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Dirichlet-to-Neumann operator for diffraction problems in stratified anisotropic acoustic waveguides

Opérateur Dirichlet-to-Neumann pour des problèmes de diffraction dans un guide d'ondes acoustique anisotrope stratifié

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ABSTRACT

The purpose of this note is to construct a Dirichlet-to-Neumann operator for the diffraction problem in stratified anisotropic acoustic waveguides. The key idea consists in using an adapted change of coordinates that enables to recover the *completeness* and the *orthogonality* of the modes on "deformed" cross-sections of the waveguide.

RÉSUMÉ

Le but de cette note est de construire un opérateur Dirichlet-to-Neumann pour le problème de diffraction dans un guide d'ondes acoustique anisotrope stratifié. Le point clé consiste à utiliser un changement de coordonnées adapté qui permet de retrouver à la fois des propriétés de *complétude* et d'*orthogonalité* des modes sur une section « déformée » du guide d'ondes.

Version française abrégée

Nous nous intéressons ici à l'étude et la résolution numérique d'un problème de diffraction dans un guide d'ondes bi-dimensionnel infini acoustique anisotrope et stratifié (représenté sur la Fig. 1). Autrement dit, on cherche à calculer la solution *sortante* des équations (1), où la matrice A est définie par (2). Afin de réduire les calculs à une zone bornée, il existe dans la littérature plusieurs approches (voir, par exemple, [2,4,5]). L'une d'elle, développée dans le cas *isotrope* (i.e. $A = Id$), consiste à écrire des conditions dites transparentes sur des frontières artificielles $\Sigma_a^\pm$ bornant le domaine de calculs (voir [4,5]). Pour cela, on exploite la représentation modale de la solution dans les parties non perturbées $\Omega_a^\pm$ du guide. En effet, la géométrie séparable des demi-guides $\Omega_a^\pm$ permet de montrer, par une technique de séparation de variables, que

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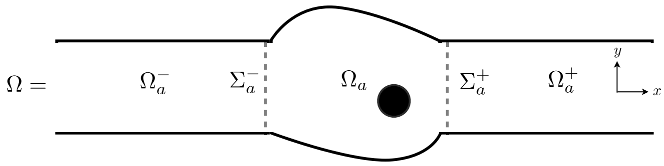

Fig. 1. Geometry of the waveguide and notations. The black dot represents a defect.

la solution p se décompose sous la forme d'une série modale (3). À l'aide de cette forme explicite de la solution, on peut obtenir un opérateur Dirichlet-to-Neumann (DtN) qui relie la trace $p|_{\Sigma_a^\pm}$ à la dérivée normale $\partial_\nu p|_{\Sigma_a^\pm}$ de la solution sortante associée dans $\Omega_a^\pm$.

Dans le cas où $A \neq Id$, les champs transverses des modes ne sont plus les éléments propres d'un problème auto-adjoint, mais sont solutions du problème aux valeurs propres quadratique (4). On montre qu'ils ne sont alors plus *orthogonaux* entre eux (on peut même montrer qu'ils ne forment plus une base de Riesz de $L^2(\Sigma_a^\pm)$), ce qui pose problème pour justifier la décomposition modale (3) dans les demi-guides $\Omega_a^\pm$. L'objectif de cette note est de montrer que la solution se décompose en fait sur des modes dans des demi-guides déformés $\Omega_{a,\alpha}^\pm$, et de prouver qu'on peut obtenir via la décomposition modale (12) un opérateur DtN. Pour cela, l'idée est d'exploiter un changement de coordonnées $(X,Y) = (x + \alpha(y), y)$ qui préserve la géométrie du guide, où $\alpha(y)$ est choisi en fonction de A, voir l'équation (7). Il est connu que changer le système de coordonnées revient à changer l'anisotropie du milieu, voir [3]. Ainsi, on peut réécrire les équations dans les parties non perturbées du guide d'ondes sous la forme (8). Dans ces nouvelles coordonnées, le calcul des modes revient alors à la résolution d'un problème aux valeurs propres linéaires auto-adjoint, ce qui permet de retrouver des propriétés de *complétude* et d'*orthogonalité*, non plus sur les sections droites $\Sigma_a^\pm$ du guide, mais sur des sections déformées $\Sigma_{a,\alpha}^\pm$, voir Fig. 2. On peut alors obtenir des conditions transparentes, réécrire le problème en domaine borné, et montrer comme dans [5] un résultat d'existence et d'unicité, voir théorème 3.1.

