## Differential geometry

# E-Bochner curvature tensor on generalized Sasakian space forms 

# Tenseur de courbure de type E-Bochner sur les espaces formes sasakiens généralisés 

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#### Abstract

Generalized Sasakian space forms have become today a rather specialized subject, but many contemporary works are concerned with the study of their properties and of their related curvature tensors. The goal of this paper is to study the E-Bochner curvature tensor on generalized Sasakian space forms, and to characterize the situations when it is, respectively: $E$-Bochner symmetric ( $\nabla B^{e}=0$ ); $E$-Bochner semisymmetric ( $R \cdot B^{e}=0$ ); $E$-Bochner recurrent; $E$-Bochner pseudosymmetric; such that $B^{e}(\xi, X) \cdot S=0$; such that $B^{e}(\xi, X) \cdot R=0$.


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## RÉS U M É

Les espaces formes sasakiens généralisés sont devenus aujourd'hui un sujet assez spécialisé, mais de nombreux travaux contemporains s'attachent à l'étude de leurs propriétés et des tenseurs de courbure associés. Le but de cette note est d'étudier le tenseur de courbure de type $E$-Bochner sur les espaces formes sasakiens généralisés, et de caractériser les conditions pour qu'il soit respectivement : $E$-Bochner symétrique ( $\nabla B^{e}=0$ ); $E$-Bochner semi-symétrique ( $R \cdot B^{e}=0$ ); $E$-Bochner récurrent ; $E$-Bochner pseudo-symétrique ; tel que $B^{e}(\xi, X) \cdot S=0$; tel que $B^{e}(\xi, X) \cdot R=0$.
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## 1. Introduction

The notion of generalized Sasakian space forms was introduced and studied by Alegre et al. [1] with several examples. A generalized Sasakian space form is an almost contact metric manifold ( $M, \phi, \xi, \eta, g$ ) whose curvature tensor is given by

[^0]\[

$$
\begin{align*}
R(X, Y) Z= & f_{1}\{g(Y, Z) X-g(X, Z) Y\}  \tag{1}\\
& +f_{2}\{g(X, \phi Z) \phi Y-g(Y, \phi Z) \phi X+2 g(X, \phi Y) \phi Z\} \\
& +f_{3}\{\eta(X) \eta(Z) Y-\eta(Y) \eta(Z) X \\
& +g(X, Z) \eta(Y) \xi-g(Y, Z) \eta(X) \xi\},
\end{align*}
$$
\]

where $f_{1}, f_{2}, f_{3}$ are differentiable functions on $M$ and $X, Y, Z$ are vector fields on $M$. In such a case, we will write the manifold as $M\left(f_{1}, f_{2}, f_{3}\right)$. This kind of manifolds appears as a natural generalization of Sasakian space forms by taking: $f_{1}=\frac{c+3}{4}$ and $f_{2}=f_{3}=\frac{c-1}{4}$, where $c$ denotes a constant $\phi$-sectional curvature. The $\phi$-sectional curvature of a generalized Sasakian space form $M\left(f_{1}, f_{2}, f_{3}\right)$ is $f_{1}+3 f_{2}$. Moreover, cosymplectic and Kenmotsu space forms are also considered as particular types of generalized Sasakian space forms. The generalized Sasakian space forms have also been studied in [2-4, 9,14,15,21,22] and many other instances.

On the other hand, Bochner [7] introduced a complex analogue to the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor. A geometric meaning of the Bochner curvature tensor is given by Blair [6]. By using the Boothby-Wang's fibration [8], Matsumoto and Chuman [17] introduced the notion of C-Bochner curvature. As an extension of C-Bochner curvature tensor, in [13] Endo defined E-Bochner curvature tensor $B^{e}$ as

$$
\begin{equation*}
B^{\mathrm{e}}(X, Y) Z=B(X, Y) Z-\eta(X) B(\xi, Y) Z-\eta(Y) B(X, \xi) Z-\eta(Z) B(X, Y) \xi \tag{2}
\end{equation*}
$$

