Differential geometry

On the volume of the Sp(n) \cdot Sp(1) shadow of a compact set

Sur le volume de la Sp(n) \cdot Sp(1)-ombre d’un ensemble compact

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ABSTRACT

Let $F : \mathbb{R}^{4k} \rightarrow \mathbb{R}^{4k}$ be an element of the quaternionic unitary group $\text{Sp}(n) \cdot \text{Sp}(1)$, let $K$ be a compact subset of $\mathbb{R}^{4k}$, and let $V$ be a 4$k$-dimensional quaternionic subspace of $\mathbb{R}^{4k} \cong \mathbb{H}^n$. The 4$k$-dimensional shadow of the image under $F$ of $K$ is its orthogonal projection $P(F(K))$ onto $V$. We show that there exists a 4$k$-dimensional quaternionic subspace $W$ of $\mathbb{R}^{4k}$ such that the volume of the shadow $P(F(K))$ is the same as the volume of the section $K \cap W$. This is a quaternionic analogue of the symplectic linear non-squeezing result recently obtained by Abbondandolo and Matveyev.

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RÉSUMÉ

Soit $F : \mathbb{R}^{4k} \rightarrow \mathbb{R}^{4k}$ un élément du groupe unitaire quaternionnier $\text{Sp}(n) \cdot \text{Sp}(1)$, soit $K$ un ensemble compact dans $\mathbb{R}^{4k}$, et soit $V$ un sous-espace vectoriel quaternionnier de dimension 4$k$ dans $\mathbb{R}^{4k} \cong \mathbb{H}^n$. L’ombre 4$k$-dimensionnelle de l’image par $F$ de $K$ est sa projection orthogonale $P(F(K))$ sur $V$. Nous montrons qu’il existe un sous-espace vectoriel quaternionnier $W \subset \mathbb{R}^{4k}$ de dimension 4$k$ tel que le volume de l’ombre $P(F(K))$ est égal au volume de la section $K \cap W$. Ceci est un analogue quaternionnier du résultat de non-squeezing lineaire symplectique obtenu récemment par Abbondandolo et Matveyev.

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1. Introduction

Consider $\mathbb{R}^{2n}$ with the canonical symplectic form $\Omega$. Let $B_R \subset \mathbb{R}^{2n}$ be the 2$n$-dimensional ball of radius $R$, and let $\omega_{2k}$ denote the volume of the unit 2$k$-dimensional ball, $1 \leq k \leq n$. Recently, Abbondandolo and Matveyev [1] have proved the following.

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Theorem 1 (Linear non-squeezing). (See [1].) Let $F$ be a linear symplectic automorphism of $\mathbb{R}^{2n} \cong \mathbb{C}^n$, and let $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the orthogonal projection onto a complex linear subspace $V \subset \mathbb{R}^{2n}$ of real dimension $2k$, $1 \leq k \leq n$. Then

$$\text{vol}_{2k} P (F (B_R)) \geq \omega_{2k} R^{2k},$$

and equality holds if and only if the image of $V$ under the adjoint of $F$ is a complex linear subspace.

For $k = 1$, this result is a reformulation of Gromov's non-squeezing theorem for linear symplectic maps [8]. The proof relies essentially on Wirtinger's inequality for the $k$th powers $\Omega^k = \Omega \wedge \cdots \wedge \Omega$, $1 \leq k \leq n$, or equivalently on the fact that the $2k$-forms $\Omega^k/k!$ give rise to calibrations in the sense of Harvey-Lawson [10].

The purpose of this note is to address a quaternionic analogue of this result. In the quaternionic case, the symplectic 2-form $\omega$ is replaced by the standard quaternionic Kähler 4-form

$$\Phi = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3,$$

on $\mathbb{R}^{4n} \cong \mathbb{H}^n$, identified by

$$(x_1^1, x_1^2, x_2^1, \ldots, x_0^1, x_1^2, x_2^2) \mapsto (x_1^1 + x_1^2 + x_2^1 + x_2^2, \ldots, x_0^1 + x_1^2 + x_2^1 + x_2^2),$$

where

$$\begin{aligned}
\omega_1 &= \sum_{i=1}^n (dx_i^1 \wedge dx_i^2 - dx_i^2 \wedge dx_i^1), \\
\omega_2 &= \sum_{i=1}^n (dx_i^3 \wedge dx_i^4 - dx_i^4 \wedge dx_i^3), \\
\omega_3 &= \sum_{i=1}^n (dx_i^5 \wedge dx_i^6 - dx_i^6 \wedge dx_i^5). 
\end{aligned}$$

For $n \geq 2$, it is known that the subgroup of $\text{GL}(4n, \mathbb{R})$ that fixes $\Phi$ is the quaternionic unitary group $\text{Sp}(n) \cdot \text{Sp}(1) \subset \text{SO}(4n)$ (cf. [4,9,12,13]). We will prove the following.

