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Analytic torsion, dynamical zeta functions and orbital integrals

Torsion analytique, fonctions zêta dynamiques et intégrales orbitales

Shu Shen

Johann von Neumann-Haus, Rudower Chaussee 25, 12489 Berlin, Germany

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ABSTRACT

The purpose of this Note is to prove an identity between the analytic torsion and the value at zero of a dynamical zeta function associated with an acyclic unitarily flat vector bundle on a closed locally symmetric reductive manifold, which solves a conjecture of Fried.

RÉSUMÉ

L'objet de cette Note est de démontrer une égalité entre la torsion analytique et la valeur en zéro d'une fonction zêta dynamique associée à un fibré vectoriel unitairement plat sur une variété compacte localement symétrique réductive. Nous démontrons aussi une conjecture de Fried.

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1. Introduction

Let Z be a closed manifold. Let (F, ∇^F) be a complex unitarily flat vector bundle over Z. Let $H^{\cdot}(Z, F)$ be the cohomology of the sheaf of locally flat sections of F. Assume that $H^{\cdot}(Z, F) = 0$.

Let g^{TZ} be a Riemannian metric on Z, and let g^F be a flat metric on F. Ray and Singer [14] constructed the analytic torsion $\tau_Z(F)$ as a spectral invariant of the associated Hodge Laplacian $\square^Z$ acting on the space $\Omega^{\cdot}(Z, F)$ of differential forms with values in F. It was conjectured by Ray and Singer [14], and it was proved by Cheeger [5] and Müller [12] that the analytic torsion is equal to the Reidemeister torsion, which is constructed using the combinatorial complex associated with a triangulation. Müller [13] generalized this equality to unimodular flat vector bundles. Bismut and Zhang [4] extended the Cheeger–Müller Theorem to non-unitarily flat vector bundles.

Milnor [9] pointed out a remarkable similarity between the Reidemeister torsion and the Weil dynamical zeta function. A quantitative description of their relation was formulated by Fried [6] when Z is a closed hyperbolic manifold. Namely, the value at zero of the Ruelle dynamical zeta function, constructed using the geodesic flow of Z and the holonomy of F, is equal to $\exp(\tau_Z(F))$. Fried [7] conjectured that a similar result holds true for general closed locally homogeneous manifolds.

E-mail address: shenshu@math.hu-berlin.de.

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In this note, we announce a proof, detailed in [16], of the Fried conjecture for closed locally symmetric reductive manifolds. We introduce a Ruelle-type dynamical zeta function, and we show that it is meromorphic on $\mathbf{C}$, holomorphic at 0, and that its value at 0 is equal to $\exp(\tau_Z(F))$.

Let us point out that the proof of the above result, which was given by Moscovici and Stanton [10], based on the Selberg trace formula, does not seem to be complete. We try to provide the proper argument to make it correct. Also, our proof is based on the explicit formula given by Bismut for semisimple orbital integrals [2, Theorem 6.1.1].

2. Analytic torsion as V-invariant

We use the notation in the Introduction, but we do not assume $H^{\cdot}(Z,F)=0$. Let $N^{\Lambda^{\cdot}(T^*Z)}$ be the number operator of $\Lambda^{\cdot}(T^*Z)$. Let P^Z be the orthogonal projection to $\ker(\Box^Z)$. We write $\mathrm{Tr_s}$ for the super trace. For $s\in\mathbf{C}$, $\mathrm{Re}(s)>\frac{1}{2}\dim(Z)$, set

$$\theta(s)=-\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr_s}\left[N^{\Lambda^{\cdot}(T^*Z)}\exp(-t\Box^Z)(1-P^Z)\right]t^{s-1}dt. \tag{1}$$

Classically, $\theta(s)$ has a meromorphic extension to $\mathbf{C}$, which is holomorphic at $s=0$. The analytic torsion is defined by $\tau_Z(F)=\theta'(0)\in\mathbf{R}$.

