On an elliptic equation of \( p \)-Kirchhoff type with convection term

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Abstract

In this paper, by using Galerkin’s approach with a priori estimates, we establish the existence of solutions to a class of elliptic problems given by a system of nonlinear equations of \( p \)-Kirchhoff type with a convection term.

Résumé

Dans ce travail, on utilise la méthode de Galerkin avec une estimation a priori pour montrer l’existence de solutions à une classe de problèmes elliptiques, donnée par un système d’équations non linéaires de type \( p \)-Kirchhoff en présence d’un terme de gradient.

1. Introduction

In this work, we study the existence of a solution for the following problem

\[
-M \left( \int_{\Omega} |\nabla u|^p \, dx \right) \Delta_p u = f(x, u, \nabla u) \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega,
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary, \( 1 < p < N \). The functional \( M \) verifies,

\[
( M_1 ) \ M : (0, +\infty) \rightarrow (0, +\infty) \text{ continuous and } m_0 = \inf_{s > 0} M(s) > 0.
\]

Because of the term \( \int_{\Omega} |\nabla u|^p \, dx \), a problem like (1.1) is nonlocal, which provokes some mathematical difficulties and also makes the study of such a problem particularly interesting. In [9], Kirchhoff proposed a model of equation extending the classical d’Alembert wave equation for free vibrations of elastic strings by considering the effect of a change in the length of the string during the vibration. Many problems of Kirchhoff type have been studied; we just quote some recent references [1–4,6,7,10–12]...
An other distinguished feature in studying such kind of equations lies in the presence of the gradient term \( \nabla u \) (or convection term), which makes the problem (1.1) nonlocal and nonvariational. To overcome the difficulties brought, our approach was motivated by [5] with the use of the method of Galerkin and, as a consequence, by the Brouwer fixed-point theorem.

Assume that

\[
  f(x, u, \nabla u) = h(x, u) + g(x, \nabla u)
\]

where \( h \) is sublinear function and \( g \) is bounded (from above) by a gradient term. For these functions, we set these hypotheses:

\[ (H_1) \quad h : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is locally Holder continuous, there exist positive constants } a_1 \in L^p(\mathbb{R}^N), b_1 \in L^{\frac{p}{p-(\gamma+1)}}(\mathbb{R}^N) \text{ and } r_1 \in (0, p - 1), \text{ such that } 0 < h(x, t) \leq a_1 + b_1|t|^{r_1}, \quad \forall (x, t) \in \Omega \times \mathbb{R}, \]

where \( p' \) is the conjugate of \( p \).

\[ (H_2) \quad g : \Omega \times \mathbb{R}^N \to \mathbb{R} \text{ is locally Holder continuous, there exist positive constants } a_2 \in L^p(\mathbb{R}^N), b_2 \in L^{\frac{p}{p-2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ and } r_2 \in (0, p - 1), \text{ such that } 0 \leq g(x, \lambda) \leq a_2 + b_2|\lambda|^{r_2}. \quad \forall (x, \lambda) \in \Omega \times \mathbb{R}^N. \]

Now, we set the main result of the present paper (Theorem 1.1).

**Theorem 1.1.** Under the conditions \((M_1), (H_1)-(H_2)\), the problem (1.1) has at least a nontrivial solution.

2. **Proof of the main result**

Let \( L^s(\Omega) \) be the Lebesgue space equipped with the norm \( |u|_s = (\int_\Omega |u|^s \, dx)^{\frac{1}{s}} \), \( 1 \leq s < \infty \), and let \( W_0^{1,p}(\Omega) \) be the usual Sobolev space with respect to the norm

\[
  \|u\| = \left( \int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}.
\]

We recall that \( u \in W_0^{1,p}(\Omega) \) is a weak solution to the problem (1.1) if it verifies

\[
  M(\|u\|) \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx - \int_\Omega \left( g(x, \nabla u) + h(x, u) \right) v \, dx = 0,
\]

for all \( v \in W_0^{1,p}(\Omega) \).

**Lemma 2.1.** (See [8].) Let \( F : \mathbb{R}^N \to \mathbb{R}^N \) be a continuous function with \( \langle F(x), x \rangle \geq 0 \), for all \( x \) verifying \( |x| = \rho > 0 \), where \( \langle ., . \rangle \) is the usual inner product of \( \mathbb{R}^N \). Then there exists \( \gamma \in B_\rho(0) \) such that \( F(\gamma) = 0 \).

**Proof of Theorem 1.1.** Let \( \Gamma = \{e_1, \ldots, e_n, \ldots\} \subset W_0^{1,p}(\Omega) \) such that

\[
  W_0^{1,p}(\Omega) = \text{span}\{e_1, \ldots, e_n\},
\]

since \( W_0^{1,p}(\Omega) \) is a reflexive and separable Banach space.

Define \( V_n = \text{span}\{e_1, \ldots, e_n\} \). It is known that \( V_n \) and \( \mathbb{R}^N \) are isomorphic and for \( \eta \in \mathbb{R}^N \), we have an unique \( v \in V_n \) by the identification \( \eta \mapsto \sum_{i=1}^N \eta_i e_i = v \).

