Harmonic analysis/Functional analysis

Strong convergence in the weighted setting of operator-valued Fourier series defined by the Marcinkiewicz multipliers

*Fonctions de la classe de Marcinkiewicz et la convergence forte des séries d’opérateurs de Fourier associées*

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**A R T I C L E  I N F O**

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**A B S T R A C T**

Suppose that $1 < p < \infty$ and let $w$ be a bilateral weight sequence satisfying the discrete Muckenhoupt $A_p$ weight condition. We show that every Marcinkiewicz multiplier $\psi : \mathbb{T} \rightarrow \mathbb{C}$ has an associated operator-valued Fourier series which serves as an analogue in $\mathfrak{B} (\ell^p (w))$ of the usual Fourier series of $\psi$, and this operator-valued Fourier series is everywhere convergent in the strong operator topology. In particular, we deduce that the partial sums of the usual Fourier series of $\psi$ are uniformly bounded in the Banach algebra of Fourier multipliers for $\ell^p (w)$. These results transfer to the framework of invertible, modulus mean-bounded operators acting on $L^p$ spaces of sigma-finite measures.

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**R É S U M É**

Soient $1 < p < \infty$ et $w$ un poids dans la classe $A_p (\mathbb{Z})$. Cette note établit (dans la topologie forte des opérateurs) la convergence des séries de Fourier (à valeurs dans $\mathfrak{B} (\ell^p (w))$) pour les « convolutions de Stieltjes », où ces convolutions sont déterminées par les fonctions $\psi$ appartenant à la classe de Marcinkiewicz $\mathfrak{M}_1 (\mathbb{T})$. Les propriétés de convergence pour ces séries de Fourier ayant valeurs dans $\mathfrak{B} (\ell^p (w))$ révèlent des propriétés de convergence des séries de Fourier traditionnelles pour les fonctions $\psi \in \mathfrak{M}_1 (\mathbb{T})$. En particulier, les sommes partielles de la série de Fourier traditionnelle pour un $\psi \in \mathfrak{M}_1 (\mathbb{T})$ quelconque sont uniformément bornées dans la norme des $p$-multiplicateurs pour $\ell^p (w)$. Ces résultats se transforment immédiatement au cadre d’une bijection linéaire arbitraire $T$ telle que $T$ soit un opérateur préservant la disjonction dont le module linéaire est à moyennes bornées.

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1. Introduction

The symbol \( K \) with (a possibly empty) set of subscripts denotes a constant which depends only on those subscripts, and which may change in value from one occurrence to another. The characteristic function of an arc \( \mathbb{A} \subseteq \mathbb{T} \) will be symbolized by \( \chi_\mathbb{A} \). For our treatment of Marcinkiewicz multipliers we shall make free use of the standard notation for the sequence \( \{t_k\}_{k=0}^{\infty} \) of dyadic points of the interval \((0, 2\pi)\), which are defined as \( 2^k t_k \pi \) if \( k \leq 0 \), and \( 2\pi - 2^{-k} t_k \pi \) if \( k > 0 \). For \( 1 < p < \infty \), a weight sequence \( w = \{w_k\}_{k=\infty}^{\infty} \) belongs to the class \( A_p (\mathbb{Z}) \) provided that there is a real constant \( C \) (called an \( A_p (\mathbb{Z}) \) weight constant for \( w \)) such that

\[
\left( \frac{1}{M - L + 1} \sum_{k=1}^{M} w_k \right) \left( \frac{1}{M - L + 1} \sum_{k=L}^{M} w_k^{-1/(p-1)} \right)^{p-1} \leq C,
\]

whenever \( L \in \mathbb{Z} \), \( M \in \mathbb{Z} \), and \( L < M \). We denote the corresponding sequence space by \( \ell^p (w) \). We say that \( \psi \in L^\infty (\mathbb{T}) \) is a multiplier for \( \ell^p (w) \) (in symbols, \( \psi \in M_{p,w} (\mathbb{T}) \)) provided that convolution by the inverse Fourier transform of \( \psi \) defines a bounded operator on \( \ell^p (w) \). Specifically, we require:

