Differential geometry

Extremal metrics for the $Q'$-curvature in three dimensions

Métriques extrêmes pour la $Q'$-courbure en dimension 3

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\textbf{A B S T R A C T}

We construct contact forms with constant $Q'$-curvature on compact three-dimensional CR manifolds that admit a pseudo-Einstein contact form and satisfy some natural positivity conditions. These contact forms are obtained by minimizing the CR analogue of the $ll$-functional from conformal geometry. Two crucial steps are to show that the $P'$-operator can be regarded as an elliptic pseudodifferential operator and to compute the leading-order terms of the asymptotic expansion of the Green's function for $\sqrt{P'}$.

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\textbf{R É S U M É}

On construit des formes de contact à $Q'$-courbure constante sur les variétés de Cauchy-Riemann de dimension 3 qui admettent une pseudo-forme de contact d'Einstein et satisfont certaines conditions naturelles de positivité. Ces formes sont obtenues en minimisant l'analogue en CR-géométrie de la $ll$-fonctionnelle en géométrie conforme. Cette construction repose sur deux étapes cruciales. On montre que le $P'$-opérateur peut être vu comme un opérateur pseudo-différentiel elliptique et on calcule les termes dominants du développement asymptotique de la forme de Green pour $\sqrt{P'}$.

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\section{1. Introduction}

On an even-dimensional manifold $(M^{2n}, g)$, the pair $(P, Q)$ of the (critical) GJMS operator $P$ and the (critical) $Q$-curvature $Q$ possesses many of the same properties of the pair $(-\Delta, K)$ on surfaces, where $K$ is the Gauss curvature. For example, $P$ is a conformally covariant formally self-adjoint operator with leading-order term $(-\Delta)^{n/2}$ that annihilates...
constants \([14,15]\), and \(Q\) is a Riemannian invariant with leading-order term \(c_n(-\Delta)^{\frac{n-2}{2}} R\), where \(R\) is the scalar curvature, which transforms in a particularly simple way within a conformal class \([4]\): if \(\tilde{g} = e^{2u} g\), then
\[
e^{2u} \tilde{Q} = Q + Pu.
\]
In particular, \(\int Q\) is conformally invariant on closed even-dimensional manifolds; indeed, it computes the Euler characteristic modulo integrals of pointwise conformal invariants \([1]\). It also follows that metrics of constant \(Q\)-curvature within a conformal class are in one-to-one correspondence with critical points of the functional
\[
II[u] = \int_M u Pu + 2 \int_M Q u - \frac{2}{n} \left( \int_M Q \right) \log \left( \frac{1}{\text{Vol}(M)} \int_M e^{nu} \right).
\]
This functional can always be minimized on the two-sphere \([21]\) and on four-manifolds with positive Yamabe constant and nonnegative Paneitz operator \([2,9,16]\), with important applications to logarithmic functional determinants \([5,21]\) and sharp Onofri-type inequalities \([2]\). Due to the parallels between conformal and CR geometry, it is interesting to determine whether a similar pair exists in the latter setting.

Works by Graham and Lee \([13]\) and Hirachi \([17]\) identified CR analogues of the Paneitz operator and \(Q\)-curvature in dimension three. However, the kernel of the Paneitz operator contains the (generally infinite-dimensional) space \(\mathcal{P}\) of CR pluriharmonic functions, and the total \(Q\)-curvature is always zero. In particular, an Onofri-type inequality involving the CR Paneitz operator cannot be satisfied. Branson, Fontana and Morpurgo overcame this latter issue on the CR spheres by introducing a formally self-adjoint operator \(P'\), which is CR covariant on CR pluriharmonic functions and in terms of which one has the sharp Onofri-type inequality
\[
\int_{S^{2n+1}} u P'u + 2 \int_{S^{2n+1}} Q'u - \frac{2}{n+1} \left( \int_{S^{2n+1}} Q' \right) \log \left( \frac{1}{\text{Vol}(S^{2n+1})} \int_{S^{2n+1}} e^{(n+1)u} \right) \geq 0
\]
for all \(u \in W^{n+1,2} \cap \mathcal{P}\), where \(Q'\) is an explicit dimension constant \([6]\). The construction of \(P'\) is analogous to the construction of the \(Q\)-curvature from the GJMS operators by analytic continuation in the dimension.