Define the function \( F = (F_1, \ldots, F_N) : \mathbb{R}^N \to \mathbb{R} \) by

\[
  F_i(u) = M \left( \int_\Omega |\nabla u|^p \, dx \right) \int_\Omega |\nabla u|^{p-2} \nabla u \nabla e_i \, dx - \int_\Omega \left( g(x, \nabla u) + h(x, u) \right) e_i \, dx, \quad u \in V_i.
\]

Our method consists in considering a class of auxiliary problems,

\[
  M \left( \int_\Omega |\nabla u_n|^p \, dx \right) \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla e_i \, dx = \int_\Omega \left( g(x, \nabla u_n) + h(x, u_n) \right) e_i \, dx, \quad (2.1)
\]

we show the existence of weak solutions \( u_n \in V_n \) for the problem (2.1).
For $u \in V_n$, we have that

$$
\langle F(u), u \rangle \geq C_1 \|u\|^p - \int h(x, u) u \, dx - \int g(x, \nabla u) u \, dx
$$

$$
\geq C_1 \|u\|^p - |a_1| u_p \|u\|_p - |b_1| \frac{p-1+1}{p} \|u\|^r_1 - |a_2| u_p \|u\|_p - \left( \int \frac{b_2^p}{p^{r_1+1}} \|u\|^r_1 \right)^{\frac{p}{r_1+1}} - |a_2| u_p \|u\|_p - C b_2^p \frac{p}{p^{r_2+1}} \|u\|^r_2.
$$

we noticed here that $pp' = p + p'$.

As we know that

$$
W^{1,p+q'}(\Omega) \hookrightarrow W^{1,p}(\Omega),
$$

then there is $C_5 > 0$ such that

$$
\|u\| \leq C_5 \|\nabla u\|_{p+p'},
$$

thus, there exist $C_6, C_7 > 0$ such that

$$
\langle F(u), u \rangle \geq C_1 \|u\|^p - |a_1| u_p \|u\|_p - C_6 b_1 \frac{p}{p-(r_1+1)} \|u\|^r_1 - |a_2| u_p \|u\|_p - C_7 b_2^p \frac{p}{p^{r_2+1}} \|u\|^r_2 + 1.
$$

Since $r_i + 1 < p, i = 1, 2$, there exist positive numbers $\rho$ and $R$ such that

$$
\langle F(u), u \rangle \geq \rho > 0 \text{ on } \|u\| = R.
$$

$F$ is continuous, so, by Lemma 2.1, the system (2.1) has a solution $u_n \in V_n \subset W^{1,p}_0(\Omega)$ with $\|u_n\| \leq R$.

Furthermore, up to a subsequence, we may assume that there exists $u \in W^{1,p}_0(\Omega)$ such that

$$
\begin{align*}
    u_n & \rightarrow u \text{ in } W^{1,p}_0(\Omega), \\
    u_n & \rightharpoonup u \text{ a.e } x \in \Omega.
\end{align*}
$$

By using the Dominated Convergence Theorem, we get

$$
\int h(x, u_n) \omega \, dx \rightarrow \int h(x, u) \omega \, dx \text{ for } \omega \in V_k.
$$

Now, we set $A_n(x) = g(x, \nabla u_n(x))$.

In view of condition (H2), we have

$$
|A_n(x)| \leq |g| + \left| \frac{b_2^p}{p} \right| \|\nabla u_n\|^r_2,
$$

$$
\leq c_1 + c_2 R^2.
$$

From the reflexivity of $L^{\frac{p}{r}}(\Omega)$, passing to a subsequence if necessary; there is $A \in L^{\frac{p}{r}}(\Omega)$ such that

$$
\int A_n \varphi \, dx \rightarrow \int A \varphi \, dx, \forall \varphi \in L^{q}(\Omega)
$$

with

$$
\frac{r_2}{p} + \frac{1}{q} = 1.
$$

Therefore, $W^{1,p}_0(\Omega) \hookrightarrow L^{q}(\Omega)$, because $q < p^*$, it yields

$$
M \left( \int \|u_n\|^r_1 \, dx \right) \int \|u_n\|^r_2 \nabla u_n \nabla \varphi \, dx = \int \left( g(x, \nabla u_n) + h(x, u_n) \right) \varphi \, dx, \forall \varphi \in W^{1,p}_0(\Omega).
$$

On the other hand we have

$$
u_n \rightarrow u \text{ (weakly)},$$

so when \( n \to \infty \),
\[
\int_\Omega \left( h(x, u_n) - h(x, u) \right) (u_n - u) \, dx + \int_\Omega \left( A_n(x) - A(x) \right) (u_n - u) \, dx \to 0,
\]
then
\[
m_0 \int_\Omega \left( |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) (\nabla u_n - \nabla u) \, dx \to 0.
\]

Using the following inequality
\[
\forall x, y \in \mathbb{R}^N
\]
\[
|x - y|^\gamma \leq 2^\gamma (|x|^\gamma - x - |y|^\gamma - y) \cdot (x - y) \quad \text{if } \gamma \geq 2,
\]
\[
|x - y|^\gamma \leq \frac{1}{\gamma - 1} (|x| + |y|)^2 - \gamma (|x|^{\gamma - 2} \, x - |y|^{\gamma - 2} \, y) \cdot (x - y) \quad \text{if } 1 < \gamma < 2,
\]
where \( x, y \) is the inner product in \( \mathbb{R}^N \), we get
\[
\int_\Omega |\nabla u_n - \nabla u|^p \, dx \leq \int_\Omega \left( |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) (\nabla u_n - \nabla u) \, dx.
\]

Hence,
\[
\| u_n - u \| \to 0,
\]
and then
\[ u_n \to u \text{ in } W^{1,p}_0(\Omega). \]

Since \( M \) is continuous,
\[
M \left( \int_\Omega |\nabla u_n|^p \, dx \right) \to \int_\Omega M \left( \int_\Omega |\nabla u|^p \right) \, dx.
\]

So we obtain that \( u \) is a weak solution to the problem (1.1) and from (H1), \( u \neq 0 \). \( \square \)

**Remark 2.2.** By a standard argument and straightforward computations, there exists \( C > 0 \) such that
\[
\| u \| \leq C,
\]
so we can conclude that the solution \( u \) is a decay solution of our problem, i.e
\[
\lim_{|x| \to \infty} u(x) = 0.
\]

**References**


