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Hochschild cohomology of poset algebras and Steenrod operations

La cohomologie de Hochschild des algèbres posets et les opérations de Steenrod

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ABSTRACT

We show that the isomorphism from the Hochschild cohomology of a poset algebra A to the simplicial cohomology of the classifying space of the category associated with A maps Gerstenhaber's pre-Lie product to Steenrod's cup-one product. On cochains, this map becomes an isomorphism of differential graded homotopy commutative algebras.

RÉSUMÉ

On démontre que l'isomorphisme de la cohomologie de Hochschild d'une algèbre poset A à la cohomologie simpliciale du classifiant de la catégorie associé à A applique le produit pré-Lie de Gerstenhaber au produit *cup-one* de Steenrod. Sur les cochaînes, cette application devient un isomorphisme des algèbres différentielles graduées commutatives à homotopie près.

1. Introduction

Hochschild cohomology, HH^*, is a topic of current research as a target for topological quantum field theories [2,10,12] with the cobordism between two disjoint copies of the unit circle, S^1, to one copy of S^1 corresponding to the product in HH^*. The product on HH^* is known to be graded commutative via two possible cochain homotopies, namely the pre-Lie products [3]. These two homotopies are themselves not cochain homotopic, which mirrors the structure for the homology of a double loop space. The simplicial (or singular) cohomology of a topological space is graded commutative under the simplicial cup-product via two possible cochain homotopies, namely the cup-one products [11]. In this case, however, these two cochain homotopies are themselves cochain homotopic via the cup-two product. In this note, we study poset algebras

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for which the Hochschild cochain complex is isomorphic as a differential graded homotopy commutative algebra to the simplicial cochains on the classifying space of the category given by the poset algebra.

Let $P = \{i, j, \dots\}$ be a finite poset of cardinality N containing no cycles and partial order denoted $\preccurlyeq$. Let k be a commutative ring with unit and let $A = A(P)$ be the poset algebra of upper triangular matrices with k-module basis given by e_{ij}, $i \preccurlyeq j$, subject to the relations

$$e_{ij}e_{k\ell} = e_{i\ell},\ j = k, \qquad e_{ij}e_{k\ell} = 0,\ j \neq k.$$

For P two types of cohomology can be computed, the Hochschild cohomology, $HH^*(A;\, A)$, and the simplicial cohomology, $H^*(BC;\, k)$, where BC is the classifying space of the category C with objects given by the elements of P and morphisms

$$\mathrm{Mor}(i,\ j) = e_{ij},\ i \preccurlyeq j, \qquad \mathrm{Mor}(i,\ j) = \emptyset,\ \text{otherwise}.$$

Composition of morphisms agrees with the product above. Note that A contains a separable subalgebra

$$S = \langle e_{11}, e_{22}, \dots, e_{NN}\rangle,$$

namely the k-module generated by the elements e_{ii}, $i = 1, 2, \dots, N$.

Let $\mathrm{Hom}_k(A^{\otimes *},\ A;\ S)$ denote the Hochschild cochains relative to the subalgebra S. Let C^* denote the cochain complex for $H^*(BC;\, k)$. There is map of cochain complexes

$$\Phi_n : \mathrm{Hom}_k(A^{\otimes n},\ A;\ S) \to C^n$$

that induces an isomorphism on cohomology [4,5]

$$\Phi^* : HH^*(A;\ A) \to H^*(BC;\ k).$$

Moreover, $\Phi = \{\Phi_n\}_{n \geq 0}$ preserves products, i.e., Φ maps the Gerstenhaber product on $\mathrm{Hom}_k(A^{\otimes *},\ A;\ S)$ to the simplicial cup product on C^*. In this note, we show that Φ maps Gerstenhaber's pre-Lie product on $\mathrm{Hom}_k(A^{\otimes *},\ A;\ S)$ to Steenrod's cup-one product on C^*. Thus, Φ becomes an isomorphism between the differential graded homotopy commutative algebras (in fact E_∞-algebras) $\mathrm{Hom}_k(A^{\otimes *},\ A;\ S)$ and C^*.