Theorem 2. Let $F \in \text{Sp}(n) \cdot \text{Sp}(1)$ be a linear quaternionic unitary automorphism of $\mathbb{R}^{4n} \cong \mathbb{H}^n$, and let $P : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$ be the orthogonal projection onto a quaternionic linear subspace $V \subset \mathbb{R}^{4n}$ of real dimension $4k$, $k = 1, \ldots, n$. Then, for every compact set $K \subset \mathbb{R}^{4n}$, there exists a $4k$-dimensional quaternionic linear subspace $W \subset \mathbb{R}^{4n}$ such that

$$\text{vol}_{4k} P (F (K)) = \text{vol}_{4k} (K \cap W).$$

2. Preliminaries

Let $\mathbb{H}$ be the real noncommutative algebra of quaternions, with the standard basis $\{1, i, j, k\}$. Multiplication is determined by the rules

$$i^2 = j^2 = k^2 = ijk = -1,$$

which imply $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. If $q \in \mathbb{H}$, we write

$$q = q_0 + q_1 i + q_2 j + q_3 k.$$

The real and imaginary parts of $q$ are $\text{Re} q = q_0$ and $\text{Im} q = q_1 i + q_2 j + q_3 k$, respectively. The conjugate of $q$ is given by $\bar{q} = \text{Re} (q) - \text{Im} (q)$ and the square norm of $q$ is defined by $N(q) = q \bar{q}$. As a vector space, $\mathbb{H}$ identifies with $\mathbb{R}^4$ under the isomorphism $q_0 + q_1 i + q_2 j + q_3 k \mapsto (q_0, q_1, q_2, q_3)$, which in turn induces an isomorphism between the subspace of imaginary quaternions $\mathbb{H}^n = \text{span} \{i, j, k\}$ and $\mathbb{R}^3$. In particular, $1, i, j, k$ identify with the elements of the canonical basis $e_0, e_1, e_2, e_3$ of $\mathbb{R}^4$, respectively. For $p, q \in \mathbb{H}$, we have $\bar{pq} = q \bar{p}$ and $pq + \bar{q} \bar{p} = 2 \text{Re} (pq) = 2 (p, q)$, where $(p, q)$ is the standard Euclidean inner product on $\mathbb{R}^4 \cong \mathbb{H}$, under the above identification.

Let $\mathbb{H}^n$ be the space of column $n$-tuples of quaternions endowed with its right $\mathbb{H}$-vector space structure, $x \cdot q = \bar{q} (x_0) \cdot q + q_0 (x)$, its standard quaternionic Hermitian product

$$h(x, y) = \bar{x} y = \sum_{i=1}^n x_i y_i,$$

and its real scalar product $(x, y) := \text{Re} h(x, y)$, which equals the standard Euclidean inner product on $\mathbb{R}^{4n} \cong \mathbb{H}^n$.

The subgroup of $\text{GL}(n, \mathbb{H})$ that fixes $h$ is the symplectic group

$$\text{Sp}(n) = \{ A \in \text{GL}(n, \mathbb{H}) \mid h(Ax, Ay) = h(x, y) \}.$$
The group $\text{Sp}(1) = S^3 = \{q \in \mathbb{H} | \bar{q}q = 1\}$ of unit quaternions acts on $\mathbb{H}^n \cong \mathbb{R}^{4n}$ by right multiplication and the corresponding $\mathbb{R}$-linear maps preserve the Euclidean inner product, i.e.,

$$\langle x, y \rangle = \langle x \cdot q, y \cdot q \rangle.$$  

(4)

In fact, for $x, y \in \mathbb{H}^n$ and $q \in \text{Sp}(1)$,

$$2\langle xq, yq \rangle = h(xq, yq) + h(yq, xq) = \bar{q}h(x, y)q + \bar{q}h(y, x)q = 2\bar{q}\langle x, y \rangle q = 2\langle x, y \rangle .$$

If $S^2 = \{u \in \mathbb{R}^4 | \bar{u}u = -u^2 = 1\}$ is the sphere of unit imaginary quaternions, right multiplication by $u \in S^2$, $R_u x := x \cdot u$, defines an orthogonal complex structure on $\mathbb{R}^{4n}$, i.e., $R_u^2 = -\text{Id}$.

