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Differential geometry

Extremal metrics for the Q'-curvature in three dimensions

Métriques extrémales pour la Q'-courbure en dimension 3

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ABSTRACT

We construct contact forms with constant Q'-curvature on compact three-dimensional CR manifolds that admit a pseudo-Einstein contact form and satisfy some natural positivity conditions. These contact forms are obtained by minimizing the CR analogue of the *II*-functional from conformal geometry. Two crucial steps are to show that the P'-operator can be regarded as an elliptic pseudodifferential operator and to compute the leading-order terms of the asymptotic expansion of the Green's function for $\sqrt{P'}$.

RÉSUMÉ

On construit des formes de contact à Q'-courbure constante sur les variétés de Cauchy–Riemann de dimension 3 qui admettent une pseudo-forme de contact d'Einstein et satisfont certaines conditions naturelles de positivité. Ces formes sont obtenues en minimisant l'analogue en CR-géométrie de la *II*-fonctionelle en géométrie conforme. Cette construction repose sur deux étapes cruciales. On montre que le P'-opérateur peut être vu comme un opérateur pseudo-differentiel elliptique et on calcule les termes dominants du développement asymtotique de la forme de Green pour $\sqrt{P'}$.

1. Introduction

On an even-dimensional manifold (M^{2n}, g), the pair (P, Q) of the (critical) GJMS operator P and the (critical) Q-curvature Q possesses many of the same properties of the pair $(-\Delta, K)$ on surfaces, where K is the Gauss curvature. For example, P is a conformally covariant formally self-adjoint operator with leading-order term $(-\Delta)^{n/2}$ that annihilates

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constants [14,15], and Q is a Riemannian invariant with leading-order term $c_n(-\Delta)^{\frac{n-2}{2}}R$, where R is the scalar curvature, which transforms in a particularly simple way within a conformal class [4]: if $\widehat{g}=e^{2u}g$, then

$$e^{n\sigma}\widehat{Q}=Q+Pu.$$

In particular, $\int Q$ is conformally invariant on closed even-dimensional manifolds; indeed, it computes the Euler characteristic modulo integrals of pointwise conformal invariants [1]. It also follows that metrics of constant Q-curvature within a conformal class are in one-to-one correspondence with critical points of the functional

$$II[u]=\int_M u\,Pu+2\int_M Q\,u-\frac{2}{n}\left(\int_M Q\right)\log\left(\frac{1}{\mathrm{Vol}(M)}\int_M e^{nu}\right).$$

This functional can always be minimized on the two-sphere [21] and on four-manifolds with positive Yamabe constant and nonnegative Paneitz operator [2,9,16], with important applications to logarithmic functional determinants [5,21] and sharp Onofri-type inequalities [2]. Due to the parallels between conformal and CR geometry, it is interesting to determine whether a similar pair exists in the latter setting.

Works by Graham and Lee [13] and Hirachi [17] identified CR analogues of the Paneitz operator and Q-curvature in dimension three. However, the kernel of the Paneitz operator contains the (generally infinite-dimensional) space $\mathcal{P}$ of CR pluriharmonic functions, and the total Q-curvature is always zero. In particular, an Onofri-type inequality involving the CR Paneitz operator cannot be satisfied. Branson, Fontana and Morpurgo overcame this latter issue on the CR spheres by introducing a formally self-adjoint operator P', which is CR covariant on CR pluriharmonic functions and in terms of which one has the sharp Onofri-type inequality

$$\int_{S^{2n+1}} u\,P'u+2\int_{S^{2n+1}} Q'u-\frac{2}{n+1}\left(\int_{S^{2n+1}} Q'\right)\log\left(\frac{1}{\mathrm{Vol}(S^{2n+1})}\int_{S^{2n+1}} e^{(n+1)u}\right)\geq 0$$

for all $u\in W^{n+1,2}\cap\mathcal{P}$, where Q' is an explicit dimensional constant [6]. The construction of P' is analogous to the construction of the Q-curvature from the GJMS operators by analytic continuation in the dimension.