2. Hochschild cohomology

Let A be an associative algebra over a unital commutative and associative ring k. To establish notation and set sign conventions, recall that Hochschild's original definition [8,9] for $HH^*(A;\ A)$, the Hochschild cohomology of A with coefficients in the bimodule A is given as the homology of the cochain complex:

$$\mathrm{Hom}_k(k,\ A) \xrightarrow{\delta} \mathrm{Hom}_k(A,\ A) \xrightarrow{\delta} \dots\dots \xrightarrow{\delta} \mathrm{Hom}_k(A^{\otimes n},\ A) \xrightarrow{\delta} \mathrm{Hom}_k(A^{\otimes (n+1)},\ A) \xrightarrow{\delta} \dots .$$

For a k-linear map $f : A^{\otimes n} \to A$, the coboundary $\delta f : A^{\otimes (n+1)} \to A$ is given by

$$\begin{aligned}&(\delta f)(a_1, a_2, \dots, a_{n+1})\\ &\quad = a_1 f(a_2, \dots, a_{n+1}) + \Big(\sum_{i=1}^{n} (-1)^i f(a_1, a_2, \dots, a_i a_{i+1}, \dots, a_n)\Big) + (-1)^{n+1} f(a_1, a_2, \dots, a_n)a_{n+1}.\end{aligned}$$

For $f \in \mathrm{Hom}_k(A^{\otimes p},\ A)$ and $g \in \mathrm{Hom}_k(A^{\otimes q},\ A)$, the Gerstenhaber (cup) product [3]

$$f \underset{G}{\cdot} g \in \mathrm{Hom}_k(A^{\otimes (p+q)},\ A)$$

is given by

$$(f \underset{G}{\cdot} g)(a_1, a_2, \dots, a_{p+q}) = f(a_1, \dots, a_p) \cdot g(a_{p+1}, \dots, a_{p+q}).$$

Gerstenhaber proves that $f \underset{G}{\cdot} g$ induces a graded commutative product on $HH^*(A;\ A)$. The cochain homotopy between $f \underset{G}{\cdot} g$ and $(-1)^{pq} g \underset{G}{\cdot} f$ is given in terms of the partial compositions $f \underset{(j)}{\circ} g \in \mathrm{Hom}_k(A^{\otimes (p+q-1)},\ A)$, where

$$(f \underset{(j)}{\circ} g)(a_1, a_2, \dots, a_{p+q-1}) = f(a_1, \dots, a_j,\ g(a_{j+1}, \dots, a_{j+q}),\ a_{j+q+1}, \dots, a_{p+q-1}).$$

We use the following sign convention for the pre-Lie product of f and g:

$$f \circ g = \sum_{j=0}^{p-1} (-1)^{(p-1-j)(q-1)} f \underset{(j)}{\circ} g. \tag{1}$$

For any separable algebra S, $HH^n(S; S) = 0$ for $n \geq 1$ and when S is a separable subalgebra of A, $HH^*(A; A)$ can be computed from a subcomplex of $\mathrm{Hom}_k(A^{\otimes *}, A)$ consisting of S-relative cochains. Specifically, a cochain $\varphi \in \mathrm{Hom}_k(A^{\otimes n}, A)$ is S-relative if for all $a_1, a_2, \ldots, a_n \in A$ and $s \in S$, we have

$$\varphi(a_1, \ldots, a_i s, a_{i+1}, \ldots, a_n) = \varphi(a_1, \ldots, a_i, s a_{i+1}, \ldots, a_n), \; 1 \leq i \leq n-1,$$

$$\varphi(s a_1, \ldots, a_n) = s\varphi(a_1, \ldots, a_n), \;\; \varphi(a_1, \ldots, a_n s) = \varphi(a_1, \ldots, a_n)s.$$