Note that $\text{Sp}(1)$ as a group of right multiplications by unit quaternions is not a subgroup of $\text{Sp}(n)$. However, $\text{Sp}(1)$ and $\text{Sp}(n)$ consist of $\mathbb{R}$-linear maps of $\mathbb{H}^n$ which preserve the Euclidean inner product of $\mathbb{H}^n \cong \mathbb{R}^{4n}$ and their intersection is $\mathbb{Z}_2 \cong \{\pm \text{Id}\}$. The quaternionic unitary group is the enhancement

$$\text{Sp}(n) \cdot \text{Sp}(1) := \text{Sp}(n) \times \text{Sp}(1)/\mathbb{Z}_2$$

of $\text{Sp}(n)$ by $\text{Sp}(1)$. It consists of the $\mathbb{R}$-linear automorphisms $T_{A,q}$ of $\mathbb{H}^n$ defined by

$$T_{A,q}(x) := Ax \cdot q, \quad x \in \mathbb{H}^n,$$

where $A \in \text{Sp}(n)$ and $q \in \text{Sp}(1)$. The group $\text{Sp}(n) \cdot \text{Sp}(1)$ is a subgroup of $\text{SO}(4n)$. In particular, for each $\lambda \in \mathbb{H}$, $T_{A,q}(x \cdot \lambda) = T_{A,q}(x) \cdot q^{-1} \lambda q$.

**Remark 1.** If $F \in \text{Sp}(n) \cdot \text{Sp}(1)$, its adjoint $F^T$ with respect to the Euclidean inner product is still an element of $\text{Sp}(n) \cdot \text{Sp}(1)$. In fact, for $F = T_{A,q} \in \text{Sp}(n) \cdot \text{Sp}(1)$ and $x, y \in \mathbb{H}^n$,

$$\langle Fx, y \rangle = \langle Ax \cdot q, y \rangle = \langle x \cdot q, 1_4 y \rangle = \langle x, 1_4 \bar{A} y \cdot \bar{q} \rangle$$

by (4). Thus, $F^T = T_{A^*, \bar{q}}$, where $A^* = 1_4 \bar{A} \in \text{Sp}(n)$ and $\bar{q} \in \text{Sp}(1)$.

**Remark 2.** If $V \subseteq \mathbb{R}^{4n}$ is a quaternionic subspace, i.e., a right $\mathbb{H}$-linear subspace of $\mathbb{R}^{4n}$, and $F \in \text{Sp}(n) \cdot \text{Sp}(1)$, then the image $F(V)$ is a quaternionic subspace. Now, for each $\lambda \in \mathbb{H}$, $T_{A,q}(x) \cdot \lambda = Axq \lambda = A(xq \lambda^{-1})q = T_{A,q}(xq \lambda^{-1}) \mathbb{H}$, where $xq \lambda^{-1} \in V$, and hence the claim. Alternatively, observe that $F$ is a quaternionic $\mathbb{R}$-linear map (cf. [2], Definition 1.6), i.e., for each $u \in S^2$, $F \circ R_u = R_u' \circ F$, for some $u' \in S^2$. As such, it clearly takes quaternionic subspaces to quaternionic subspaces.

Let $\omega_u(x, y) := \text{Re} h(R_u x, y) = \langle R_u x, y \rangle$ be the Kähler 2-form corresponding to the complex structure $R_u$, for $u \in S^2$. Then, for all $x, y \in \mathbb{H}^n$,

$$h(x, y) = \langle x, y \rangle + i\omega_l(x, y) + j\omega_j(x, y) + k\omega_k(x, y).$$

In fact, $(1, h(x, y)) = \text{Re} h(x, y)$, and for $u \in S^2$,

$$2(u, h(x, y)) = \bar{u}h(x, y) + h(x, y)u = h(x \cdot u, y) + h(y, x \cdot u) = 2\text{Re} h(R_u x, y).$$

According to the identification (1), the Kähler 2-forms $\omega_l, \omega_j, \omega_k$ are expressed as in (2).