where $B$ is the $C$-Bochner curvature tensor defined by

$$
\begin{align*}
B(X, Y) Z= & R(X, Y) Z+\frac{1}{2(n+2)}[S(X, Z) Y-S(Y, Z) X+g(X, Z) Q Y \\
& -g(Y, Z) Q X+S(\phi X, Z) \phi Y-S(\phi Y, Z) \phi X+g(\phi X, Z) Q \phi Y  \tag{3}\\
& -g(\phi Y, Z) Q \phi X+2 S(\phi X, Y) \phi Z+2 g(\phi X, Y) Q \phi Z \\
& -S(X, Z) \eta(Y) \xi+S(Y, Z) \eta(X) \xi-\eta(X) \eta(Z) Q Y+\eta(Y) \eta(Z) Q X] \\
& -\frac{\tau+2 n}{2(n+2)}[g(\phi X, Z) \phi Y-g(\phi Y, Z) \phi X+2 g(\phi X, Y) \phi Z] \\
& -\frac{\tau-4}{2(n+2)}[g(X, Z) Y-g(Y, Z) X] \\
& +\frac{\tau}{2(n+2)}[g(X, Z) \eta(Y) \xi-g(Y, Z) \eta(X) \xi+\eta(X) \eta(Z) Y-\eta(Y) \eta(Z) X]
\end{align*}
$$

where $\tau=\frac{r+2 n}{2(n+2)}, S$ is the Ricci tensor, $Q$ is the Ricci-operator, i.e. $g(Q X, Y)=S(X, Y)$ for all $X$ and $Y$, and $r$ is the scalar curvature of the manifold. In [20], contact manifolds with a C-Bochner curvature tensor have been studied. Also, an E-Bochner curvature tensor for a $(k, \mu)$-contact metric manifold has been studied in the papers [10,16]. Again, De and Ghosh [11] studied an E-Bochner curvature tensor on $N(k)$-contact metric manifolds.

In the context of generalized Sasakian space forms, Kim [15] studied conformally flat and locally symmetric generalized Sasakian space forms. De and Sarkar [9] studied some symmetric properties of generalized Sasakian space forms with a projective curvature tensor. In [21], Prakasha has shown that every generalized Sasakian space form is Weyl-pseudosymmetric. The symmetric properties of generalized Sasakian space forms have also been studied in [14] with a $W_{2}$-curvature tensor. Also, Prakasha and Nagaraja [22] studied quasi-conformally flat and quasi-conformally semisymmetric generalized Sasakian space forms. As a continuation of this study, in this paper we plan to study generalized Sasakian space forms satisfying certain curvature conditions on an E-Bochner curvature tensor.

The paper is organized as follows: Section 2 is devoted to preliminaries. In section 3, we characterize E-Bochner symmetric, E-Bochner semisymmetric, E-Bochner recurrent and E-Bochner pseudosymmetric generalized Sasakian space forms. Section 4 deals with the study of a generalized Sasakian space form satisfying the condition $B^{\mathrm{e}}(\xi, X) \cdot S=0$ and $B^{\mathrm{e}}(\xi, X) \cdot R=0$. Finally, some examples of generalized Sasakian space forms with $f_{1}=f_{3}$ and/or $f_{1}-f_{3}=1$ are given.

## 2. Preliminaries

An odd-dimensional Riemannian manifold $(M, g)$ is said to be an almost contact metric manifold [5] if there exist on $M$ a $(1,1)$ tensor field $\phi$, a vector field $\xi$ (called the structure vector field), and a 1 -form $\eta$ such that $\eta(\xi)=1, \phi^{2}(X)=$ $-X+\eta(X) \xi$ and $g(\phi X, \phi Y)=g(X, Y)-\eta(X) \eta(Y)$, for any vector fields $X, Y$ on $M$. In particular, in an almost contact metric manifold, we also have $\phi \xi=0$ and $\eta \circ \phi=0$.

Such a manifold is said to be a contact metric manifold if $\mathrm{d} \eta=\Phi$, where $\Phi(X, Y)=g(X, \phi Y)$ is called the fundamental 2 -form of $M$. If, in addition, $\xi$ is a Killing vector field, then $M$ is said to be a $K$-contact manifold. It is well known that a contact metric manifold is a $K$-contact manifold if and only if $\nabla_{X} \xi=-\phi X$, for any vector field $X$ on $M$. On the other hand,
the almost contact metric structure of $M$ is said to be normal if $[\phi, \phi](X, Y)=-2 \mathrm{~d} \eta(X, Y) \xi$, for any $X, Y$, where $[\phi, \phi]$ denotes the Nijenhuis torsion of $\phi$. A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if $\left(\nabla_{X} \phi\right) Y=g(X, Y) \xi-\eta(Y) X$, for any $X, Y$.