Let $\mathrm{Hom}_k(A^{\otimes n}, A; S)$ denote the submodule of S-relative cochains. For a poset algebra $A = A(P)$, and $S = \langle e_{11}, e_{22}, \ldots, e_{NN}\rangle$, an element $\varphi \in \mathrm{Hom}_k(A^{\otimes n}, A; S)$ is determined by

$$\varphi(e_{i_0 i_1}, e_{i_1 i_2}, \ldots, e_{i_{n-1} i_n}),$$

where $i_0 \preccurlyeq i_1 \preccurlyeq i_2 \preccurlyeq \ldots \preccurlyeq i_n$ [5]. Moreover,

$$\varphi(e_{i_0 i_1}, e_{i_1 i_2}, \ldots, e_{i_{n-1} i_n}) = \lambda e_{i_0 i_n}$$

for some $\lambda \in k$. We shall introduce the notation

$$e_{i_0, i_1, i_2, \ldots, i_n} = (e_{i_0 i_1}, e_{i_1 i_2}, \ldots, e_{i_{n-1} i_n}) = e_{i_0 i_1} \otimes e_{i_1 i_2} \otimes \ldots \otimes e_{i_{n-1} i_n},$$

$$\varphi(e_{i_0, i_1, i_2, \ldots, i_n}) = (\lambda_{i_0, i_1, i_2, \ldots, i_n}) e_{i_0 i_n}.$$

Elements $\varphi \in \mathrm{Hom}_k(k, A; S)$ are of the form $\varphi(1) = \sum_{i=1}^N \lambda_i e_{ii}$. For $\varphi \in \mathrm{Hom}_k(A^{\otimes n}, A; S)$, $n = 0, 1, 2, \ldots$, a direct calculation yields

$$(\delta\varphi)(e_{i_0, i_1, i_2, \ldots, i_{n+1}}) = \sum_{j=0}^{n+1} (-1)^j \lambda_{i_0, i_1, \ldots, \hat{i}_j, \ldots, i_{n+1}} e_{i_0 i_{n+1}},$$

where $\hat{i}_j$ means that the index i_j is deleted.

3. Simplicial cohomology

For a finite poset $P = \{i, j, \ldots\}$ of cardinality N, consider the category C with objects given by the elements of P. The morphisms $\mathrm{Mor}(i, j) = \mathrm{Hom}(i, j)$ are given by the elements e_{ij}, $i \preccurlyeq j$, although to distinguish elements of $A(P)$ from morphisms, we introduce the notation

$$\mathrm{Hom}(i, j) = f_{ij}, \; i \preccurlyeq j, \qquad \mathrm{Hom}(i, j) = \emptyset, \;\; \text{otherwise}.$$

Of course, f_{ii} is the identity morphism, and the composition $f_{ij} f_{k\ell}$ is defined only when $j = k$, in which case $f_{ij} f_{j\ell} = f_{i\ell}$. The nerve $N_*(C)$ of this category is the simplicial set with $N_0 C = \{i \mid i \in P\}$, $N_1 C = \{f_{ij} \mid i \preccurlyeq j\}$,

$$N_n C = \{(f_{i_0 i_1}, f_{i_1 i_2}, \ldots, f_{i_{n-1} i_n}) \mid i_k \in P, \; i_0 \preccurlyeq i_1 \preccurlyeq \ldots \preccurlyeq i_n\}.$$

The face maps $d_0, d_1 : N_1 C \to N_0 C$ are $d_0(f_{ij}) = j$ and $d_1(f_{ij}) = i$. We introduce the notation $f_{i_0, i_1, \ldots, i_n} = (f_{i_0 i_1}, f_{i_1 i_2}, \ldots, f_{i_{n-1} i_n})$. For $n \geq 2$, the face maps $d_k : N_n C \to N_{n-1} C$, $k = 0, 1, 2, \ldots, n$, are given by

$$d_k(f_{i_0, i_1, \ldots, i_n}) = f_{i_0, i_1, \ldots, \hat{i}_k, \ldots, i_n}.$$