The V-invariant was introduced by Bismut and Goette [3] for a manifold S equipped with an action of a compact Lie group L, with Lie algebra $\mathfrak{l}$. Indeed, if $a\in\mathfrak{l}$, the V-invariant $V_a(S)\in\mathbf{R}$ is defined. If $B_f\subset S$ is the critical submanifold of an L-invariant Morse–Bott function f on S, a formula relating $V_a(S)$ and $V_a(B_f)$ is given in [3, Theorem 4.10].

Assume for the moment F to be the trivial line bundle $\mathbf{R}$. As explained in [1, Introduction], if $S=LZ$ is the free loop space of Z equipped with its canonical S^1-action, if a is the generator of the Lie algebra of S^1, then at least formally, we have

$$V_a(LZ)=-\frac{1}{2}\tau_Z(\mathbf{R}). \tag{2}$$

Consider the energy functional $E: x_{\cdot}\in LZ\to\frac{1}{2}\int_0^1|\dot{x}_s|^2ds$. The critical submanifold B_E is formed by the closed geodesics. Let $[\Gamma]$ be the set of conjugacy classes in the fundamental group $\Gamma=\pi_1(Z)$. We identify $[\Gamma]$ with the free homotopy space of Z. For $[\gamma]\in[\Gamma]$, let $B_{E,[\gamma]}$ be the set of closed geodesics whose free homotopy class is $[\gamma]$. Then $B_E=\coprod_{[\gamma]\in[\Gamma]}B_{E,[\gamma]}$. Assume that Z is of nonpositive sectional curvature. Then, all the critical points in B_E are local minima, and $B_{E,[1]}\simeq Z$. By [3, Theorem 4.10], if the Euler number of Z vanishes, we have the formal equality

$$V_a(LZ)=\sum_{[\gamma]\in[\Gamma]\backslash\{1\}}V_a(B_{E,[\gamma]}). \tag{3}$$

From (2) and (3), we can expect that, for general acyclic flat vector bundle F, $-\frac{1}{2}\tau_Z(F)$ is a regularized sum of the product of $V_a(B_{E,[\gamma]})$ by the holonomy of F along γ.

3. Our main result

Let G be a connected reductive group, let $\mathfrak{g}$ be its Lie algebra, and let θ be the Cartan involution. Let K be the maximal compact subgroup of G of the points of G that are fixed by θ, and let $\mathfrak{k}$ be its Lie algebra. Let $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$ be the Cartan decomposition. Let B be a nondegenerate bilinear symmetric form on $\mathfrak{g}$ which is invariant under the adjoint action and θ. Assume that B is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Set $X=G/K$. Then B induces a Riemannian metric on the tangent bundle $TX=G\times_K\mathfrak{p}$, such that X is of nonpositive sectional curvature. Let $d_X: X\times X\to\mathbf{R}$ be the distance function on X.

We follow [2, Chapter 3]. Let $\gamma\in G$ be semisimple. After conjugating γ, we can write γ in the form $\gamma=e^ak^{-1}$, $a\in\mathfrak{p}$, $k\in K$ and $\mathrm{Ad}(k)a=a$. Let $Z(\gamma)$ be the centralizer of γ with Lie algebra $\mathfrak{z}(\gamma)$. Then $Z(\gamma)$ is a reductive Lie group with maximal compact subgroup $K(\gamma)=K\cap Z(\gamma)$. Let $Z^0(\gamma)$, $K^0(\gamma)$ be the connected components of the identity in $Z(\gamma)$, $K(\gamma)$. Let $X(\gamma)$ be the subset of X where the displacement function $x\in X\to d_X(x,\gamma x)$ takes its minimum value. The groups $Z(\gamma)$ and $Z^0(\gamma)$ act on $X(\gamma)$, and $X(\gamma)\simeq Z(\gamma)/K(\gamma)=Z^0(\gamma)/K^0(\gamma)$.