1. Introduction

In this note, we are interested in the study and the numerical resolution of the diffraction problem in a stratified anisotropic acoustic bi-dimensional waveguide. In other words, we want to solve the following equations:

$$\begin{cases} -\mathrm{div}(A(x,y)\nabla p) - \omega^2 p = f & \text{in} \quad \Omega, \\ A(x,y)\nabla p \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where p denotes the pressure field, $\omega > 0$ the frequency and ν the outward unitary normal. The geometry Ω of the waveguide is represented in Fig. 1, and can be decomposed into three parts, $\Omega = \Omega_a^- \cup \Omega_a \cup \Omega_a^+$, where $\Omega_a^\pm = \{\pm x \geq a\} \times [-h,h]$, $\Omega_a = \{|x| \leq a\} \cap \Omega$, and $a, h > 0$. h represents the height of the waveguide, and $2a$ the length (in the x-direction) of the bounded subdomain Ω_a. This subdomain Ω_a may be perturbed and may contain defects, as represented in Fig. 1. Moreover, we suppose that $\mathrm{Supp}(f) \subset \Omega_a$. We also denote by $\Sigma_a^\pm = \{x = a\} \times [-h,h]$ the interfaces between Ω_a and $\Omega_a^\pm$. Besides, $A(x,y)$ is a symmetric positive definite matrix that verifies $A(x,y) = A(y)$ if $(x,y) \notin \Omega_a$ and

$$A(y) = \begin{bmatrix} c_1(y) & c_3(y) \\ c_3(y) & c_2(y) \end{bmatrix}, \quad \text{where} \quad \begin{cases} c_1(y), c_2(y) > 0, \\ c_1(y)c_2(y) - [c_3(y)]^2 > 0. \end{cases} \tag{2}$$

In the sequel, for simplicity reasons, we will not recall the dependency of $A(y)$ (and its coefficients) on y, and we will simply denote $A(y)$ by A.

Classically, to define the diffraction problem, we must impose a *radiation condition* at infinity (see, for instance, [4]). This condition ensures that the solution is *outgoing* (that is to say "propagates" toward infinity), and is defined thanks to the modal decomposition of the solution in the half-guides $\Omega_a^\pm$. Modes are particular solutions to the homogeneous equations (1) (i.e. $f = 0$) of the form $e^{i\beta x}\psi(y)$ in a perfect waveguide of separable geometry $\mathbb{R} \times [-h,h]$. In the *isotropic* case (i.e. $A = Id$), simple calculations show that there are two families of modes $\{p_n^\pm\}_{n\in\mathbb{N}}$, which are given by:

$$p_n^\pm(x,y) = e^{\pm i\beta_n x}\psi_n(y), \quad \text{where } \psi_n(y) = N_n \cos\left(\frac{n\pi y}{2h}\right) \text{ and } \beta_n = i\sqrt{\omega^2 - \left(\frac{\pi n}{2h}\right)^2}, \quad \forall n \in \mathbb{N}.$$

We consider the complex square-root with branch-cut on $\mathbb{R}^-$. N_n is a normalization coefficient chosen so that the family $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2([-h,h])$. Supposing that ω is not a cut-off frequency (that is $\beta_n \neq 0, \forall n \in \mathbb{N}$), the modes p_n^+ (resp. p_n^-) are rightgoing (resp. leftgoing). Therefore, we say that p is *outgoing* if it expands in Ω_a^+ (resp. Ω_a^-) as a series of modes p_n^+ (resp. p_n^-). Thus, except at the cut-off frequency, we define the diffraction problem by Eqs. (1) and the *radiation condition*.