It was observed by the first- and third-named authors in dimension three \([7]\) and by Hirachi in general dimension \([18]\) that one can define the \(P'\)-operator on general pseudohermitian manifolds \((M^{2n+1}, T^{1,0}, \theta)\). Roughly speaking, if \(P_{2n+2}^N\) is the CR GJMS operator of order \(2n + 2\) on a \((2N + 1)\)-dimensional manifold, one defines \(P'\) as the limit of \(\frac{2}{N-1} P_{2n+2}^N\) as \(N \to n\). This is made rigorous by explicit computation in dimension three \([7]\) and via the ambient metric in general dimension \([18]\). Regarded as a map from \(\mathcal{P}\) to \(C^\infty(M)/\mathcal{P}^\perp\), the \(P'\)-operator is CR covariant: if \(\tilde{\theta} = e^{\theta} \theta\), then \(e^{(n+1)\theta} P' = P'\).

If \(\theta\) is a pseudo-Einstein contact form (cf. \([7,18,20]\)), then the \(P'\)-operator is formally self-adjoint and annihilates constants. Note that if \(M^{2n+1}\) is the boundary of a domain in \(C^{\infty,1}\), then the defining functions constructed by Fefferman \([11]\) induce pseudo-Einstein contact forms on \(M\). One can construct a pseudohermitian invariant \(Q'\) on pseudo-Einstein manifolds by formally considering the limit \(\left( \frac{2}{n-1} \right)^{2n(N+1)} P_{2n+2}^N\) as \(N \to n\); this can be made rigorous by direct computation in dimension three \([7]\) and via the ambient metric in general dimension \([18]\). Regarded as \(C^\infty(M)/\mathcal{P}^\perp\)-valued, the \(Q'\)-curvature transforms linearly with a change of contact form: if \(\tilde{\theta} = e^{\theta} \theta\) is also pseudo-Einstein, then
\[
e^{2(n+1)\theta} \tilde{Q}' = Q' + P'(\sigma).
\]
Since \(\tilde{\theta}\) is pseudo-Einstein if and only if \(e^{\theta} \theta \in \mathcal{P}\) \([17,20]\), this makes sense. It follows from the properties of \(P'\) that \(\int Q'\) is independent of the choice of the pseudo-Einstein contact form. Direct computation on \(S^{2n+1}\) shows that it is a nontrivial invariant; indeed, in dimension three it is a nonzero multiple of the Burns–Epstein invariant \([7]\). In particular, the pair \((P', Q')\) on pseudo-Einstein manifolds has the same properties as the pair \((P, Q)\) on Riemannian manifolds.

If \((M^{2n+1}, T^{1,0}, \theta)\) is a compact pseudo-Einstein manifold, the self-adjointness of \(P'\) and (1.1) imply that critical points of the functional \(II: \mathcal{P} \to \mathbb{R}\) defined by
\[
II[u] = \int_M u P'u + 2 \int_M Q'u - \frac{2}{n+1} \left( \int_M Q' \right) \log \left( \frac{1}{\text{Vol}(M)} \int_M e^{(n+1)u} \right)
\]
are in one-to-one correspondence with pseudo-Einstein contact forms with constant \(Q'\)-curvature (still regarded as \(C^\infty(M)/\mathcal{P}^\perp\)-valued). The existence and classification of minimizers of the \(II\)-functional on the standard CR spheres was given by Branson, Fontana, and Morpurgo \([6]\). In this note, we discuss the main ideas used by the authors to give criteria that guarantee that minimizers exist for the \(II\)-functional on a given pseudo-Einstein three-manifold \([8]\).

**Theorem 1.1.** Let \((M^3, T^{1,0}, \theta)\) be a compact, embeddable pseudo-Einstein three-manifold such that \(P' \geq 0\) and \(\ker P' = \mathbb{R}\). Suppose additionally that
\[ \int_M Q' \theta \wedge d\theta < 16\pi^2. \]

(1.3)

Then there exists a function \( w \in \mathcal{P} \) that minimizes the II-functional. Moreover, the contact form \( \hat{\theta} := e^w \theta \) is such that \( \hat{Q}' \) is constant.

The assumptions on \( P' \) mean that the pairing \( (u, v) := \int u P' v \) defines a positive definite quadratic form on \( \mathcal{P} \). It is important to emphasize that the conclusion is that \( \hat{Q}' \) is constant as a \( C^\infty(M)/\mathcal{P}' \)-valued invariant; a local formula for the \( Q' \)-curvature was given by the first- and third-named authors \([7]\), while we observe that, on \( S^1 \times S^2 \) with any of its locally spherical contact structures, there is no pseudo-Einstein contact form with \( Q' \) pointwise zero; see \([8, \text{Section 5}]\).