In addition to the relation (1), for a $(2 n+1)$-dimensional $(n>1)$ generalized Sasakian space form $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$, the following relations also hold [1]:

$$
\begin{align*}
& R(X, Y) \xi=\left(f_{1}-f_{3}\right)\{\eta(Y) X-\eta(X) Y\}  \tag{4}\\
& R(\xi, X) Y=\left(f_{1}-f_{3}\right)\{g(X, Y) \xi-\eta(Y) X\}  \tag{5}\\
& \eta(R(X, Y) Z)=\left(f_{1}-f_{3}\right)\{g(Y, Z) \eta(X)-g(X, Z) \eta(Y)\}  \tag{6}\\
& S(X, Y)=\left(2 n f_{1}+3 f_{2}-f_{3}\right) g(X, Y)-\left\{3 f_{2}+(2 n-1) f_{3}\right\} \eta(X) \eta(Y) \tag{7}
\end{align*}
$$

In view of (4)-(7), it can be easily constructed that in a $(2 n+1)$-dimensional ( $n \geq 2$ ) generalized Sasakian space form $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$, the $E$-Bochner curvature tensor satisfies the following conditions:

$$
\begin{align*}
B^{\mathrm{e}}(X, Y) \xi & =\frac{2\left(f_{1}-f_{3}-1\right)}{n+2}\{\eta(X) Y-\eta(Y) X\},  \tag{8}\\
B^{\mathrm{e}}(\xi, Y) Z & =\eta(Z) \frac{2\left(f_{1}-f_{3}-1\right)}{n+2}\{Y-\eta(Y) \xi\},  \tag{9}\\
\eta\left(B^{\mathrm{e}}(X, Y) Z\right) & =0 \tag{10}
\end{align*}
$$

by using (1) and the well-known fact that establishes that in a $K$-contact manifold, the sectional curvature of any plane section containing $\xi$ is equal to 1 .

Moreover, it is well known that any Sasakian manifold is a $K$-contact manifold. For a generalized Sasakian space form, the converse is also true.

## 3. E-Bochner semisymmetric and E-Bochner pseudosymmetric generalized Sasakian space forms

A Riemannian manifold $M$ is called locally symmetric if its curvature tensor $R$ is parallel, that is, $\nabla R=0$, where $\nabla$ denotes the Levi-Civita connection. As a proper generalization of locally symmetric manifolds, the notion of semisymmetric manifolds was defined by

$$
\begin{equation*}
(R(X, Y) \cdot R)(U, V) W=0, \quad X, Y, U, V, W \in \chi(M) \tag{11}
\end{equation*}
$$

and studied by many authors (e.g., [18,19,24]). A complete intrinsic classification of these spaces was given by Z.I. Szabó [23].

An almost contact manifold is said to be E-Bochner symmetric if $\nabla B^{e}=0$, and it is called E-Bochner semisymmetric if

$$
\begin{equation*}
\left(R(X, Y) \cdot B^{\mathrm{e}}\right)(U, V) W=0 \tag{12}
\end{equation*}
$$

Let $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$ be a E-Bochner semisymmetric generalized Sasakian space form. Then from (12), we have:

$$
\begin{equation*}
R(X, \xi) B^{\mathrm{e}}(U, V) W-B^{\mathrm{e}}(R(X, \xi) U, V) W-B^{\mathrm{e}}(U, R(X, \xi) V) W-B^{\mathrm{e}}(U, V) R(X, \xi) W=0 \tag{13}
\end{equation*}
$$

In view of (5) the above expression becomes

$$
\begin{align*}
& \left(f_{1}-f_{3}\right)\left[g\left(\xi, B^{\mathrm{e}}(U, V) W\right) X-g\left(X, B^{\mathrm{e}}(U, V) W\right) \xi\right.  \tag{14}\\
& \quad-\eta(U) B^{\mathrm{e}}(X, V) W+g(X, U) B^{\mathrm{e}}(\xi, V) W-\eta(V) B^{\mathrm{e}}(U, X) W \\
& \left.\quad+g(X, V) B^{\mathrm{e}}(U, \xi) W-\eta(W) B^{\mathrm{e}}(U, V) X+g(X, W) B^{\mathrm{e}}(U, V) \xi\right]=0
\end{align*}
$$

