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Differential geometry

Area minimizing projective planes on the projective space of dimension 3 with the Berger metric

Plans projectifs minimisant l'aire dans l'espace projectif à dimension 3 muni de la métrique de Berger

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ABSTRACT

We show that, among the projective planes embedded into the real projective space $\mathbb{R}P^3$ endowed with the Berger metric, those of least area are exactly the ones obtained by projection of the equatorial spheres of S^3. This result generalizes a classical result for the projective spaces with the standard metric.

RÉSUMÉ

On démontre que, parmi les plans projectifs dans l'espace projectif réel $\mathbb{R}P^3$, muni de la métrique de Berger, ceux qui réalisent le minimum de l'aire sont exactement ceux qu'on obtient par la projection des sphères équatoriales de S^3. Le résultat généralise un résultat classique pour l'espace projectif muni de la métrique ordinaire.

1. Introduction

A classical result due to M. Berger [2] and A.T. Fomenko [6] is that totally geodesic projective planes are the only area minimizer surfaces of the real projective 3-dimensional space, in their homology class.

A Berger sphere is the Riemannian manifold (S^3, g_μ) where the metric g_μ for $\mu > 0$ is defined as follows: we denote by J the complex structure of $\mathbb{R}^4 \simeq \mathbb{C}^2$ and by H the Hopf vector field JN, where $N(p) = p$ is the outwards unit normal of S^3, then

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$$g_\mu(H,H) = \mu g(H,H) = \mu, \quad (g_\mu)|_{H^\perp} = g|_{H^\perp}, \quad \text{and} \quad g_\mu(H,H^\perp) = 0,$$

where $\perp$ means orthogonal with respect to the standard metric g. Berger spheres are one of the five classes of simply connected homogeneous 3-manifolds with 4-dimensional group of isometries.

In [3], the author and V. Borrelli have shown that if on the 3-dimensional sphere we consider the family of metrics g_μ and we consider, after passing to the quotient, the corresponding Berger metrics on 3-dimensional real projective space, then the projections of the equatorial spheres are not totally geodesic surfaces, except for the round sphere $\mu = 1$, but they are always minimal surfaces. The aim of this paper is to extend the Berger and Fomenko result to projective spaces endowed with Berger metrics. We show that *area-minimizing projective planes in* $(\mathbb{R}P^3, g_\mu)$ *are exactly the equatorial projective planes.* The key point for the proof is the existence of area-minimizing projective planes in 3-manifolds proved by Bray, Brendle, Eichmar, and Neves in [4].

2. Area-minimizing projective planes

Let $\pi : (S^3, g_\mu) \longrightarrow (\mathbb{R}P^3, g_\mu)$ be the usual projection. This 2-fold covering map is a Riemannian submersion for every value of the parameter μ. For each $v \in S^3$, we consider the equatorial sphere $\tilde{\Sigma}_v = S^3 \cap v^\perp$ determined by v, and we denote by $\Sigma_v = \pi(S^3 \cap v^\perp)$ the *equatorial projective plane* obtained by projection.

Theorem 1. *The only projective planes minimally embedded in real projective Berger space* $(\mathbb{R}P^3, g_\mu)$ *are the equatorial projective planes. Moreover, the equatorial projective planes in* $(\mathbb{R}P^3, g_\mu)$ *are exactly the area-minimizing projective planes.*

Proof. It has been shown in [3] that for all $v \in S^3$, the surface Σ_v is minimal in $(\mathbb{R}P^3, g_\mu)$ for all $\mu \in \mathbb{R}$. Now, let us fix μ and let Σ be any minimal embedded surface homeomorphic to $\mathbb{R}P^2$. We represent by $\tilde{\Sigma}$ the embedded surface of the 3-dimensional sphere defined as $\tilde{\Sigma} = \pi^{-1}(\Sigma)$. Since π is a Riemannian submersion and its restriction to $\tilde{\Sigma}$ is a 2-fold covering of the projective space Σ, the surface $\tilde{\Sigma}$ is a 2-sphere minimally embedded in the Berger sphere (S^3, g_μ). The 2-fold covering $\tilde{\Sigma}$ is not given by two copies of Σ because there is no embedded projective plane in S^3.

Now, the use of generalized Hopf differentials of [1] allows us to conclude that $\tilde{\Sigma}$ is a standard rotational sphere (see Theorem 4.5 of [5] for an alternative proof) and by Theorem 1 part i) in [7], $\tilde{\Sigma}$ must be one of the equatorial spheres $\tilde{\Sigma}_v$ and, consequently, Σ is an equatorial projective plane as stated.

For the second part, let us represent by $\mathcal{P}$ the collection of all embedded surfaces of $\mathbb{R}P^3$ that are homeomorphic to $\mathbb{R}P^2$. Since $\Sigma_v \in \mathcal{P}$ the set $\mathcal{P}$ is nonempty and we can define

$$\mathcal{A}(\mathbb{R}P^3, g_\mu) = \inf\{\mathrm{area}_\mu(\Sigma)\,;\ \Sigma \in \mathcal{P}\},$$

where the $\mathrm{area}_\mu(\Sigma)$ is the area of the surface Σ computed in the metric g_μ.