It was observed by the first- and third-named authors in dimension three [7] and by Hirachi in general dimension [18] that one can define the P'-operator on general pseudohermitian manifolds $(M^{2n+1},T^{1,0},\theta)$. Roughly speaking, if P^N_{2n+2} is the CR GJMS operator of order $2n+2$ on a $(2N+1)$-dimensional manifold, one defines P' as the limit of $\frac{2}{(N-n)}P^N_{2n+2}|_{\mathcal{P}}$ as $N\to n$. This is made rigorous by explicit computation in dimension three [7] and via the ambient metric in general dimension [18]. Regarded as a map from $\mathcal{P}$ to $C^\infty(M)/\mathcal{P}^\perp$, the P'-operator is CR covariant: if $\widehat{\theta}=e^\sigma\theta$, then $e^{(n+1)\sigma}\widehat{P}'=P'$.

If θ is a pseudo-Einstein contact form (cf. [7,18,20]), then the P'-operator is formally self-adjoint and annihilates constants. Note that if M^{2n+1} is the boundary of a domain in $\mathbb{C}^{n+1}$, then the defining functions constructed by Fefferman [11] induce pseudo-Einstein contact forms on M. One can construct a pseudohermitian invariant Q' on pseudo-Einstein manifolds by formally considering the limit $\left(\frac{2}{N-n}\right)^2P^N_{2n+2}(1)$ as $N\to n$; this can be made rigorous by direct computation in dimension three [7] and via the ambient metric in general dimension [18]. Regarded as $C^\infty(M)/\mathcal{P}^\perp$-valued, the Q'-curvature transforms linearly with a change of contact form: if $\widehat{\theta}=e^\sigma\theta$ is also pseudo-Einstein, then

$$e^{2(n+1)}\widehat{Q}'=Q'+P'(\sigma). \tag{1.1}$$

Since $\widehat{\theta}$ is pseudo-Einstein if and only if $\sigma\in\mathcal{P}$ [17,20], this makes sense. It follows from the properties of P' that $\int Q'$ is independent of the choice of the pseudo-Einstein contact form. Direct computation on S^{2n+1} shows that it is a nontrivial invariant; indeed, in dimension three it is a nonzero multiple of the Burns–Epstein invariant [7]. In particular, the pair (P',Q') on pseudo-Einstein manifolds has the same properties as the pair (P,Q) on Riemannian manifolds.

If $(M^{2n+1},T^{1,0},\theta)$ is a compact pseudo-Einstein manifold, the self-adjointness of P' and (1.1) imply that critical points of the functional $II\colon\mathcal{P}\to\mathbb{R}$ defined by

$$II[u]=\int_M u\,P'u+2\int_M Q'u-\frac{2}{n+1}\left(\int_M Q'\right)\log\left(\frac{1}{\mathrm{Vol}(M)}\int_M e^{(n+1)u}\right) \tag{1.2}$$

are in one-to-one correspondence with pseudo-Einstein contact forms with constant Q'-curvature (still regarded as $C^\infty(M)/\mathcal{P}^\perp$-valued). The existence and classification of minimizers of the II-functional on the standard CR spheres was given by Branson, Fontana, and Morpurgo [6]. In this note, we discuss the main ideas used by the authors to give criteria that guarantee that minimizers exist for the II-functional on a given pseudo-Einstein three-manifold [8].

Theorem 1.1. *Let $(M^3,T^{1,0},\theta)$ be a compact, embeddable pseudo-Einstein three-manifold such that $P'\geq 0$ and $\ker P'=\mathbb{R}$. Suppose additionally that*

$$\int_M Q'\theta \wedge d\theta < 16\pi^2. \tag{1.3}$$

Then there exists a function $w \in \mathcal{P}$ that minimizes the II-functional. Moreover, the contact form $\widehat{\theta} := e^w\theta$ is such that $\widehat{Q}'$ is constant.

The assumptions on P' mean that the pairing $(u, v) := \int u P'v$ defines a positive definite quadratic form on $\mathcal{P}$. It is important to emphasize that the conclusion is that $\widehat{Q}'_4$ is constant as a $C^\infty(M)/\mathcal{P}^\perp$-valued invariant: a local formula for the Q'-curvature was given by the first- and third-named authors [7], while we observe that, on $S^1 \times S^2$ with any of its locally spherical contact structures, there is no pseudo-Einstein contact form with Q' pointwise zero; see [8, Section 5].