Putting $V=\xi$ in (14) and using (9) and (10), we have

$$
\begin{equation*}
\left(f_{1}-f_{3}\right)\left[B^{\mathrm{e}}(U, X) W-\frac{2\left(f_{1}-f_{3}-1\right)}{n+2}\{g(X, W) U-g(U, W) X\}\right]=0 \tag{15}
\end{equation*}
$$

This implies either $f_{1}-f_{3}=0$, or

$$
\begin{equation*}
B^{\mathrm{e}}(U, X) W=\frac{2\left(f_{1}-f_{3}-1\right)}{n+2}\{g(X, W) U-g(U, W) X\} \tag{16}
\end{equation*}
$$

Contracting $U$ in the above equation, we conclude that

$$
\begin{equation*}
\frac{2\left(f_{1}-f_{3}-1\right)}{n+2} 2 n g(\phi X, \phi W)=0 \tag{17}
\end{equation*}
$$

Since $g(\phi X, \phi W) \neq 0$, in general, therefore we obtain from (17) that $\frac{2\left(f_{1}-f_{3}-1\right)}{n+2}=0$, that is,

$$
\begin{equation*}
f_{1}-f_{3}=1 \tag{18}
\end{equation*}
$$

Now with the help of (18), equation (16) reduces to

$$
B^{\mathrm{e}}(U, X) W=0
$$

That is, $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$ is $E$-Bochner flat. Hence we conclude the following:
Theorem 3.1. A $(2 n+1)$-dimensional $(n \geq 2)$ E-Bochner semisymmetric generalized Sasakian space form is either E-Bochner flat (then $f_{1}-f_{3}=1$ ) or $f_{1}=f_{3}$.

It is clear that $\nabla B^{\mathrm{e}}=0 \Rightarrow R \cdot B^{\mathrm{e}}=0$, and from Theorem 3.1 we get:

Corollary 3.2. A $(2 n+1)$-dimensional $(n \geq 2)$ E-Bochner symmetric generalized Sasakian space form is either $E$-Bochner flat (then $f_{1}-f_{3}=1$ ) or $f_{1}=f_{3}$.

Remark 3.3. A Riemannian manifold is said to be $E$-Bochner recurrent if $\nabla B^{\mathrm{e}}=A \otimes B^{\mathrm{e}}$, where $A$ is a non-zero 1 -form. It can be easily shown that a $E$-Bochner recurrent manifold satisfies $R \cdot B^{e}=0$. Hence we immediately get the following:

Corollary 3.4. A $(2 n+1)$-dimensional $(n \geq 2)$ E-Bochner recurrent generalized Sasakian space form is either E-Bochner flat or $f_{1}=f_{3}$.

In particular, for a Sasakian space form, $f_{1}=\frac{c+3}{4}$ and $f_{3}=\frac{c-1}{4}$. So, $f_{1} \neq f_{3}$. Hence we have the following corollary:
Corollary 3.5. A $(2 n+1)$-dimensional $(n \geq 2)$ Sasakian space form is $E$-Bochner semisymmetric if and only if it is $E$-Bochner flat.
Next, for a ( $0, k$ )-tensor field $T$ on $M, k \geq 1$, and a symmetric ( 0,2 )-tensor field $A$ on $M$, we define the $(0, k+2)$-tensor fields $R \cdot T$ and $Q(A, T)$ by

$$
(R . T)\left(X_{1}, \ldots, X_{K} ; X, Y\right)=-T\left(R(X, Y) X_{1}, X_{2}, \ldots, X_{k}\right)-\ldots-T\left(X_{1}, \ldots, X_{k-1}, R(X, Y) X_{k}\right)
$$

and

$$
Q(A, T)\left(X_{1}, \ldots, X_{K} ; X, Y\right)=-T\left(\left(X \wedge_{A} Y\right) X_{1}, X_{2}, \ldots, X_{k}\right)-\ldots-T\left(X_{1}, \ldots, X_{k-1},\left(X \wedge_{A} Y\right) X_{k}\right)
$$

respectively, where $X \wedge_{A} Y$ is the endomorphism given by

$$
\begin{equation*}
\left(X \wedge_{A} Y\right) Z=A(Y, Z) X-A(X, Z) Y \tag{19}
\end{equation*}
$$