**Definition 1.1.**

(i) For each \( x = \{x_k\}_{k=-\infty}^{\infty} \in \ell^p (w) \) and each \( j \in \mathbb{Z} \), the series

\[
(\psi^\vee \ast x) (j) = \sum_{k=-\infty}^{\infty} \psi^\vee (j - k) x_k
\]

converges absolutely, and

(ii) the mapping \( T_{(p,w)} : x \in \ell^p (w) \rightarrow \psi^\vee \ast x \) is a bounded linear mapping of \( \ell^p (w) \) into \( \ell^p (w) \).

We then call \( T_{(p,w)} \) the multiplier transform corresponding to \( \psi \), and define the multiplier norm by setting

\[
\|\psi\|_{M_{p,w} (\mathbb{T})} = \left\| T_{(p,w)} \psi \right\|_{M_{p,w} (\mathbb{T})} .
\]

In particular, it is well-known that \( \mathcal{M}_1 (\mathbb{T}) \subseteq M_{p,w} (\mathbb{T}) \), where \( \mathcal{M}_1 (\mathbb{T}) \) is the Banach algebra of periodic Marcinkiewicz multipliers, consisting of all functions \( \psi : \mathbb{T} \rightarrow \mathbb{C} \) such that

\[
\|\psi\|_{\mathcal{M}_1 (\mathbb{T})} = \sup_{z \in \mathbb{T}} |\psi (z)| + \sup_{k \in \mathbb{Z}} \text{var} (\psi, \Delta_k) < \infty
\]

(here \( \Delta_k \) is the dyadic arc of \( \mathbb{T} \) specified by \( \Delta_k = \{e^{i\theta} : \theta \in [t_k, t_{k+1}]\} \)). Moreover, \( \|\psi\|_{M_{p,w} (\mathbb{T})} \leq K_{p, \mathcal{C}} \|\psi\|_{\mathcal{M}_1 (\mathbb{T})} \). A key structural example of an element of \( M_{p,w} (\mathbb{T}) \) is furnished, for each \( k \in \mathbb{Z} \), by the function \( \chi_k (z) = e^{i k z} \), whose multiplier transform is \( \ell^k \), where \( \ell \) designates the left bilateral shift on \( \ell^p (w) \). In particular, for each \( \phi \in L^1 (\mathbb{T}) \), the \( n^d \) partial sum of its Fourier series \( S_n (\phi, e^{i k}) \equiv \sum_{k=-n}^{n} \phi (k) e^{i k z} \) belongs to \( M_{p,w} (\mathbb{T}) \), with multiplier transform expressed by

\[
T_{(p,w)} (S_n (\phi, \cdot)) = \sum_{k=-n}^{n} \phi (k) \ell^k.
\]

For further background items concerning our framework, the reader is referred to \([1–3]\). Our main result can now be stated as follows.

**Theorem 1.2.** Suppose that \( \psi \in \mathcal{M}_1 (\mathbb{T}) \). Then whenever \( 1 < p < \infty \), and \( w \in A_p (\mathbb{Z}) \) with an \( A_p (\mathbb{Z}) \) weight constant \( C \), we have:

\[
\sup_{n \geq 0} \left\| S_n (\psi_z (\cdot), (-)) \right\|_{M_{p,w} (\mathbb{T})} \leq K_{p, \mathcal{C}} \|\psi\|_{\mathcal{M}_1 (\mathbb{T})} ,
\]

where \( \psi_z (\cdot) \equiv \psi ((-\cdot) z) \). Consequently, \( \sum_{k=-\infty}^{\infty} e^{i k z} \psi (O) \ell^k \), the Fourier series of \( T_{\psi_z}^{(p,w)} \) relative to the strong operator topology of \( \mathcal{B} (\ell^p (w)) \), converges in the strong operator topology to \( T_{\psi_z}^{(p,w)} \) at each \( z \in \mathbb{T} \).

