Complex analysis

Faber polynomial coefficients of bi-subordinate functions

Polynômes de Faber et coefficients des fonctions bi-subordonnées

Samaneh G. Hamidi\textsuperscript{a}, Jay M. Jahangiri\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Brigham Young University, Provo, UT 84604, USA
\textsuperscript{b} Department of Mathematical Sciences, Kent State University, Burton, OH 44021, USA

\textbf{A R T I C L E  I N F O}

\textbf{A B S T R A C T}

A function is said to be bi-univalent in the open unit disk $\mathbb{D}$ if both the function and its inverse map are univalent in $\mathbb{D}$. By the same token, a function is said to be bi-subordinate in $\mathbb{D}$ if both the function and its inverse map are subordinate to certain given function in $\mathbb{D}$. The behavior of the coefficients of such functions are unpredictable and unknown. In this paper, we use the Faber polynomial expansions to find upper bounds for the $n$-th ($n \geq 3$) coefficients of classes of bi-subordinate functions subject to a gap series condition as well as determining bounds for the first two coefficients of such functions.

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\textbf{R É S U M É}

Une fonction est dite bi-univalente dans le disque unité ouvert $\mathbb{D}$ si elle et son inverse sont univalentes dans $\mathbb{D}$. Dans le même ordre, une fonction est dite bi-subordonnée dans $\mathbb{D}$ si elle et son inverse sont subordonnées à une fonction donnée dans $\mathbb{D}$. Le comportement des coefficients de telles fonctions est imprévisible et inconnu. Dans cette Note, nous utilisons les développements en polynômes de Faber afin d’établir une borne supérieure pour le $n$\textsuperscript{th} ($n \geq 3$) coefficient d’une fonction bi-subordonnée, lorsque les $n-2$ précédents coefficients sont nuls. Nous donnons également des bornes plus précises pour les deux premiers coefficients de telles fonctions.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the class of functions $f$ that are analytic and univalent in $\mathbb{D}$ and are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  \hspace{1cm} (1)

\textit{E-mail addresses:} s.hamidi_61@yahoo.com (S.G. Hamidi), jjahangi@kent.edu (J.M. Jahangiri).

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For \( f(z) \) and \( F(z) \) analytic in \( \mathbb{D} \), we say that \( f(z) \) is subordinate to \( F(z) \), written \( f \prec F \), if there exists a Schwarz function \( \varphi(z) \) with \( \varphi(0) = 0 \) and \( |\varphi(z)| < 1 \) in \( \mathbb{D} \) such that \( f(z) = F(\varphi(z)) \). We note that \( f(\mathbb{D}) \subset F(\mathbb{D}) \) if \( f \) and \( F \) are in \( S \). For real numbers \( A \) and \( B \) so that \(-1 \leq B < A \leq 1\) we let \( S[A, B] \) consist of functions \( f \in S \) satisfying the subordination condition
\[
\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}, \quad |z| < 1.
\]

For classes related to \( S[A, B] \), see Janowski [18,19].

If \( f \) is given by the power series (1), then \( g = f^{-1} \), the inverse map of the function \( f \), has a Maclaurin series expansion in some disk about the origin (e.g. see Duren [10]). A function \( f \in S \) is said to be bi-univalent if \( g = f^{-1} \) also belongs to \( S \). Certain coefficient bounds for subclasses of bi-univalent functions were obtained by several authors including Ali et al. [4], Altinkaya and Yalçın [6,5,7], Bulut [8], Deniz [9], Frasin and Aouf [12], Hamidi and Jahangiri ([13,14]), Jahangiri and Hamidi [16], Jahangiri et al. [17], Magesh and Yamini [20], Srivastava et al. ([21,22]) and Zaprawa [23]. The bi-univalency condition imposed on the function \( f \) makes the behavior of the coefficients of bi-univalent functions unpredictable. Not much is known about the higher coefficients of bi-univalent functions as Ali et al. [4] also declared the bounds for the \( n \)-th \( (n \geq 4) \) coefficients of bi-univalent functions an open problem. We use the Faber polynomial expansions to obtain bounds for the \( n \)-th \( (n \geq 3) \) coefficients of bi-univalent functions subject to a gap series condition. We then demonstrate the unpredictability of the early coefficients \( a_2 \) and \( a_3 \) of such bi-subordinate functions. A function \( f \) in \( S[A, B] \) is said to be bi-subordinate if its inverse map \( g = f^{-1} \) is also in \( S[A, B] \).