A Riemannian manifold $M$ is said to be pseudosymmetric (in the sense of R. Deszcz [12]) if

$$
R \cdot R=L_{R} Q(g, R)
$$

holds on $U_{R}=\left\{x \in M \left\lvert\, R-\frac{r}{n(n-1)} G \neq 0\right.\right.$ at $\left.x\right\}$, where $G$ is the $(0,4)$-tensor defined by $G\left(X_{1}, X_{2}, X_{3}, X_{4}\right)=g\left(\left(X_{1} \wedge X_{2}\right) X_{3}, X_{4}\right)$ and $L_{R}$ is some smooth function on $U_{R}$. A Riemannian manifold $M$ is said to be E-Bochner pseudosymmetric if

$$
\begin{equation*}
\left(R(X, Y) \cdot B^{\mathrm{e}}\right)(U, V) W=L_{B^{\mathrm{e}}} Q\left(g, B^{\mathrm{e}}\right)(U, V, W ; X, Y) \tag{20}
\end{equation*}
$$

holds on the set $U_{B^{e}}=\left\{x \in M: B^{\mathrm{e}} \neq 0\right\}$ at $x$, where $L_{B^{e}}$ is some function on $U_{B^{e}}$ and $B^{e}$ is the E-Bochner curvature tensor.
Let $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$ be a $(2 n+1)$-dimensional $(n \geq 2)$ E-Bochner pseudosymmetric generalized Sasakian space form. Then, from (19) and (20), we have

$$
\begin{equation*}
\left(R(\xi, Y) \cdot B^{\mathrm{e}}\right)(U, V) W=L_{B}\left[\left((\xi \wedge Y) \cdot B^{\mathrm{e}}\right)(U, V) W\right] . \tag{21}
\end{equation*}
$$

If $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$ be a $(2 n+1)$-dimensional ( $n \geq 2$ ) generalized Sasakian space form, from (5) and (19) we get

$$
\begin{equation*}
R(\xi, X) Y=\left(f_{1}-f_{3}\right)(\xi \wedge X) Y \tag{22}
\end{equation*}
$$

In view of (21) in (22), it is easy to see that

$$
\begin{equation*}
L_{B^{\mathrm{e}}}=\left(f_{1}-f_{3}\right) \tag{23}
\end{equation*}
$$

Hence, by taking into account previous calculations and discussions, we conclude the following:

Theorem 3.6. Let $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$ be a $(2 n+1)$-dimensional ( $n \geq 2$ ) generalized Sasakian space form. If $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$ is E-Bochner pseudosymmetric, then $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$ is either $E$-Bochner flat, in which case $f_{1}-f_{3}=1$ or $L_{B^{e}}=f_{1}-f_{3}$ holds on $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$.

But $L_{B^{e}}$ needs not be zero, in general and hence there exist E-Bochner pseudosymmetric manifolds which are not EBochner semisymmetric. Thus the class of E-Bochner pseudosymmetric manifolds is a natural extension of the class of E-Bochner semisymmetric manifolds. Thus, if $L_{B^{e}} \neq 0$, then it is easy to see that $R \cdot B^{\mathrm{e}}=\left(f_{1}-f_{3}\right) Q\left(g, B^{\mathrm{e}}\right)$, which implies that the pseudosymmetric function $L_{B^{e}}=f_{1}-f_{3}$. Therefore, we able to state the following result:

Theorem 3.7. Every generalized Sasakian space form is E-Bochner pseudosymmetric of the form $R \cdot B^{\mathrm{e}}=\left(f_{1}-f_{3}\right) Q\left(g, B^{\mathrm{e}}\right)$.
4. Generalized Sasakian space forms satisfying the conditions $B^{\mathrm{e}}(\xi, X) \cdot S=0$ and $B^{\mathrm{e}}(\xi, X) \cdot R=0$

In this section, we study a generalized Sasakian space form $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)(n>1)$ satisfying the conditions $B^{\mathrm{e}}(\xi, X)$. $S=0$ and $B^{\mathrm{e}}(\xi, X) \cdot R=0$ as in the following subcases.

