|^p \int_0^{2\pi} \left| e^{i\phi} + \left| \frac{Q(\theta)}{G(\theta)} \right| \right|^p d\phi \\ &\ge |G(\theta)|^p \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi. \end{aligned}$$

For $G(\theta) = 0$, this inequality is trivially true. Using this in (3.6), we conclude that, for real ϕ,

$$\int_0^{2\pi} |G(\theta)|^p d\theta \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \le \int_0^{2\pi} \Lambda^p d\phi \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.$$

This implies

$$\left\{\int_0^{2\pi}\left|a_n e^{in\theta}+\frac{a_k}{\binom{n}{k}}e^{ik\theta}\right|^p d\theta\right\}^{1/p}\le\frac{\left\{\int_0^{2\pi}\Lambda^p d\phi\right\}^{1/p}\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}}{\left\{\int_0^{2\pi}|1+e^{i\phi}|^p d\phi\right\}^{1/p}}$$

which in conjunction with Lemma 2.5, gives

$$|a_n|+\frac{|a_k|}{\binom{n}{k}}\le 2c_p\frac{\|P(z)\|_p}{\|1+z\|_p}$$

where $c_p=\begin{cases}1 & \text{if } k=0\\ \dfrac{1}{\|1+z\|_p} & \text{if } k>0.\end{cases}$ □

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