Moreover, given a Dirichlet data $g^\pm$ on $\Sigma_a^\pm$, we can express the *outgoing* solution p in the half-guide $\Omega_a^\pm$ verifying $p = g^\pm$ on $\Sigma_a^\pm$. Indeed, thanks to the *orthogonality* of the ψ_n, it comes that

$$p(x,y)=\sum_{n\geq 0}\left(g^{\pm},\psi_n\right)_{L^2([-h,h])}\mathrm{e}^{\pm \mathrm{i}\beta_n(x-a)}\psi_n(y),\quad \text{in}\quad \Omega_a^{\pm}. \tag{3}$$

Using this modal decomposition, we derive the so-called Dirichlet-to-Neumann (DtN) operators $T^{\pm}$ that map the Dirichlet data $g^{\pm}$ to the normal derivatives $\partial_\nu p(g^{\pm})$, where $p(g^{\pm})$ denotes the above function. Thanks to the DtN operators $T^{\pm}$, we can formulate the diffraction problem in the bounded domain Ω_a by writing "transparent" boundary conditions (TBC) $\partial_\nu p = T(p|_{\Sigma_a^{\pm}})$ on $\Sigma_a^{\pm}$ (see [5]).

Now, what about the *anisotropic* case ($A \neq Id$)? We can still compute the modes, at least numerically. Indeed, injecting the form of the solution $\mathrm{e}^{\mathrm{i}\beta x}\psi(y)$ into the equations (1) leads to solve:

$$\begin{cases} -\left(c_2\psi'\right)' - \mathrm{i}\,\beta\left(c_3\psi' + (c_3\psi)'\right) - \left(\omega^2 - c_1\beta^2\right)\psi = 0 & \text{in}\ \ [-h,h], \\ c_2\psi' + \mathrm{i}\,c_3\beta\psi = 0 & \text{on}\ \ \{-h,h\}. \end{cases} \tag{4}$$

In the case where $c_3 = 0$, this problem is a linear self-adjoint eigenvalue problem where β^2 represents the eigenvalue and ψ the eigenfunction. We can then extend the classical results of the isotropic case (that is to say proving *orthogonality* and *completeness* of the transverse part of the modes). The difficulty arises when $c_3 \neq 0$. Indeed, (4) is then a quadratic eigenvalue problem where β is the eigenvalue, and the major point is that the transverse part of the modes are no longer *orthogonal*.

The purpose of this note is to propose a method that enables to compute the modes by solving a *linear* self-adjoint eigenvalue problem, and that enables to recover the *completeness* and the *orthogonality* relations of the transverse parts of the modes, not on the straight section $[-h,h]$ of the waveguide, but on the "deformed" section of the waveguide, which will be clarified in the sequel. Thus, we will be able to derive DtN operators and reformulate the diffraction problem in a bounded domain. A numerical validation is presented at the end of the note.

2. Computation modes

The key idea consists in using an appropriate change of coordinates. Indeed, changing the coordinates amounts to change the anisotropy of the medium. This idea has been intensively used in acoustic cloaking, see for instance [3]. Another application that we can mention is the construction of water-wave guide shifter, see [1]. Let us consider a straight waveguide of geometry $\mathbb{R}\times[-h,h]$. In order to compute the modes, we propose a particular change of coordinates of this form:

$$(X,Y) = (x+\alpha(y), y), \tag{5}$$

where $\alpha(y)$ is a function of the transverse coordinate y. Let us point out that this change of coordinates (5) is not the usual one associated with the anisotropy axes. The advantage of this mapping is that it "eliminates" the extra-diagonal term c_3 while preserving the geometry of the waveguide (because $Y = y$). Indeed, in these new coordinates, the gradient operator ∇ rewrites:

$$\nabla = R\,\widehat{\nabla},\quad R = \begin{bmatrix} 1 & 0 \\ \alpha' & 1 \end{bmatrix},\quad \text{where}\quad \alpha' = \frac{\mathrm{d}\alpha}{\mathrm{d}y}\quad \text{and}\quad \widehat{\nabla} = \begin{bmatrix} \partial_X \\ \partial_Y \end{bmatrix}.$$