As in the study of Riemannian four-manifolds (cf. [9,16]), the hypotheses of Theorem 1.1 can be replaced by the nonnegativity of the pseudohermitian scalar curvature and of the CR Paneitz operator. Indeed, Chanillo, Chiu and the third-named author proved that these assumptions imply that $(M^3, T^{1,0})$ is embeddable [10]; the first- and third-named authors proved that these assumptions imply both that $P' \geq 0$ with $\ker P' = \mathbb{R}$ and that $\int Q' \leq 16\pi^2$ with equality if and only if $(M^3, T^{1,0})$ is CR equivalent to the standard CR three-sphere [7]; and Branson, Fontana and Morpurgo showed that minimizers of the *II*-functional exist on the standard CR three-sphere [6].

Corollary 1.2. *Let $(M^3, T^{1,0}, \theta)$ be a compact pseudo-Einstein manifold with nonnegative scalar curvature and nonnegative CR Paneitz operator. Then there exists a function $w \in \mathcal{P}$ which minimizes the II-functional. Moreover, the contact form $\widehat{\theta} := e^w\theta$ is such that $\widehat{Q}'$ is constant.*

2. Sketch of the proof of Theorem 1.1

The proof of Theorem 1.1 proceeds analogously to the proof of the corresponding result on four-dimensional Riemannian manifolds [9] with one important difference: P' is defined as a $C^\infty(M)/\mathcal{P}^\perp$-valued operator; in particular, it is a nonlocal operator. Let $\tau : C^\infty(M) \to \mathcal{P}$ be the orthogonal projection with respect to the standard L^2-inner product. A key observation is that the operator $\overline{P}' := \tau P' : \mathcal{P} \to \mathcal{P}$ is a self-adjoint elliptic pseudodifferential operator of order -2; see [8, Theorem 9.1]. This follows from the observation that, while the sub-Laplacian Δ_b is subelliptic, the Toeplitz operator $\tau\Delta_b\tau$ is a classical elliptic pseudodifferential operator of order -1. This is achieved by writing $\Delta_b = 2\Box_b + iT$, relating τ to the Szegő projector S, and using well-known properties of the latter operator (cf. [3,19]).

Since $\int u P'v = \int u \overline{P}'v$ for all $u, v \in \mathcal{P}$, it follows that $\overline{P}'$ is a nonnegative operator with $\ker \overline{P}' = \mathbb{R}$. In particular, the positive square root $\left(\overline{P}'\right)^{1/2}$ of $\overline{P}'$ is well defined and such that $\ker \left(\overline{P}\right)^{1/2} = \mathbb{R}$. Using the pseudodifferential calculus and the fact that, as a local operator, P' equals Δ_b^2 plus lower-order terms [7], we then observe that the Green's function of $\left(\overline{P}'\right)^{1/2}$ is of the form $c\rho^{-2} + O(\rho^{-1-\varepsilon})$ for $\rho^4(z,t) = |z|^4 + t^2$ the Heisenberg pseudo-distance, $\varepsilon \in (0,1)$, and c the same constant as the computation on the three-sphere [6]; for a more precise statement, see [8, Theorem 1.3].

From this point, the remaining argument is fairly standard. The above fact about the Green's function of $\left(\overline{P}'\right)^{1/2}$ allows us to apply the Adams-type theorem of Fontana and Morpurgo [12] to conclude that the former operator satisfies an Adams-type inequality with the same constant as on the standard CR three-sphere. This has two important effects. First, it implies that *II*-functional is coercive under the additional assumption $\int Q' < 16\pi^2$; see [8, Theorem 4.1]. Second, it implies that if $w \in W^{2,2} \cap \mathcal{P}$ satisfies

$$\tau\left(P'w + Q' - \lambda e^{2w}\right) = 0,$$

then $w \in C^\infty(M)$; see [8, Theorem 4.2]. The former assumption allows us to minimize *II* within $W^{2,2} \cap \mathcal{P}$ and the latter assumption yields the regularity of the minimizers. The final conclusion follows from the transformation formula (1.1) for the Q'-curvature.

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