2. Main results

In the following theorem we use the Faber polynomials introduced by Faber [11] to obtain a bound for the general coefficients \( |a_n| \) of the bi-univalent functions in \( S[A, B] \) subject to a gap series condition.

**Theorem 2.1.** For \(-1 \leq B < A \leq 1\) if both functions \( f \) and its inverse map \( g = f^{-1} \) are in \( S[A, B] \), then
\[
|a_n| \leq \frac{A - B}{n - 1} \text{ for } a_k = 0; \quad 2 \leq k \leq n - 1.
\]

**Proof.** An application of Faber polynomial expansion to the power series \( f \in S[A, B] \) (e.g. see [2] or [3, equation (1.6)]) yields
\[
\frac{zf'(z)}{f(z)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, \ldots, a_n)z^{n-1}
\]
where
\[
F_{n-1}(a_2, a_3, \ldots, a_n) = \sum_{i_1+2i_2+\cdots+(n-1)i_{n-1}=n-1} A(i_1, i_2, \ldots, i_{n-1})(a_2^{i_1}a_3^{i_2} \cdots a_n^{i_{n-1}})
\]
and
\[
A(i_1, i_2, \ldots, i_{n-1}) := (-1)^{(n-1)+2i_1+\cdots+ni_{n-1}} \frac{(i_1+i_2+\cdots+i_{n-1}-1)!}{(i_1!)^2(i_2!) \cdots (i_{n-1}!)^n}.
\]
The first few terms of \( F_{n-1}(a_2, a_3, \ldots, a_n) \) are
\[
F_1 = -a_2,
F_2 = a_2^2 - 2a_3,
F_3 = -a_2^3 + 3a_2a_3 - 3a_4,
F_4 = a_2^4 - 4a_2^3a_3 + 6a_2a_4 + 4a_2^2 - 4a_5,
F_5 = -a_2^5 + 5a_2^3a_3 + 5a_2a_4 - 5(a_2^2 - a_3)a_2 + 5a_3a_4 - 4a_6,
F_6 = a_2^6 - 6a_2^4a_3 + 6a_2^3a_4 - 6(2a_2a_3a_4 - a_6)a_2 - 2a_3^3 + 9a_2^2a_4^2 + 6a_3a_5 + 3a_4^2 - 3a_2a_5 - 6a_7.
\]
By the same token, the coefficients of the inverse map \( g = f^{-1} \) may be expressed by
\[
g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{n}(a_2, a_3, \ldots, a_n)w^n = w + \sum_{n=2}^{\infty} b_n w^n
\]
where
\[
K_{n-1}^{-n} = \frac{(-n)!}{(2(-n + 1))(n - 1)!} a_2^{n-1} + \frac{(-n)!}{2(-n + 1)(n - 3)!} a_2^{n-3} a_3
\]
\[
+ \frac{(-n)!}{(2(-n + 1))(n - 4)!} a_2^{n-4} a_4
\]
\[
+ \frac{(-n)!}{2(-n + 2)(n - 5)!} a_2^{n-5} \left[ a_5 + (-2n + 2)a_3^2 \right]
\]
\[
+ \frac{(-n)!}{(2(-n + 5))(n - 6)!} a_2^{n-6} \left[ a_6 + (-2n + 5)a_3 a_4 \right] + \sum_{j=7} a_2^{n-j} V_j,
\]
and \( V_j \) for \( 7 \leq j \leq n \) is a homogeneous polynomial in the variables \( a_3, a_4, \cdots, a_n \). Obviously,
\[
\frac{w g'(w)}{g(w)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(b_2, b_3, \cdots, b_n) w^{n-1}.
\]
(3)

Since, both functions \( f \) and its inverse function \( g = f^{-1} \) are in \( S[A, B] \), by the definition of subordination, there exist two Schwarz functions \( \varphi(z) = c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots, z \in \mathbb{D} \) and \( \psi(w) = d_1 w + d_2 w^2 + \cdots + d_n w^n + \cdots, w \in \mathbb{D} \), so that
\[
\frac{zf'(z)}{f(z)} = 1 + A \varphi(z) = 1 - \sum_{n=1}^{\infty} (A - B) K_n^{-1}(c_1, c_2, \cdots, c_n, B) z^n
\]
(4)

and
\[
\frac{wg'(w)}{g(w)} = 1 + A \psi(w) = 1 - \sum_{n=1}^{\infty} (A - B) K_n^{-1}(d_1, d_2, \cdots, d_n, B) w^n.
\]
(5)