In [4], Proposition 2.3, it is shown that for every Riemannian 3-manifold, this infimum is attained by some embedded minimal projective plane and, consequently, for each $\mu > 0$, there exists at least one surface $\Sigma^\mu \in \mathcal{P}$ such that $\mathcal{A}(\mathbb{R}P^3, g_\mu) = \mathrm{area}_\mu(\Sigma^\mu)$. Any area minimizing Σ^μ must be a minimal surface and then equal to one of the equatorial projective planes. Consequently, $\mathcal{A}(\mathbb{R}P^3, g_\mu) = \mathrm{area}_\mu(\Sigma_v)$, for all $v \in S^3$. $\square$

Remark. In [3] it has been shown that a Berger projective space of dimension 3 can be realized as the manifold $(T^r S^2, g^S)$ of vectors of norm r on the unit sphere; more precisely, $(T^r S^2, g^S)$ is isometric to $(\mathbb{R}P^3, 4g_{r^2})$. The isometry preserves the S^1-bundle structure of both manifolds, namely the one determined by the fibers of the Hopf fibration in the projective space and the natural one in the tangent space.

Just by rewording the results in [3], keeping in mind this correspondence, we know that the projective planes Σ_v are sections of the Hopf fibration defined in an open set of the form $S^2 \setminus \{p_0\}$ and that if Σ is a projective plane minimally embedded in $(\mathbb{R}P^3, g_\mu)$ such that $\Sigma = \overline{\sigma(U)}$ is the closure of a section of the Hopf fibration defined in an open and dense subset of S^2, then Σ must be of the form Σ_v.

Theorem 1 above shows that the unicity result holds without the assumption of the surface being a section almost everywhere.

Corollary 2. *The minimum of the area is given by* $\mathcal{A}(\mathbb{R}P^3, g_\mu) = \pi A(\mu)$ *where* $A(\mu)$ *is the function*

$$A(\mu) = \begin{cases} 1 + \frac{\mu}{\sqrt{1-\mu}} \ln\left(\frac{\sqrt{1-\mu}+1}{\sqrt{\mu}}\right) & \text{if} \quad \mu < 1, \\ 1 + \frac{\mu}{\sqrt{\mu-1}} \arcsin\left(\frac{\sqrt{\mu-1}}{\sqrt{\mu}}\right) & \text{if} \quad \mu > 1 \quad \textit{and} \\ 2 \quad \text{if} \quad \mu = 1. \end{cases}$$

Proof. $\mathcal{A}(\mathbb{R}P^3, g_\mu) = \mathrm{area}_\mu(\Sigma_v) = (1/2)\mathrm{area}_\mu(\tilde{\Sigma}_v)$ for every unit vector v. Choose an orthonormal basis of $\mathbb{R}^4$ of the form $\{e_1, e_2 = Je_1, e_3 = -Jv, e_4 = v\}$ then the map $\varphi :]0,\pi[\times]0,2\pi[\longrightarrow S^3$ given by

$$\varphi(t,\theta) = \sin t \sin\theta\, e_1 + \sin t \cos\theta\, e_2 + \cos t\, e_3$$

is a chart of the equatorial sphere $\tilde{\Sigma}_v$ with coordinate vector fields

$$\varphi_t = \cos t \sin\theta\, e_1 + \cos t \cos\theta\, e_2 - \sin t\, e_3, \quad \varphi_\theta = \sin t \cos\theta\, e_1 - \sin t \sin\theta\, e_2.$$

Since the Hopf vector field in this chart is

$$H(\varphi(t,\theta)) = \sin t \sin\theta\, e_2 - \sin t \cos\theta\, e_1 + \cos t\, v,$$

it is easy to see that

$$\mathrm{area}_\mu(\tilde{\Sigma}_v) = 2\pi \int_0^\pi \sqrt{\cos^2 t + \mu \sin^2 t}\, \sin t\, \mathrm{d}t$$

and the conclusion follows by straightforward computation. □

References

[1] U. Abresch, H. Rosenberg, Generalized Hopf differentials, Mat. Contemp. 28 (2005) 1–28.
[2] M. Berger, Du côté de chez Pu, Ann. Sci. Éc. Norm. Supér. 5 (1972) 1–44.
[3] V. Borrelli, O. Gil-Medrano, Area-minimizing vector fields on 2-spheres, J. Reine Angew. Math. 640 (2010) 85–99.
[4] H. Bray, S. Brendle, M. Eichmar, A. Neves, Area-minimizing projective planes in 3-manifolds, Commun. Pure Appl. Math. LXIII (2010) 1237–1247.
[5] I. Fernández, P. Mira, Constant mean curvature surfaces in 3-dimensional Thurston geometries, in: Proceedings of the International Congress of Mathematicians, vol. II, Hindustan Book Agency, New Delhi, 2010, pp. 830–861.
[6] A.T. Fomenko, Minimal compacta in Riemannian manifolds and Reifensberg's conjecture, Math. USSR 6 (1972) 1037–1066.
[7] F. Torralbo, Rotationally invariant constant mean curvature surfaces in homogeneous 3-manifolds, Differ. Geom. Appl. 28 (2010) 593–607.