Thanks to the Dominated Ergodic Estimate Theorem of F.J. Martín-Reyes and A. de la Torre (in the form and notation of Theorem 2.5 in \([3]\)), one can transfer Theorem 1.2 to a broader framework, where the following outcome ensues.

**Theorem 1.3.** Suppose that \( 1 < p < \infty \), \( (\Omega, \mu) \) is a sigma-finite measure space, and \( \mathcal{U} \in \mathcal{B} (L^p (\mu)) \) is an invertible, disjoint, modulus mean-bounded operator. Let \( E (\cdot) : \mathbb{R} \rightarrow \mathcal{B} (L^p (\mu)) \) be the (idempotent-valued) spectral decomposition of \( \mathcal{U} \), and let \( \psi \in \mathcal{M}_1 (\mathbb{T}) \) be a continuous function. Then
\begin{equation}
\sup \left\{ \left\| \sum_{k=-n}^{n} z^k \hat{\psi}(k) \Delta^{k} \right\|_{\mathfrak{B}(L^p(\mu))] : n \geq 0, z \in T \right\} \leq K_{p,c} \left\| \psi \right\|_{\mathfrak{M}_1(T)},
\end{equation}

where $\mathcal{C}$ is the common $A_p(\mathbb{Z})$ weight constant of the weights $w^{(x)}$, $x \in \Omega$. Moreover, $\sum_{k=-\infty}^{\infty} z^k \hat{\psi}(k) \Delta^{k}$ the Fourier series (in the strong operator topology of $\mathfrak{B}(L^p(\mu)))$ for the Stieltjes convolution $\int_{[0,2\pi]} \psi_z(e^{it}) \, d\mathcal{E}(t)$ converges to $\int_{[0,2\pi]} \psi_z(e^{it}) \, d\mathcal{E}(t)$ in the strong operator topology at each $z \in T$.

\section{Proof of Theorem 1.2}

The key to demonstration of Theorem 1.2 resides in the following seminal forerunner.

\textbf{Theorem 2.1.} Suppose that $1 < p < \infty$, $w \in A_p(\mathbb{Z})$ with an $A_p(\mathbb{Z})$ weight constant $C$, and $\psi \in \mathfrak{M}_1(T)$. Then we have:

\begin{equation}
\sup \left\{ \left\| T_{\psi}(p,w) \right\|_{\mathfrak{B}(L^p(\mu))] : n \geq 0, m \in \mathbb{Z} \right\} \leq K_{p,c} \left\| \psi \right\|_{\mathfrak{M}_1(T)}.
\end{equation}

\textbf{Proof (Sketch).} For each non-negative integer $m$, define $I_m$ to be the arc $[e^{i\theta} : -m \leq \theta \leq m]$, and let $\chi_m$ symbolize the characteristic function, defined on $T$, of $I_m$. Define $\psi_m \in BV(T)$ by putting $\psi_m = \psi \chi_m$. Temporarily fix an arbitrary non-negative integer $n$, and observe that there is a non-negative integer $n$ (in general, depending on $n$) such that, for arbitrary $z \in T$,

\begin{equation}
\left\| s_n((\psi \chi_m) - s_n((\psi \chi_m)) \right\|_{\mathfrak{M}_p,L^p(\mu))} \leq K_{p,c} \left\| \psi \right\|_{\mathfrak{M}_1(T)}.
\end{equation}

Keeping this value of $n$ fixed, we now shift our attention to $\psi \chi_m$, in order to circumvent the dependence of $\psi \chi_m$ on $n$ by reducing matters to the pleasant operator-valued Fourier series phenomena associated with multiplier transforms defined by $BV(T)$ functions (as evinced in Theorem 4.4 of [1] and Theorem 4.1 of [2], which apply to spectral decomposability set in a broader framework that specializes to ours). Temporarily fix $z \in T$, $m \in \mathbb{Z}$, $m$ non-negative. For all $\zeta \in T$, let us consider the following expression.