Let $\mathfrak{z}^{a,\perp}(\gamma)$ be the orthogonal to a in $\mathfrak{z}(\gamma)$ with respect to B. Then $\mathfrak{z}^{a,\perp}(\gamma)$ is a Lie algebra. Let $Z^{a,\perp,0}(\gamma)$ be the connected Lie subgroup of $Z^0(\gamma)$ associated with $\mathfrak{z}^{a,\perp}(\gamma)$. Then $Z^{a,\perp,0}(\gamma)$ is still a reductive Lie group with maximal compact subgroup $K^0(\gamma)$. Set $X^{a,\perp}(\gamma)=Z^{a,\perp,0}(\gamma)/K^0(\gamma)$. By [2, (3.3.11)], if $a\neq 0$, we have

$$Z^0(\gamma)=\mathbf{R}\times Z^{a,\perp,0}(\gamma),\quad X(\gamma)=\mathbf{R}\times X^{a,\perp}(\gamma). \tag{4}$$

Let $e(TX^{a,\perp}(\gamma),\nabla^{TX^{a,\perp}(\gamma)})$ be the Euler form on $TX^{a,\perp}(\gamma)$ associated with the Levi–Civita connection $\nabla^{TX^{a,\perp}(\gamma)}$. Let $o(TX^{a,\perp}(\gamma))$ be the orientation line of $TX^{a,\perp}(\gamma)$. If $\alpha\in\Omega^{\cdot}\left(X^{a,\perp}(\gamma),o(TX^{a,\perp}(\gamma))\right)$ is a $Z^{a,\perp,0}(\gamma)$-invariant form of maximal degree, set $[\alpha]^{\max}\in\mathbf{R}$ such that $\alpha=[\alpha]^{\max}dv_{X^{a,\perp}(\gamma)}$, where $dv_{X^{a,\perp}(\gamma)}$ is the Riemannian volume on $X^{a,\perp}(\gamma)$.

Let Γ be a discrete cocompact torsion free subgroup of G. Then $\Gamma\backslash\{1\}$ consists of nonelliptic semisimple elements. Set $Z=\Gamma\backslash X$. Then Z is a closed locally symmetric manifold with $\pi_1(Z)=\Gamma$. By [11, Lemma 8.1], $\Gamma\cap Z(\gamma)$ is a discrete cocompact torsion free subgroup of $Z(\gamma)$. Also, $\Gamma \cap Z(\gamma)\backslash X(\gamma)$ is equipped with an S^1-action induced by e^{ta}, for $t \in \mathbf{R}$. For $x \in X(\gamma)$, the unique geodesic in X connecting x and γx descends to a closed geodesic in Z, whose length is equal to $|a|$. We have a canonical identification $\Gamma \cap Z(\gamma)\backslash X(\gamma) \simeq B_{E,[\gamma]}$. Let $\mathrm{vol}\big(\Gamma \cap Z(\gamma)\backslash X(\gamma)\big)$ be the volume of $\Gamma \cap Z(\gamma)\backslash X(\gamma)$ associated with the Riemannian metric induced by B. In [16], we show:

Proposition 3.1. *Assume $\gamma \in \Gamma\backslash\{1\}$. The following identity holds:*

$$V_a\big(B_{E,[\gamma]}\big) = -\frac{\mathrm{vol}\big(\Gamma \cap Z(\gamma)\backslash X(\gamma)\big)}{2|a|}\left[e\left(TX^{a,\perp}(\gamma), \nabla^{TX^{a,\perp}(\gamma)}\right)\right]^{\max}. \tag{5}$$

For $r \in \mathbf{N}$, let $\rho : \Gamma \to \mathrm{U}(r)$ be a unitary representation of Γ, and let F be the induced unitarily flat vector bundle on Z. The main result contained in this note is an announcement of the solution of the Fried conjecture. The proof is detailed in [16].

Theorem 3.1. *Assume that $H^\cdot(Z, F) = 0$. The dynamical zeta function*

$$R_\rho(\sigma) = \exp\Big(-2\sum_{[\gamma]\in[\Gamma]\backslash\{1\}} \mathrm{Tr}[\rho(\gamma)]V_a\big(B_{E,[\gamma]}\big)e^{-\sigma|a|}\Big), \tag{6}$$

is well defined for $\sigma \in \mathbf{C}$ and $\mathrm{Re}(\sigma) \gg 1$. It has a meromorphic extension to $\mathbf{C}$ which is holomorphic at 0 and is such that $R_\rho(0) = \exp(\tau_Z(F))$.