The degeneracy $s_0 : N_0 C \to N_1 C$ is $s_0(i) = f_i = f_{ii}$. For $n \geq 1$, $s_k : N_n C \to N_{n+1} C$, $k = 0, 1, 2, \ldots, n$, the degeneracies are

$$s_k(f_{i_0, i_1, \ldots, i_n}) = f_{i_0, i_1, \ldots, i_k, i_k, \ldots, i_n}.$$

By definition, the classifying space BC is the geometric realization of $N_* C$. The homology groups $H_*(BC; k)$ can be computed from the chain complex

$$k[N_0 C] \xleftarrow{d} \cdots \xleftarrow{d} k[N_{n-1} C] \xleftarrow{d} k[N_n C] \xleftarrow{d} \cdots,$$

where $d = \sum_{i=0}^n (-1)^i d_i$. The cohomology groups $H^*(BC; k)$ are computed using the Hom_k-dual complex $(\mathrm{Hom}_k(k[N_* C], k), d^*)$. For $\alpha \in \mathrm{Hom}_k(k[N_n C], k)$, we introduce the notation

$$\alpha(f_{i_0, i_1, \ldots, i_n}) = \mu_{i_0, i_1, \ldots, i_n} \in k.$$

From the definition of d^*, it follows that

$$d^*(\alpha)(f_{i_0, i_1, \ldots, i_{n+1}}) = \sum_{j=0}^{n+1} (-1)^j \mu_{i_0, i_1, \ldots, \hat{i}_j, \ldots, i_{n+1}},$$

where $\hat{i}_j$ means that i_j is deleted.

Let $C^n = \mathrm{Hom}_k(k[N_n C], k)$. Clearly there is an isomorphism of cochain complexes

$$\Phi : \mathrm{Hom}_k(A^{\otimes *}, A; S) \to C^*, \quad \Phi_n : \mathrm{Hom}_k(A^{\otimes n}, A; S) \to C^n$$

given by the k-linear map determined via $\Phi_0(\varphi)(i) = \lambda_i$, where $\varphi(1) = \sum_{j=1}^N \lambda_j e_{jj}$,

$$\Phi_n(\varphi)(f_{i_0, i_1, \dots, i_n}) = \lambda_{i_0, i_1, \dots, i_n},$$

where $\varphi(e_{i_0, i_1, \dots, i_n}) = (\lambda_{i_0, i_1, \dots, i_n}) e_{i_0 i_n}$. The inverse $\Psi : C^* \to \mathrm{Hom}_k(A^{\otimes *}, A; S)$ is determined by $\Psi_0(\alpha)(1) = \sum_{i=1}^N \mu_i e_{ii}$, where $\alpha(i) = \mu_i$, and

$$\Psi_n(\alpha)(e_{i_0, i_1, \dots, i_n}) = (\mu_{i_0, i_1, \dots, i_n}) e_{i_0 i_n},$$

where $\alpha(f_{i_0, i_1, \dots, i_n}) = \mu_{i_0, i_1, \dots, i_n}$.

The product on $H^*(BC; k)$ is the simplicial cup product. For $\alpha \in C^p$, $\beta \in C^q$, we have $\alpha \underset{S}{\cdot} \beta \in C^{p+q}$, with

$$(\alpha \underset{S}{\cdot} \beta)(\sigma) = \alpha(f_p(\sigma)) \beta(b_q(\sigma)),$$

where $f_p(\sigma)$ is the front p-face of $\sigma \in N_{p+q}C$ and $b_q(\sigma)$ is the back q-face of σ. Specifically,

$$(\alpha \underset{S}{\cdot} \beta)(f_{i_0, i_1, \dots, i_{p+q}}) = \alpha(f_{i_0, i_1, \dots, i_p}) \beta(f_{i_p, i_{p+1}, \dots, i_{p+q}}).$$