### 3. Extremal properties of Hilbert forms and proof of Theorem 2

On $\mathbb{R}^{4n} \cong \mathbb{H}^n$, consider the quaternionic Kähler 4-form $\Phi = \omega_l \wedge \omega_l + \omega_j \wedge \omega_j + \omega_k \wedge \omega_k$.

**Proposition 3.** (See Bruni [5].) The external powers of $\Phi$ and the standard powers of the polynomial $x^2 + y^2 + z^2$ have the same formal expression.

This observation is important in view of the following classical result.

**Theorem 4.** (See Hilbert [11].) For every $m \in \mathbb{N}$, the $m$th power of the sum of squares $x^2 + y^2 + z^2$ is a linear combination with coefficients in $\mathbb{Q}^+$ of $2m$th powers of first degree polynomials with integer coefficients, i.e.,

$$(x^2 + y^2 + z^2)^m = \sum_{t=1}^{s} p_t(a_t x + b_t y + c_t z)^{2m}, \quad p_t \in \mathbb{Q}^+, \quad a_t, b_t, c_t \in \mathbb{Z}.$$

This theorem suggests the following.
Definition 1. (See Bruni [5].) Let \( r_1, \ldots, r_s \in \mathbb{R}^+ \) and \( q_1, \ldots, q_s \in S^2 \subset \text{Im} \mathbb{H} \), with \( q_t = q_t^1 i + q_t^2 j + q_t^3 k \) \( (t = 1, \ldots, s) \). For \( p < n \), a Hilbert form is the \( 4p \)-forms given by

\[
\Psi = \sum_{t=1}^s r_t \left( \bigwedge^{2p} \omega_{q_t} \right) = \sum_{t=1}^s r_t \left( \bigwedge^{2p} (q_t^1 \omega_i + q_t^2 \omega_j + q_t^3 \omega_k) \right).
\]

The height of \( \Psi \) is the real number \( h_\Psi := (\sum_{t=1}^s r_t) (2p)! \).

Remark 3. Examples of Hilbert forms include the exterior powers of the quaternionic Kähler 4-form \( \Phi \). For instance,

\[
\Phi^2 = \frac{9}{12} \left( \left( \frac{1}{\sqrt{3}} \omega_i + \frac{1}{\sqrt{3}} \omega_j + \frac{1}{\sqrt{3}} \omega_k \right)^4 + \left( \frac{1}{\sqrt{3}} \omega_i + \frac{1}{\sqrt{3}} \omega_j - \frac{1}{\sqrt{3}} \omega_k \right)^4 + \left( \frac{1}{\sqrt{3}} \omega_i - \frac{1}{\sqrt{3}} \omega_j + \frac{1}{\sqrt{3}} \omega_k \right)^4 \right)
\]

Moreover, the heights of \( \Phi \), \( \Phi^2 \), and \( \Phi^3 \) are, respectively, \( 3 \cdot (2)! \), \( 5 \cdot (4)! \), and \( 7 \cdot (6)! \) (cf. [5]).

Remark 4. In [6], it is shown that every linear combination of \( \omega_i \), \( \omega_j \) and \( \omega_k \) is invariant under the action of \( \text{Sp}(n) \). This implies that all Hilbert forms are \( \text{Sp}(n) \)-invariant. However, note that the \( k \)th powers \( \Phi^k \), \( 1 \leq k \leq n \), of the quaternionic Kähler form \( \Phi \) are invariant under \( \text{Sp}(n) \cdot \text{Sp}(1) \).

For the Hilbert forms, the following holds.

Theorem 5. (See Bruni [5].) Let \( \xi \) be a simple \( 4p \)-vector and let \( \Psi \) be a Hilbert form. Then

\[
|\Psi(\xi)| \leq h_\Psi \text{vol}(\xi),
\]

and equality holds if and only if the linear subspace corresponding to \( \xi \) is quaternionic.

Remark 5. From the previous theorem, it follows that any Hilbert form \( \Psi \) defines a calibration \( \Psi/h_\Psi \). In particular, since \( h_\Phi = 6 \), the 4-form \( \frac{1}{6} \Phi \) is a calibration (cf. [3,6,14]).

Following the idea of proof of Theorem 1 (cf. [1]), we now prove Theorem 2.