Proof. We have the identification $\Omega^\cdot(X) = C^\infty(G, \Lambda^\cdot(\mathfrak{p}^*))^K$. The group Γ acts on $C^\infty(G, \Lambda^\cdot(\mathfrak{p}^*))^K \otimes_{\mathbf{R}} \mathbf{C}^r$ so that if $\gamma \in \Gamma$, $s \in C^\infty(G, \Lambda^\cdot(\mathfrak{p}^*))^K \otimes_{\mathbf{R}} \mathbf{C}^r$, then $\gamma s(g) = \rho(\gamma)s(\gamma^{-1}g)$. Its Γ-invariant subspace is identified with $\Omega^\cdot(Z, F)$. By [2, Proposition 7.8.1], the Casimir element $C^{\mathfrak{g}}$ of G acting on $\Omega^\cdot(Z, F)$ coincides with the Hodge Laplacian $\square^Z$. The trace of the heat operator of $\square^Z$ can be computed using Selberg's trace formula. It contains orbital integrals that will be evaluated using Bismut's formula for semisimple orbital integrals [2, Theorem 6.1.1, Sections 7.8 and 7.9].

Let $\delta(G) \in \mathbf{N}$ be the difference between the complex ranks of G and K. We carry out the proof for the different values of $\delta(G)$. If $\delta(G) = 0$, by showing that the Euler number of Z never vanishes, we conclude that there are no acyclic flat vector bundles. If $\delta(G) \geqslant 2$, using (5), we show that for all $\gamma \in \Gamma\backslash\{1\}$, $V_a\big(B_{E,[\gamma]}\big) = 0$. Also, it is known [2, Theorem 7.9.1] or [10] that $\tau_Z(F) = 0$. We shall explain a detailed proof when $\delta(G) = 1$ in the following two sections. □

4. A first step of the proof of Theorem 3.1 in the case $\delta(G) = 1$

We assume $\delta(G) = 1$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. Set $\mathfrak{b} = \{a \in \mathfrak{p} : [a, \mathfrak{t}] = 0\}$. Then $\dim\mathfrak{b} = 1$. For $a \in \mathfrak{b}$ nonzero, put $M = Z^{a,\perp,0}(e^a)$, $K_M = K^0(e^a)$ and $\mathfrak{m} = \mathfrak{z}^{a,\perp}(e^a)$ with Cartan decomposition $\mathfrak{m} = \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}$. Let $\mathfrak{p}_{\mathfrak{m}}^\perp \subset \mathfrak{p}$, $\mathfrak{k}_{\mathfrak{m}}^\perp \subset \mathfrak{k}$ be such that the orthogonal decompositions $\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{p}_{\mathfrak{m}}^\perp$, $\mathfrak{k} = \mathfrak{k}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}^\perp$ hold. The dimension of $\mathfrak{p}_{\mathfrak{m}}^\perp$ is even. Put $l = \dim(\mathfrak{p}_{\mathfrak{m}}^\perp)/2$.

The group K_M acts on $\mathfrak{p}_{\mathfrak{m}}$ and $\mathfrak{p}_{\mathfrak{m}}^\perp$. A key observation in [16], which remedies the gaps in [10], is that the representations $\mathfrak{p}_{\mathfrak{m}}$ and $\mathfrak{p}_{\mathfrak{m}}^\perp$ of K_M lift to elements in the real representation ring $R(K)$ of K. Therefore, for $0 \leqslant j \leqslant 2l$, there exist $\mathbf{Z}/2\mathbf{Z}$-representations $E_j = E_j^+ \oplus E_j^-$ of K so that the identities in $R(K)$ hold:

$$\sum_{i=0}^{\dim(\mathfrak{p}_{\mathfrak{m}})} (-1)^i\Lambda^i(\mathfrak{p}_{\mathfrak{m}}^*) \otimes \Lambda^j(\mathfrak{p}_{\mathfrak{m}}^{\perp,*}) = E_j^+ - E_j^-, \qquad \sum_{i=1}^{\dim(\mathfrak{p})} (-1)^{i-1} i\Lambda^i(\mathfrak{p}^*) = \sum_{j=0}^{2l}(-1)^j(E_j^+ - E_j^-). \tag{7}$$