It can be easily checked that Φ is a map of (differential graded) algebras, i.e., $\Phi(\varphi \underset{G}{\cdot} \xi) = \Phi(\varphi) \underset{S}{\cdot} \Phi(\xi)$. To show that Φ is a map of (differential graded) homotopy commutative algebras, we must first specify the cochain homotopy between $\alpha \underset{S}{\cdot} \beta$ and $(-1)^{pq} \beta \underset{S}{\cdot} \alpha$, namely the cup-one product [11]. For $\alpha \in C^p$, $\beta \in C^q$, recall that the cup-one product, $\alpha \underset{1,S}{\cdot} \beta \in C^{p+q-1}$, can be written in terms of the face maps d_i as

$$(\alpha \underset{1,S}{\cdot} \beta)(\sigma) = \sum_{j=0}^{p-1} (-1)^{(p-1-j)(q-1)} (\alpha \underset{1,S}{\cdot} \beta)_j(\sigma), \quad \sigma \in N_{p+q-1}C,$$
$$(\alpha \underset{1,S}{\cdot} \beta)_j(\sigma) = \alpha((d_{j+1} d_{j+2} \dots d_{j+q-1})(\sigma)) \cdot \beta((d_0 d_1 \dots d_{j-1} d_{j+q+1} d_{j+q+2} \dots d_{p+q-1})(\sigma)). \tag{2}$$

4. A map of homotopy commutative algebras

In this section we show that the cochain map

$$\Phi : \mathrm{Hom}_k(A^{\otimes *}, A; S) \to C^*$$

takes Gerstenhaber's pre-Lie product to Steenrod's cup-one product.

Theorem 4.1. *Let $A = A(P)$ be the poset algebra for a finite poset P over a commutative unital ground ring k. Let $\varphi \in \mathrm{Hom}_k(A^{\otimes p}, A; S)$ and $\xi \in \mathrm{Hom}_k(A^{\otimes q}, A; S)$. Then*

$$\Phi(\varphi \underset{(j)}{\circ} \xi) = (\Phi(\varphi) \underset{1,S}{\cdot} \Phi(\xi))_j.$$

Proof. First, we offer a computational proof. For $m_0 \preccurlyeq m_1 \preccurlyeq \dots \preccurlyeq m_p$ and $\ell_0 \preccurlyeq \ell_1 \preccurlyeq \dots \preccurlyeq \ell_q$, let

$$\varphi(e_{m_0, m_1, \dots, m_p}) = (\lambda_{m_0, m_1, \dots, m_p}) e_{m_0 m_p}, \quad \xi(e_{\ell_0, \ell_1, \dots, \ell_q}) = (\mu_{\ell_0, \ell_1, \dots, \ell_q}) e_{\ell_0 \ell_q}.$$

Then

$$(\varphi \underset{(j)}{\circ} \xi)(e_{i_0, i_1, \dots, e_{p+q-1}}) = \varphi(e_{i_0, i_1, \dots, i_j} \otimes \xi(e_{i_j, i_{j+1}, \dots, i_{j+q}}) \otimes e_{i_{j+q}, \dots, i_{p+q-1}})$$
$$= (\lambda_{i_0, i_1, \dots, i_j, i_{j+q}, i_{j+q+1}, \dots, i_{p+q-1}})(\mu_{i_j, i_{j+1}, \dots, i_{j+q}}) e_{i_0 i_{p+q-1}}.$$

Thus,

$$\Phi(\varphi \underset{(j)}{\circ} \xi)(f_{i_0, \dots, i_{p+q-1}}) = (\lambda_{i_0, i_1, \dots, i_j, i_{j+q}, i_{j+q+1}, \dots, i_{p+q-1}})(\mu_{i_j, i_{j+1}, \dots, i_{j+q}}).$$