Noting that $\mathrm{div} = \nabla^{\mathrm{t}}$, it comes

$$\mathrm{div}\,(A\nabla\cdot) = \widehat{\nabla}^{\mathrm{t}} R^{\mathrm{t}} A R \widehat{\nabla}\,(\cdot)\,.$$

After simple computations, we get that

$$R^{\mathrm{t}}AR = \begin{bmatrix} c & c_3+\alpha' c_2 \\ c_3+\alpha' c_2 & c_2 \end{bmatrix},\quad \text{where}\ \ c = c_1 + 2\alpha' c_3 + \left(\alpha'\right)^2 c_2 > 0. \tag{6}$$

To check that $c>0$, we simply consider the polynomial $P(z) = c_1 + 2c_3 z + c_2 z^2$. Its discriminant $4\left((c_3)^2 - c_1 c_2\right)$ is strictly negative because of (2), and therefore the sign of $P(z)$ for all z is the same as c_2 (or c_1).

Given (6), in order to eliminate the extra-diagonal terms in $R^{\mathrm{t}}AR$, the "smart" choice of α' is $\alpha' = \frac{-c_3}{c_2}$, so that

$$\alpha = \int_{-h}^{y} \frac{-c_3(s)}{c_2(s)}\mathrm{d}s + C^{ste}. \tag{7}$$

The constant term C^{ste} simply corresponds to a shift in the x-direction and can be chosen arbitrarily. In the rest, we will suppose $C^{ste} = 0$. With this choice of α, the equations (1) simplify to

$$-c\partial^2_{XX}\widehat{p} - \partial_Y\left(c_2\partial_Y\widehat{p}\right) - \omega^2\widehat{p} = 0\ \ \text{in}\ \ \mathbb{R}\times[-h,h], \tag{8}$$

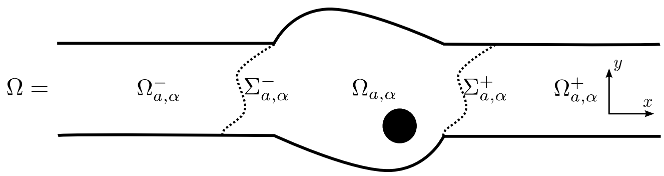

Fig. 2. Geometry of the waveguide and new notations.

where $\widehat{p}(X,Y) = p(x,y)$. We recall that the coefficients c and c_2 depend on the transverse coordinate $Y = y$. Let us remark that, since $c > 0$ and $c_2 > 0$, we are still led to solve a Helmholtz-type equation. Concerning the free boundary conditions $A\nabla p \cdot \nu = 0$ on $\mathbb{R} \times \{-h, h\}$, we have:

$$A\nabla p = AR\widehat{\nabla}\widehat{p} = \begin{bmatrix} c_1 + \alpha' c_3 & c_3 \\ 0 & c_2 \end{bmatrix} \widehat{\nabla}\widehat{p} = \begin{bmatrix} c\partial_X \widehat{p} + c_3 \partial_Y \widehat{p} \\ c_2 \partial_Y \widehat{p} \end{bmatrix} \tag{9}$$

Let us underline that, for our choice of α', $c_1 + \alpha' c_3 = c$. Therefore, the boundary conditions become $c_2 \partial_Y \widehat{p} = 0$, or equivalently $\partial_Y \widehat{p} = 0$. To sum up, this change of coordinates $(X,Y) = (x + \alpha(y), y)$ with α given by (7) enables to

(i) eliminate the cross-derivative in the operator $\mathrm{div}(A\nabla \cdot)$,
(ii) get boundary conditions that only involve derivative with respect to Y,
(iii) preserve the separable geometry $\mathbb{R} \times [-h, h]$ of the waveguide.