In general (e.g., see [1] and [2]), the coefficients \( K_n^p(k_1, k_2, \cdots, k_n, B) \) are given by
\[
K_n^p(k_1, k_2, \cdots, k_n, B) = \frac{p!}{(p - n)!}\left[ k_1^p B^{n-1} + \frac{p!}{(p - n + 1)!\cdot 2} k_1^{p-2} k_2 B^{n-2}
\right.
\]
\[
+ \frac{p!}{(p - n + 2)!\cdot 3} k_1^{p-3} k_3 B^{n-3}
\]
\[
+ \frac{p!}{(p - n + 4)!\cdot 4} k_1^{p-4} \left[ k_4 B^{n-4} + \frac{p - n + 3}{2} k_2^2 B^3 \right]
\]
\[
+ \frac{p!}{(p - n + 4)!\cdot 5} k_1^{p-5} \left[ k_5 B^{n-5} + (p - n + 4) k_2 k_4 B^{n-5} \right] + \sum_{j=6}^n k_1^{n-j} X_j,
\]
where \( X_j \) is a homogeneous polynomial of degree \( j \) in the variables \( k_2, k_3, \cdots, k_n \).

For the coefficients of the Schwarz functions \( \varphi(z) \) and \( \psi(w) \) we have \( |c_n| \leq 1 \) and \( |d_n| \leq 1 \) (e.g., see Duren [10]). Comparing the corresponding coefficients of (2) and (4) yields
\[
F_{n-1}(a_2, a_3, \cdots, a_n) = (A - B) K_{n-1}^{-1}(c_1, c_2, \cdots, c_{n-1}, B)
\]
(6)

which under the assumption \( a_k = 0, \ 2 \leq k \leq n - 1 \) we get
\[
-(n - 1)a_n = -(A - B)c_{n-1}.
\]
(7)

Similarly, comparing the corresponding coefficients of (3) and (5) gives
\[
F_{n-1}(b_2, b_3, \cdots, b_n) = (A - B) K_{n-1}^{-1}(d_1, d_2, \cdots, d_{n-1}, B)
\]
(8)

which by the hypothesis, we obtain \( -(n - 1)b_n = -(A - B)d_{n-1} \).

Note that, for \( a_k = 0, \ 2 \leq k \leq n - 1 \) we have \( b_n = -a_n \) and therefore
\[
(n - 1)a_n = -(A - B)d_{n-1}.
\]
(9)

Taking the absolute values of either of the equations (7) or (9) and dividing by \( (n - 1) \) we obtain the required bound
\[
|a_n| \leq \frac{A - B}{n - 1}.
\]

To prove our next theorem, we shall need the following well-known lemma (e.g., see Jahangiri [15] or Duren [10]).
Lemma 2.2. Let \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{A} \) be a positive real function so that \( \text{Re}(p(z)) > 0 \) for \( |z| < 1 \). If \( \alpha \geq -1/2 \) then
\[
\left| p_2 + \alpha p_1^2 \right| \leq 2 + \alpha |p_1|^2. \tag{10}
\]

Corollary 2.3. Let \( \varphi(z) = \sum_{n=1}^{\infty} \varphi_n z^n \in \mathcal{A} \) be a Schwarz function so that \( |\varphi(z)| < 1 \) for \( |z| < 1 \). If \( \gamma \geq 0 \) then
\[
|\varphi_2 + \gamma \varphi_1^2 | \leq 1 + (\gamma - 1)|\varphi_1|^2. \tag{11}
\]

**Proof.** Set \( p(z) = [1 + \varphi(z)]/[1 - \varphi(z)] \) where \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \) is so that \( \text{Re}(p(z)) > 0 \) for \( |z| < 1 \). Comparing the corresponding coefficients of powers of \( z \) yields \( p_1 = 2\varphi_1 \) and \( p_2 = 2(\varphi_2 + \varphi_1^2) \). Substituting for \( p_1 \) and \( p_2 \) in (10), we obtain
\[
2(\varphi_2 + \varphi_1^2) + \alpha (2\varphi_1)^2 \leq 2 + \alpha |2\varphi_1|^2
\]
or
\[
|\varphi_2 + (1 + 2\alpha)\varphi_1^2 | \leq 1 + 2\alpha |\varphi_1|^2.
\]

Now (11) follows upon substitution of \( \gamma = 1 + 2\alpha \geq 0 \) in the above inequality. \( \square \)

In the following theorem, we see how relaxing the restrictions imposed on Theorem 2.1 reveals the unpredictability of the coefficients of bi-univalent functions in \( S[A, B] \).