Case (i): Generalized Sasakian space forms satisfying $B^{\mathrm{e}}(\xi, X) \cdot S=0$.

The condition $B^{\mathrm{e}}(\xi, X) \cdot S=0$ is equivalent to

$$
\begin{equation*}
S\left(B^{\mathrm{e}}(\xi, X) U, \xi\right)+S\left(U, B^{\mathrm{e}}(\xi, X) \xi\right)=0 \tag{24}
\end{equation*}
$$

For a $(2 n+1)$-dimensional generalized Sasakian space form $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$, it is well known that

$$
\begin{equation*}
S(X, \xi)=2 n\left(f_{1}-f_{3}\right) \eta(X) \tag{25}
\end{equation*}
$$

In view of (9), (25) gives

$$
\begin{equation*}
S\left(B^{\mathrm{e}}(\xi, X) U, \xi\right)=2 n\left(f_{1}-f_{3}\right) \eta\left(B^{\mathrm{e}}(\xi, X) U\right) \tag{26}
\end{equation*}
$$

Using (9) in the above equation, we get

$$
\begin{equation*}
S\left(B^{\mathrm{e}}(\xi, X) U, \xi\right)=0 \tag{27}
\end{equation*}
$$

Again, in view of (9) we have

$$
\begin{equation*}
S\left(B^{\mathrm{e}}(\xi, X) \xi, U\right)=\frac{2\left(f_{1}-f_{3}-1\right)}{(n+2)}\left(S(X, U)-2 n\left(f_{1}-f_{3}\right) \eta(X) \eta(U)\right) \tag{28}
\end{equation*}
$$

Substituting (27) and (28) in (24) followed by a simple calculation gives

$$
\begin{equation*}
\frac{2\left(f_{1}-f_{3}-1\right)}{(n+2)}\left(S(X, U)-2 n\left(f_{1}-f_{3}\right) \eta(X) \eta(U)\right)=0 \tag{29}
\end{equation*}
$$

which implies that either $f_{1}-f_{3}=1$ or

$$
\begin{equation*}
S(X, U)=2 n\left(f_{1}-f_{3}\right) \eta(X) \eta(U) \tag{30}
\end{equation*}
$$

Again, if $f_{1}-f_{3}=1$ then we can easily obtain (9) that $B^{\mathrm{e}}(\xi, X) \cdot S=0$.
And, if the space form satisfies the relation (30), then in view of (9), we have:

$$
\begin{aligned}
B^{\mathrm{e}}(\xi, X) \cdot S & =-S\left(B^{\mathrm{e}}(\xi, X) Y, V\right)-S\left(U, B^{\mathrm{e}}(\xi, X) V\right) \\
& =-2 n\left(f_{1}-f_{3}\right)\left[\eta\left(B^{\mathrm{e}}(\xi, X) U\right) \eta(V)+\eta(U) \eta\left(B^{\mathrm{e}}(\xi, X) V\right)\right] \\
& =0
\end{aligned}
$$

In view of the above discussion, we state the following:
Theorem 4.1. Let $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$ be a $(2 n+1)$-dimensional ( $n \geq 2$ ) generalized Sasakian space form. Then $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$ satisfies $B^{\mathrm{e}}(\xi, X) \cdot S=0$ if and only if either $f_{1}-f_{3}=1$ or the Ricci tensor satisfies the relation $S(X, U)=2 n\left(f_{1}-f_{3}\right) \eta(X) \eta(U)$.

## Case (ii): Generalized Sasakian space forms satisfying $B^{\mathrm{e}}(\xi, X) \cdot R=0$.

The condition $B^{\mathrm{e}}(\xi, X) \cdot R=0$ gives

$$
\begin{align*}
& B^{\mathrm{e}}(\xi, U) R(X, Y) Z-R\left(B^{\mathrm{e}}(\xi, U) X, Y\right) Z  \tag{31}\\
& \quad-R\left(X, B^{\mathrm{e}}(\xi, U) Y\right) Z-R(X, Y) B^{\mathrm{e}}(\xi, U) Z=0,
\end{align*}
$$

which in view of (9) provides

$$
\begin{align*}
& \frac{2\left(f_{1}-f_{3}-1\right)}{n+2}[\eta(R(X, Y) Z)(U-\eta(U) \xi)-\eta(X) R(U-\eta(U) \xi, Y) Z  \tag{32}\\
& \quad+\eta(Y) R(X, U-\eta(U) \xi) Z-\eta(Z) R(X, Y)(U-\eta(U) \xi)]=0
\end{align*}
$$