Then, we can look for particular solutions the form $e^{i\beta X}\widehat{\psi}(Y)$, which leads to solve the following generalized *linear* and self-adjoint eigenvalue problem:

$$\begin{cases} -\left(c_2 \widehat{\psi}'\right)' = \left(\omega^2 - c\beta^2\right)\widehat{\psi} & \text{in } [-h,h], \\ \widehat{\psi}' = 0 & \text{on } \{\pm h\}, \end{cases} \tag{10}$$

where β^2 is the eigenvalue and ψ the associated eigenvector. Because the operator $-(c_2(\cdot)')'$ is symmetric and positive, we get by classical spectral theory results the following proposition.

Proposition 2.1. *There exists a sequence of positive eigenvalues $\{\lambda_n = \omega^2 - c\beta_n^2\}_{n\in\mathbb{N}}$ and associated eigenfunctions $\{\widehat{\psi}_n\}_{n\in\mathbb{N}}$ solutions to (10). Moreover, the eigenvalues λ_n tend to $+\infty$ with respect to n, and the eigenfunctions $\widehat{\psi}_n$ form a basis of $L^2([-h,h])$ that verifies the orthogonality relations:*

$$\left(\widehat{\psi}_n, \widehat{\psi}_m\right)_c := \int_{-h}^{h} c(s)\widehat{\psi}_n(s)\widehat{\psi}_m(s)\,\mathrm{d}s = 0, \quad \textit{if} \quad \lambda_n \neq \lambda_m.$$

In what follows, we consider that the $\widehat{\psi}_n$ are normalized with respect to the scalar product $(\cdot,\cdot)_c$. Then, the two families of modes $p_n^\pm$ are given by (using the coordinates (x, y))

$$p_n^\pm(x,y) = \widehat{\psi}_n(y)e^{\pm i\beta_n(x+\alpha(y))}, \text{ where } \beta_n = \sqrt{(\omega^2 - \lambda_n)/c}. \tag{11}$$

Supposing that ω is not a cut-off frequency, the family p_n^+ (resp. p_n^-) corresponds to the rightgoing (resp. leftgoing) modes. Therefore, except at the cut-off frequencies that form a countable set, we can define the *radiation condition* for the diffraction problem (1). Let us remark that at a given x, let say $x = 0$, the *evanescent* modes (for which $\beta_n = i|\beta_n|$) are exponentially growing (or decaying, depending on the sign of α) in the cross-section y of the waveguide because of the term $e^{i\alpha\beta_n y} = e^{-\alpha|\beta_n|y}$. Due to this exponential behavior, we can prove that the transverse parts of the modes on straight cross-sections $\Sigma_a^\pm$ are not Riesz bases of $L^2(\Sigma_a^\pm)$.

Now, when considering "deformed" cross-sections, everything goes fine since we recover the *orthogonality* and *completeness* of the transverse parts of the modes.

3. Derivation of the Dirichlet-to-Neumann operator

Let us now come back to our initial problem (1) where Ω is a locally perturbed waveguide. We denote by $\Sigma^\pm_{a,\alpha} = \Omega \cap \{\pm X = a\}$ two deformed cross-sections of the waveguide (see Fig. 2). Let us remark that if A is constant, then $\Sigma^\pm_{a,\alpha}$ are oblique straight lines because $\alpha(y) = -c_3 y/c_2$, and therefore $X = a \Leftrightarrow x - c_3 y/c_2 = a$. Given the traces $p|_{\Sigma^\pm_{a,\alpha}}$ on $\Sigma^\pm_{a,\alpha}$ and recalling that $\{\widehat{\psi}_n\}_{n\in\mathbb{N}}$ defined in Proposition 2.1 is an orthonormal basis of $L^2([-h,h])$, we can get the *outgoing modal decomposition* associated with the solution in $\Omega^\pm_{a,\alpha}$:

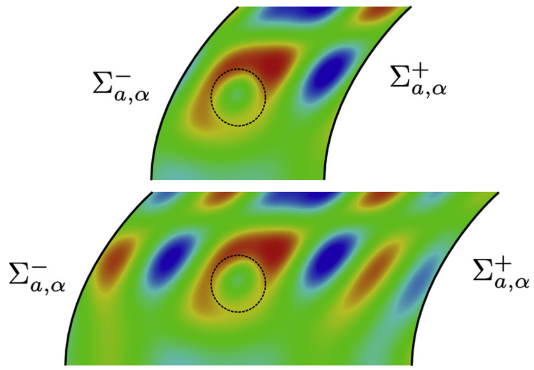

Fig. 3. Solution p^a of the problem (14) for two sizes of $\Omega^\pm_a$. The dotted circle represented the support of f.

$$p(x,y) = \sum_{n\geq 0} \left(p|_{\Sigma^\pm_{a,\alpha}}, \widehat{\psi}_n \right)_c e^{i\beta_n(x+\alpha y)} \widehat{\psi}_n(y) \text{ in } \Omega^\pm_{a,\alpha}. \tag{12}$$

Using these explicit expressions of the solution in $\Omega^\pm_{a,\alpha}$ and noticing that $A\nabla p(x,y)\cdot \nu|_{\Sigma^\pm_{a,\alpha}} = \pm c\partial_X \widehat{p}(X,Y)$, we can derive the DtN operators that map the traces $p|^\pm_{\Sigma_{a,\alpha}}$ to the normal derivatives $A\nabla p\cdot \nu|_{\Sigma^\pm_{a,\alpha}}$ on the boundaries $\Sigma^\pm_{a,\alpha}$:

$$T^\pm_\alpha : p|_{\Sigma^\pm_{a,\alpha}} \in H^{1/2}(\Sigma^\pm_{a,\alpha}) \to T^\pm_\alpha \left(p|_{\Sigma^\pm_{a,\alpha}} \right) = \sum_{n\geq 0} i\beta_n \left(p|_{\Sigma^\pm_{a,\alpha}}, \widehat{\psi}_n \right)_c \widehat{\psi}_n \in H^{-1/2}(\Sigma^\pm_{a,\alpha}). \tag{13}$$

Thus, we can derive TBCs and reformulate the diffraction problem (1) with the *radiation condition* in a bounded domain as follows: Find $p_a \in H^1(\Omega_{a,\alpha})$, solution to

$$\begin{cases} -\text{div}\,(A\nabla p_a) - \omega^2 p_a = f \text{ in } \Omega_{a,\alpha}, \\ A\nabla p_a \cdot \nu = T^\pm_\alpha \left(p|_{\Sigma^\pm_{a,\alpha}} \right) \quad \text{on } \Sigma^\pm_{a,\alpha}. \end{cases} \tag{14}$$

With this formulation, we can prove as in [5] the following result.

Theorem 3.1. *Let us suppose that ω is not a cut-off frequency. The problem (14) (or equivalently (1) with the radiation condition) admits a unique solution in $H^1(\Omega_{a,\alpha})$, except for at most a discrete set of frequencies ω.*

Let us emphasize that in some perturbed waveguides, there exist at some frequencies (which are not the cut-off frequencies) trapped mode solutions to the homogeneous problem (14). Thanks to formulation (14), we can numerically compute the solution in $\Omega^\pm_{a,\alpha}$. To illustrate these "new" TBC, we have considered a particular case of stratified medium:

$$\Omega = \mathbb{R}\times[-h,h], \quad A = \begin{bmatrix} 1 & c_3 \\ c_3 & 1 \end{bmatrix}, \text{ where } c_3(y) = \frac{y+h}{2h(1+0.001)}, \quad h=1, \text{ and } \omega = 10.$$

In Fig. 3, we have represented the real part of the computed solution for two sizes of $\Omega_{a,\alpha}$. As we can see, the conditions on $\Sigma^\pm_{a,\alpha}$ are "transparent" because they have no influence on the computed solution p_a.

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