**Theorem 2.4.** For \(-1 < A \leq B \leq 1 \) if both functions \( f \) and its inverse map \( g = f^{-1} \) are in \( S[A, B] \), then

\[
(i). |a_2| \leq \begin{cases} \frac{A-B}{\sqrt{1+A}} & \text{if } B \leq 0 < A; \\ A - B & \text{otherwise}. \end{cases}
\]

\[
(ii). |a_3 - a_2^2| \leq \begin{cases} \frac{A-B}{2} \left[ 1 - \frac{1+A}{(A-B)^2} |a_2|^2 \right] & \text{if } A \leq 0; \\ \frac{A-B}{2} & \text{if } A > 0. \end{cases}
\]

**Proof.** For \( n = 2 \), (6) and (8) imply \( a_2 = (A - B)c_1 \) and \( b_2 = (A - B)d_1 = -a_2 \). Taking absolute values of both sides of either of these equations, we obtain
\[
|a_2| \leq A - B.
\]

For \( n = 3 \), the equations (6) and (8), respectively, imply
\[
a_2^2 - 2a_3 = (A - B)(Bc_1^2 - c_2) \tag{12}
\]
and
\[
-3a_2^2 + 2a_3 = (A - B)(Bd_1^2 - d_2). \tag{13}
\]

Adding (12) and (13) yields
\[
-2a_2^2 = -(A - B) \left[ (c_2 - Bc_1^2) + (d_2 - Bd_1^2) \right].
\]

Taking absolute values of both sides of the above equation, we obtain
\[
2|a_2|^2 \leq (A - B) \left[ |c_2 + (-B)c_1^2| + |d_2 + (-B)d_1^2| \right].
\]

If \( B \leq 0 \), then by (11) we have
\[
2|a_2|^2 \leq (A - B) \left[ 1 + (-B - 1)|c_1|^2 + 1 + (-B - 1)|d_1|^2 \right].
\]

Upon substituting \( \frac{|a_2|^2}{(A-B)^2} \) for \( |c_1|^2 \) and \( |d_1|^2 \), we obtain
\[
2|a_2|^2 \leq (A - B) \left[ 2 - \frac{2(1+B)}{(A-B)^2}|a_2|^2 \right] = 2(A-B) - \frac{2(1+B)}{(A-B)}|a_2|^2.
\]
A simple algebraic manipulation reveals that

\[ |d_2| \leq \frac{A - B}{\sqrt{1 + A}}. \]

Obviously, for \( A > 0 \) we have

\[ \frac{A - B}{\sqrt{1 + A}} < A - B. \]

For the second part of the Theorem 2.4, rewrite equation (13) as

\[ 2 \left( a_3 - a_2^2 \right) = (A - B)[Bd_1^2 - d_2] + a_2^2. \]

Upon substituting \((A - B)d_1^2\) for \(a_2^2\) we obtain

\[ 2 \left( a_3 - a_2^2 \right) = -(A - B)[d_2 - Ad_1^2]. \]

Taking the absolute values of both sides gives

\[ 2 |a_3 - a_2^2| \leq (A - B) \left[ d_2 + (-A)d_1^2 \right]. \]

If \( A \leq 0 \), then by (11) we have

\[ 2 |a_3 - a_2^2| \leq (A - B) \left( 1 + (-A - 1)|d_1|^2 \right) \]

which upon re-substituting for \(|d_1|^2 = \frac{|a_2|^2}{(A-B)^2}\) we obtain

\[ |a_3 - a_2^2| \leq \frac{A - B}{2} \left[ 1 - \frac{1 + A}{(A - B)^2}|a_2|^2 \right]. \]

For \( A > 0 \), we subtract (12) from (13) to get

\[ 4 \left( a_3 - a_2^2 \right) = (A - B) \left[ B(d_1^2 - c_1^2) + (c_2 - d_2) \right]. \]

Using the fact that \( c_1^2 = d_1^2 \) and taking the absolute values of both sides of the above equation, we obtain the desired inequality

\[ |a_3 - a_2^2| \leq \frac{(A - B)|c_2 - d_2|}{4} \leq \frac{(A - B)(|c_2| + |d_2|)}{4} \leq \frac{A - B}{2}. \]

\[ \square \]

Remark 2.5. For different values of \( A \) and \( B \), Theorem 2.4 demonstrates the unpredictability of the coefficients of the bi-subordinate functions. Determination of extremal functions for bi-univalent functions (in general) and for bi-subordinate functions (in particular) remains a challenge.

References