\begin{equation}
F(\zeta) \equiv T_{(p,w)}((\psi \chi_m) \zeta). \end{equation}

On the right-hand side of (2.3) we can apply in succession the following items from [2]: Theorem 4.1; Theorem 4.5; and (3.2). Along with careful simplifications, this procedure shows that for arbitrary fixed $z \in T$,

\begin{equation}
\sup \left\{ \left\| s_n(((\psi \chi_m) \zeta \chi_m) - s_n(((\psi \chi_m) \zeta \chi_m)) \right\|_{\mathfrak{M}_p,L^p(\mu))] : n \geq 0, \zeta \in T \right\} \leq K_{p,c} \left\| ((\psi \chi_m) \zeta \chi_m) \right\|_{BV(T)}.
\end{equation}

In order to profit from this estimate, notice that the $BV(T)$ function involved in the majorant of (2.4) – specifically, $\zeta \in T \mapsto ((\psi \chi_m) \zeta \chi_m)\zeta$ – vanishes outside at most two disjoint closed subarcs of the fixed dyadic arc $\chi_m$, and coincides with $\psi \chi_m$ on each of these subarcs. Hence if we confine $z$ to the arc $\chi_m \equiv \{e^{i\theta} : 0 \leq \theta \leq t_{m+2} - t_{m+1}\}$, straightforward reasoning yields

\begin{equation}
\left\| ((\psi \chi_m) \chi_m) \right\|_{BV(T)} \leq K \left\| \psi \right\|_{\mathfrak{M}_1(T)}.
\end{equation}

Applying this to (2.4) we infer that

\begin{equation}
\sup \left\{ \left\| s_n(((\psi \chi_m) \chi_m) - s_n(((\psi \chi_m) \chi_m)) \right\|_{\mathfrak{M}_p,L^p(\mu))] : n \geq 0, \zeta \in T \right\} \leq K_{p,c} \left\| \psi \right\|_{\mathfrak{M}_1(T)}.
\end{equation}

By specializing the result in (2.5) to the case where the parameters $\zeta \in T$ and $z \in \chi_m$ are both taken to be 1, we arrive at the following central estimate.

\begin{equation}
\sup \left\{ \left\| s_n((\psi \chi_m) - s_n((\psi \chi_m)) \right\|_{\mathfrak{M}_p,L^p(\mu))] : n \geq 0 \right\} \leq K_{p,c} \left\| \psi \right\|_{\mathfrak{M}_1(T)}.
\end{equation}

Extensive calculations proceeding from (2.6) can be carried out to show that

\begin{equation}
\sup \left\{ \left\| (\chi_m \chi_m) s_n((\psi \chi_m) - s_n((\psi \chi_m)) \right\|_{\mathfrak{M}_p,L^p(\mu))] : n \geq 0 \right\} \leq K_{p,c} \left\| \psi \right\|_{\mathfrak{M}_1(T)}.
\end{equation}

We omit the details here for expository reasons. Applying this last estimate to the fixed but arbitrary non-negative integer $n$ in (2.2), we readily deduce (2.1) with the aid of standard features of $A_p$ weighted spaces. \qed
Proof of Theorem 1.2. When Theorem 2.1 is specialized to the setting $p = 2$ and applied in conjunction with the Littlewood–Paley inequalities for weighted spaces, we easily see that (1.1) holds for all $A_2(\mathbb{Z})$ weights. By invoking a suitable version of the recent “streamlined” rendition of Rubio de Francia’s Extrapolation Theorem (see Theorem 3.1 of [4]), we readily obtain (1.1) in the full range $1 < p < \infty$. The remaining conclusion of Theorem 1.2 can now be seen from this general case of (1.1) by calculations based on the norm density in $\ell^p(w)$ of the finitely supported vectors. □

References