On the other hand, for $\sigma = f_{i_0, i_1, \dots, i_{p+q-1}}$, we have

$$(\Phi(\varphi) \underset{1,S}{\cdot} \Phi(\xi))_j(\sigma) = \Phi(\varphi)(d_{j+1} d_{j+2} \dots d_{j+q-1}(\sigma)) \cdot \Phi(\xi)(d_0 d_1 \dots d_{j-1} d_{j+q+1} d_{j+q+2} \dots d_{p+q-1}(\sigma))$$
$$= \Phi(\varphi)(f_{i_0, i_1, \dots, i_j, i_{j+q}, \dots, i_{p+q-1}}) \cdot \Phi(\xi)(f_{i_j, i_{j+1}, \dots, i_{j+q}}) = (\lambda_{i_0, i_1, \dots, i_j, i_{j+q}, i_{j+q+1}, \dots, i_{p+q-1}})(\mu_{i_j, i_{j+1}, \dots, i_{j+q}}).$$

Thus, $\Phi(\varphi \underset{(j)}{\circ} \xi) = (\Phi(\varphi) \underset{1,S}{\cdot} \Phi(\xi))_j$ for $j = 0, 1, 2, \ldots, p-1$. □

An alternative proof of Theorem 4.1 can be constructed by noting that both $\mathrm{Hom}_k(A^{\otimes n}, A; S)$ and C^* are multiplicative operads [6], and Φ is an isomorphism of multiplicative operads, which yields the above theorem.

Corollary 4.2. *With φ and ξ as in Theorem 4.1, we have*

$$\Phi(\varphi \circ \xi) = \Phi(\varphi) \underset{1,S}{\cdot} \Phi(\xi).$$

Proof. The proof follows from Theorem 4.1, the definition of the pre-Lie product (1) and the definition of the cup-one product (2). □

Since the pre-Lie product agrees with Steenrod's cup-one product for a poset algebra, the cup-two product is a cochain homotopy between $\varphi \circ \xi$ and $(-1)^{(p+1)(q+1)} \xi \circ \varphi$, or equivalently between

$$\Phi(\varphi) \underset{1,S}{\cdot} \Phi(\xi), \quad (-1)^{(p+1)(q+1)} \Phi(\xi) \underset{1,S}{\cdot} \Phi(\varphi).$$

Let $\overline{C}^*$ denote the k-module of normalized simplicial cochains of C^* and let $\overline{\mathrm{Hom}}(A^{\otimes *}, A; S)$ be the k-module of normalized, relative Hochschild cochains [5, p. 205]. Then $\overline{C}^*$ becomes an E_∞-algebra via an action of the Barratt–Eccles operad [1] or equivalently via an action of the Eilenberg–Zilber operad [7]. Since Φ and $\Psi = \Phi^{-1}$ map normalized cochains to normalized cochains, $\overline{\mathrm{Hom}}(A^{\otimes *}, A; S)$ inherits an E_∞-algebra structure via the isomorphism Φ. In the special case of a poset algebra, both the products and homotopies between the products agree on $\mathrm{Hom}(A^{\otimes *}, A; S)$ and C^* as cochains via Φ. In fact, the jth partial composition of the endomorphism operad on $\mathrm{Hom}(A^{\otimes *}, A; S)$ is mapped to the jth summand of the cup-one operation on C^* via Φ. As a result, $\overline{\Phi} : \overline{\mathrm{Hom}}(A^{\otimes *}, A; S) \to \overline{C}^*$ is an isomorphism of E_∞-algebras. Thus, the cup-$(i+1)$ product provides a cochain homotopy between the two possible compositions of cup-i products on $\mathrm{Hom}(A^{\otimes *}, A; S)$. Using the cup-two product as a cochain homotopy between the two compositions of cup-one products, we recover the following [4]:

Corollary 4.3. *For the poset algebra $A = A(P)$, the Lie bracket on $HH^*(A; A)$ is zero, i.e., for cocycles $\varphi \in \mathrm{Hom}_k(A^{\otimes p}, A; S)$ and $\xi \in \mathrm{Hom}_k(A^{\otimes q}, A; S)$,*

$$0 = [\varphi, \xi] = \varphi \circ \xi - (-1)^{(p+1)(q+1)} \xi \circ \varphi$$

as an element in $HH^{p+q-1}(A; A)$.

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