From (32) we have either $f_{1}-f_{3}=1$, or

$$
\begin{align*}
& {[\eta(R(X, Y) Z) U-\eta(U) \eta(R(X, Y) Z) \xi)-\eta(X) R(U, Y) Z+\eta(X) \eta(U) R(\xi, Y) Z}  \tag{33}\\
& \quad-\eta(Y) R(X, U) Z+\eta(Y) \eta(U) R(X, \xi) Z-\eta(Z) R(X, Y) U+\eta(U) \eta(Z) R(X, Y) \xi)]=0
\end{align*}
$$

Setting $X=Z=\xi$ in (33) and using (4), we get

$$
\left(f_{1}-f_{3}\right)[g(Y, U)-\eta(Y) \eta(U)] \xi=0
$$

The above relation yields $f_{1}-f_{3}=0$, since $g(Y, U) \neq \eta(Y) \eta(U)$ in general. Thus, we are able to state the following theorem:
Theorem 4.2. Let $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$ be a $(2 n+1)$-dimensional ( $n \geq 2$ ) generalized Sasakian space form. If $M^{2 n+1}\left(f_{1}, f_{2}, f_{3}\right)$ satisfies $B^{\mathrm{e}}(\xi, X) \cdot R=0$ then either $f_{1}-f_{3}=1$ or $f_{1}-f_{3}=0$.

## 5. Examples

In this section, we give some examples on generalized Sasakian space forms, with $f_{1}=f_{3}$ and /or with $f_{1}-f_{3}=1$.

Example 5.1. [1] A cosymplectic-space form, i.e. a cosymplectic manifold with constant $\phi$-sectional curvature $c$, is a generalized Sasakian space form with $f_{1}=f_{2}=f_{3}=\frac{c}{4}$. Hence $f_{1}=f_{3}$.

Example 5.2. [1] A non-Sasakian generalized Sasakian space form satisfying the equation

$$
R(X, Y, Z, W)=R(\phi X, \phi Y, Z, W)+R(\phi X, Y, \phi Z, W)+R(\phi X, Y, Z, \phi W)
$$

has $f_{1}=f_{3}$.
Example 5.3. [1] Let $N(a, b)$ be a generalized complex space form of dimension 4, then by the warped product $M=\mathbb{R} \times N$ endowed with the almost contact metric structure $\left(\phi, \xi, \eta, g_{f}\right)$, it is a generalized Sasakian space form $M\left(f_{1}, f_{2}, f_{3}\right)$ with

$$
f_{1}=\frac{a-\left(f^{\prime}\right)^{2}}{f^{2}}, \quad f_{2}=\frac{b}{f^{2}}, \quad f_{3}=\frac{a-\left(f^{\prime}\right)^{2}}{f^{2}}+\frac{f^{\prime \prime}}{f}
$$

where $f=f(t), t \in \mathbb{R}$ and $f^{\prime}$ denotes the derivative of $f$ with respect to $t$.
If we choose $a=0, b=1$ and $f(t)=t$ with $t>0$, then $f_{1}=-\frac{1}{t^{2}}, f_{2}=\frac{1}{t^{2}}$ and $f_{3}=-\frac{1}{t^{2}}$. Hence $f_{1}=f_{3}$.

Example 5.4. [2] A Sasakian space form, i.e. a Sasakian manifold with constant $\phi$-sectional curvature $c$ is a generalized Sasakian space form with $f_{1}=\frac{c+3}{4}, f_{2}=f_{3}=\frac{c-1}{4}$. Hence $f_{1}-f_{3}=1$.