As in the study of Riemannian four-manifolds (cf. \([9,16]\)), the hypotheses of Theorem 1.1 can be replaced by the nonnegativity of the pseudohermitian scalar curvature and of the CR Paneitz operator. Indeed, Chanillo, Chiu and the third-named author proved that these assumptions imply that \((M^3, T^{1,0})\) is embeddable \([10]\); the first- and third-named authors proved that these assumptions imply both that \( P' \geq 0 \) with \( \ker P' = \mathbb{R} \) and that \( \int Q' \leq 16\pi^2 \) with equality if and only if \((M^3, T^{1,0})\) is CR equivalent to the standard CR three-sphere \([7]\); and Branson, Fontana and Morpurgo showed that minimizers of the II-functional exist on the standard CR three-sphere \([6]\).

**Corollary 1.2.** Let \((M^3, T^{1,0}, \theta)\) be a compact pseudo-Einstein manifold with nonnegative scalar curvature and nonnegative CR Paneitz operator. Then there exists a function \( w \in \mathcal{P} \) which minimizes the II-functional. Moreover, the contact form \( \hat{\theta} := e^w \theta \) is such that \( \hat{Q}' \) is constant.

2. Sketch of the proof of Theorem 1.1

The proof of Theorem 1.1 proceeds analogously to the proof of the corresponding result on four-dimensional Riemannian manifolds \([9]\) with one important difference: \( P' \) is defined as a \( C^\infty(M)/\mathcal{P}' \)-valued operator; in particular, it is a nonlocal operator. Let \( \tau : C^\infty(M) \to \mathcal{P} \) be the orthogonal projection with respect to the standard \( L^2 \)-inner product. A key observation is that the operator \( \mathcal{P}' \) := \( \tau P' : \mathcal{P} \to \mathcal{P} \) is a self-adjoint elliptic pseudodifferential operator of order \(-2\); see \([8, \text{Theorem 9.1}]\). This follows from the observation that, while the sub-Laplacian \( \Delta_0 \) is subelliptic, the Toeplitz operator \( \tau \Delta_0 \tau \) is a classical elliptic pseudodifferential operator of order \(-1\). This is achieved by writing \( \Delta_0 = 2 e^\theta + i \tau \), relating \( \tau \) to the Szegő projector \( S \), and using well-known properties of the latter operator (cf. \([3,19])\).

Since \( \int u P' v = \int u \mathcal{P}' v \) for all \( u, v \in \mathcal{P} \), it follows that \( \mathcal{P}' \) is a nonnegative operator with \( \ker \mathcal{P}' = \mathbb{R} \). In particular, the positive square root \((\mathcal{P}')^{1/2}\) of \( \mathcal{P}' \) is well defined and such that \( \ker (\mathcal{P}')^{1/2} = \mathbb{R} \). Using the pseudodifferential calculus and the fact that, as a local operator, \( P' \) equals \( \Delta_0^2 \) plus lower-order terms \([7]\), we then observe that the Green’s function of \((\mathcal{P}')^{1/2}\) is of the form \( c \rho^{-2} + O(\rho^{-1-\varepsilon}) \) for \( \rho^2(z, t) = |z|^4 + t^2 \) the Heisenberg pseudo-distance, \( \varepsilon \in (0, 1) \), and \( c \) the same constant as the computation on the three-sphere \([6]\); for a more precise statement, see \([8, \text{Theorem 1.3}]\).

From this point, the remaining argument is fairly standard. The above fact about the Green’s function of \((\mathcal{P}')^{1/2}\) allows us to apply the Adams-type theorem of Fontana and Morpurgo \([12]\) to conclude that the former operator satisfies an Adams-type inequality with the same constant as on the standard CR three-sphere. This has two important effects. First, it implies that \( \text{II-functional is coercive under the additional assumption} \int Q' < 16\pi^2; \text{see} \ [8, \text{Theorem 4.1}] \). Second, it implies that if \( w \in W^{1,2} \cap \mathcal{P} \) satisfies

\[ \tau \left( P'w + Q' - \lambda e^{2w} \right) = 0, \]

then \( w \in C^\infty(M); \) see \([8, \text{Theorem 4.2}]\). The former assumption allows us to minimize II within \( W^{1,2} \cap \mathcal{P} \) and the latter assumption yields the regularity of the minimizers. The final conclusion follows from the transformation formula \((1.1)\) for the \( Q' \)-curvature.

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**References**