Proof of Theorem 2. For \( F \in \text{Sp}(n) \cdot \text{Sp}(1) \), let \( A := P \circ F : \mathbb{R}^{4n} \rightarrow V \subset \mathbb{R}^{4n} \) and denote by \( A^T : V \rightarrow \mathbb{R}^{4n} \) the adjoint of \( A \) with respect to the Euclidean inner products. Then, \( A^T A : \mathbb{R}^{4n} \rightarrow \mathbb{R}^{4n} \) is symmetric, positive semidefinite and with kernel of codimension \( 4k \),

\[
\text{Ker}(A^T A) = \text{Ker}A = (\text{Ran}A^T)^\perp.
\]

The application \( A^T A \) restricts to an automorphism of the \( 4k \)-dimensional space \( (\text{Ker}A)^\perp = \text{Ran}A^T \) and, as such, it is the composition of the two isomorphisms

\[
A_{|(\text{Ker}A)^\perp} : (\text{Ker}A)^\perp \rightarrow V, \quad A^T : V \rightarrow (\text{Ker}A)^\perp,
\]

which, being one the adjoint of each other, have the same determinant. Thus

\[
\det(A_{|(\text{Ker}A)^\perp}) = |\det(A_{|(\text{Ker}A)^\perp})|^2.
\]

Let \( \xi_1, \ldots, \xi_{4k} \) be a basis for \( (\text{Ker}A)^\perp = \text{Ran}A^T = F^T V \), such that \( |\xi_1 \wedge \cdots \wedge \xi_{4k}| = 1 \), where \( |\xi_1 \wedge \cdots \wedge \xi_{4k}| \) denotes the volume of the \( 4k \)-parallelepiped spanned by \( \xi_1, \ldots, \xi_{4k} \). Since \( A(K) = A(K \cap (\text{Ker}A)^\perp) = A(K \cap \text{Ran}A^T) \), we have (see, for instance, [1,7])

\[
\frac{\text{vol}_{4k}(A(K))}{\text{vol}_{4k}(K \cap (\text{Ker}A)^\perp)} = |A_{(\text{Ker}A)^\perp} A \xi_1 \wedge \cdots \wedge A \xi_{4k}| = |\text{det}(A_{|(\text{Ker}A)^\perp})| = \sqrt{\det(A^T A_{|(\text{Ker}A)^\perp})}.
\]

From this identity and Theorem 5, it follows that

\[
\left( \frac{\text{vol}_{4k}(A(K))}{\text{vol}_{4k}(K \cap (\text{Ker}A)^\perp)} \right)^2 = \det(A_{|(\text{Ker}A)^\perp}) = |A^T A_{(\text{Ker}A)^\perp} A \xi_1 \wedge \cdots \wedge A^T A \xi_{4k}| \geq \frac{1}{h_{\phi_k}} |\Phi_k(A^T A_{(\text{Ker}A)^\perp} A \xi_1 \wedge \cdots \wedge A^T A \xi_{4k})| = \frac{1}{h_{\phi_k}} |\Phi_k(F^T A_{(\text{Ker}A)^\perp} A \xi_1 \wedge \cdots \wedge F^T A \xi_{4k})|.
\]
and equality holds if and only if the subspace generated by $A^T A \xi_1, \ldots, A^T A \xi_{4k}$, i.e. the subspace $F^TV$, is quaternionic.

According to Remark 1, if $F \in \text{Sp}(n) \cdot \text{Sp}(1)$, also $F^T \in \text{Sp}(n) \cdot \text{Sp}(1)$. Hence, since $\Phi^k$ is invariant under $\text{Sp}(n) \cdot \text{Sp}(1)$,

$$
\Phi^k(F^T A \xi_1 \wedge \cdots \wedge F^T A \xi_{4k}) = \Phi^k(A \xi_1 \wedge \cdots \wedge A \xi_{4k}).
$$

(7)

Now, since the restriction of $\Phi^k$ to the quaternionic subspace $V$ is $h_{\Phi^k}$-times the standard volume form, by (5), we have

$$
\frac{1}{h_{\Phi^k}} \left| \Phi^k(A \xi_1 \wedge \cdots \wedge A \xi_{4k}) \right| = |A \xi_1 \wedge \cdots \wedge A \xi_{4k}| = \frac{\text{vol}_{4k}(A(K))}{\text{vol}_{4k}(K \cap (\text{Ker} A^\perp))}.
$$

(8)

Taking into account that, by Remark 2, the subspace $F^TV =: W$ is always a quaternionic subspace, it follows from (6), (7), and (8) that

$$
\text{vol}_{4k}(A(K)) = \text{vol}_{4k}(K \cap (\text{Ker} A^\perp)) = \text{vol}_{4k}(K \cap W).
$$

□

References