Example 5.5. [4] Let $N(c)$ is a complex space form, and by the warped product $M=\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times{ }_{f} N$ endowed with the almost contact metric structure ( $\phi, \xi, \eta, g_{f}$ ) is a generalized Sasakian space form with functions

$$
f_{1}=\frac{c-4 f^{\prime 2}}{4 f^{2}}, \quad f_{2}=\frac{c}{4 f^{2}}, \quad f_{3}=\frac{c-4 f^{\prime 2}}{4 f^{2}}+\frac{f^{\prime \prime}}{f}
$$

where $f=f(t), t \in \mathbb{R}$ and $f^{\prime}$ denotes the derivative of $f$ with respect to $t$.
If we choose $f(t)=\cos t$, then $f_{1}-f_{3}=1$.

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## References

[1] P. Alegre, D.E. Blair, A. Carriazo, Generalized Sasakian space forms, Isr. J. Math. 14 (2004) 157-183.
[2] P. Alegre, A. Carriazo, Structures on generalized Sasakian space forms, Differ. Geom. Appl. 26 (2008) 656-666.
[3] P. Alegre, A. Carriazo, Submanifolds generalized Sasakian space forms, Taiwan. J. Math. 13 (2009) 923-941.
[4] P. Alegre, A. Carriazo, Generalized Sasakian space forms and conformal change of metric, Results Math. 59 (2011) 485-493.
[5] D.E. Blair, Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics, vol. 509, Springer-Verlag, 1976.
[6] D.E. Blair, On the geometric meaning of the Bochner tensor, Geom. Dedic. 4 (1975) 33-38.
[7] S. Bochner, Curvature and Betti numbers, Ann. of Math. 50 (1949) 77-93.
[8] W.M. Boothby, H.C. Wang, On contact manifolds, Ann. of Math. (2) 68 (1958) 721-734.
[9] U.C. De, A. Sarkar, On the projective curvature tensor of generalized Sasakian space forms, Quaest. Math. 33 (2010) 245-252.
[10] U.C. De, S. Samui, E-Bochner curvature tensor on $(k, \mu)$-contact metric manifolds, Int. Electron. J. Geom. 7 (1) (2014) 143-153.
[11] U.C. De, S. Ghosh, E-Bochner curvature tensor on $N(k)$-contact metric manifolds, Hacet. J. Math. Stat. 43 (3) (2014) 365-374.
[12] R. Deszcz, On pseudosymmetric spaces, Bull. Soc. Math. Belg., Sér. A 44 (1) (1992) 1-34.
[13] H. Endo, On K-contact Riemannian manifolds with vanishing E-contact Bochner curvature tensor, Colloq. Math. 62 (1991) $293-297$.
[14] S.K. Hui, A. Sarkar, On the $W_{2}$-curvature tensor of generalized Sasakian space forms, Math. Pannon. 23 (1) (2012) 113-124.
[15] U.K. Kim, Conformally flat generalized Sasakian space forms and locally symmetric generalized Sasakian space forms, Note Mat. 26 (2006) 55-67.
[16] J.S. Kim, M.M. Tripathi, J.D. Choi, On C-Bochner curvature tensor of a contact metric manifold, Bull. Korean Math. Soc. 42 (2005) 713-724.
[17] M. Matsumoto, G. Chuman, On the C-Bochner curvature tensor, TRU Math. 5 (1969) 21-30.
[18] K. Nomizu, On hypersurfaces satisfying a certain condition on the curvature tensor, Tohoku Math. J. 20 (1968) 46.
[19] Y.A. Ogawa, Condition for a compact Kahlerian space to be locally symmetric, Nat. Sci. Rep. Ochanomizu Univ. 28 (1971) 21.
[20] G. Pathak, U.C. De, Y.H. Kim, Contact manifolds with C-Bochner curvature tensor, Bull. Calcutta Math. Soc. 96 (1) (2004) 45-50.
[21] D.G. Prakasha, On generalized Sasakian space forms with Weyl-conformal curvature tensor, Lobachevskii J. Math. 33 (3) (2012) 223-228.
[22] D.G. Prakasha, H.G. Nagaraja, On quasi-conformally flat and quasi-conformally semisymmetric generalized Sasakian space forms, CUBO 15 (3) (2013) 59-70.
[23] Z.I. Szabó, Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R=0$. The local version, J. Differ. Geom. 17 (1982) $531-582$.
[24] S. Tanno, Locally symmetric K-contact Riemannian manifold, Proc. Jpn. Acad. 43 (1967) 581.


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