Let $F_j^\pm = G \times_K E_j^\pm$ be the associated homogeneous vector bundle on X. It descends to a vector bundle on Z, which we still denote by $F_j^\pm$. The Casimir operator of G descends to an operator $C^{\mathfrak{g},Z,\rho,E_j^\pm}$ acting on $C^\infty(Z, F_j^\pm \otimes F)$. Put $r_j = \dim\ker(C^{\mathfrak{g},Z,\rho,E_j^+}) - \dim\ker(C^{\mathfrak{g},Z,\rho,E_j^-})$. Let $\alpha = \mathrm{ad}|_{\mathfrak{b}} \in \mathfrak{b}^* \otimes \mathrm{End}(\mathfrak{g})$, and let $|\alpha|$ be the norm of α relative to the scalar product B on $\mathfrak{b}$ and to the operator norm on $\mathrm{End}(\mathfrak{g})$. Set

$$C_\rho = \prod_{j=0}^{l-1}\big(-4|\alpha|^2(l-j)^2\big)^{(-1)^{j-1}r_j}, \qquad r_\rho = 2\sum_{j=0}^{l}(-1)^{j-1}r_j. \tag{8}$$

In the first step of the proof of Theorem 3.1 in the case $\delta(G) = 1$, we show the following statement, in which we do not need to assume $H^\cdot(Z, F) = 0$.

Theorem 4.1. *The dynamical zeta function $R_\rho(\sigma)$ is well defined for $\sigma \in \mathbf{C}$ and $\mathrm{Re}(\sigma) \gg 1$. It has a meromorphic extension to $\mathbf{C}$. When $\sigma \to 0$, we have*

$$R_\rho(\sigma) = C_\rho \exp(\tau_Z(F))\sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho+1}). \tag{9}$$

Proof. The proof is based on a relation obtained via the Bismut formula for a semisimple orbital integral [2, Theorem 6.1.1], between the meromorphic functions $R_\rho(\sigma)$ and the function $\det(\sigma + C^{\mathfrak{g},Z,\rho,E_j^+})/\det(\sigma + C^{\mathfrak{g},Z,\rho,E_j^-})$, $(0 \le j \le l)$, where det is a regularized determinant. □

5. A second step of the proof of Theorem 3.1 in the case $\delta(G) = 1$

Let $\pi : G \to \Gamma\backslash G$ be the natural projection. Then $L^2(\Gamma\backslash G, \pi^* F)$ is a right G-space. Let V be the subspace of $L^2(\Gamma\backslash G, \pi^* F)$ on which the action of the center $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ vanishes. The compactness of $\Gamma\backslash G$ implies that V is a finite sum of irreducible unitary representations of G. By the classification theory of irreducible unitary representations with nonzero $(\mathfrak{g}, K)$-cohomology [17,18] and with vanishing $\mathcal{Z}(\mathfrak{g})$-action [15], we deduce that $H^{\cdot}(Z, F) = 0$ if and only if $V = 0$.

Theorem 5.1. *Assume $\delta(G) = 1$. If $H^{\cdot}(Z, F) = 0$, for $0 \leqslant j \leqslant 2l$, $r_j = 0$.*

Proof. The proof is based on the Hecht–Schmid formula [8] relating r_j with certain Lie algebra cohomologies of V. □

End of the proof of Theorem 3.1. By (8) and Theorem 5.1, if $H^{\cdot}(Z, F) = 0$, we have $C_\rho = 1$ and $r_\rho = 0$. From (9), Theorem 3.1 follows. □

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