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Ordinary differential equations/Partial differential equations On an elliptic equation of p-Kirchhoff type with convection term

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ABSTRACT

In this paper, by using Galerkin's approach with a priori estimates, we establish the existence of solutions to a class of elliptic problems given by a system of nonlinear equations of p-Kirchhoff type with a convection term.

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RÉSUMÉ

Dans ce travail, on utilise la méthode de Galerkin avec une estimation a priori pour montrer l'existence de solutions à une classe de problèmes elliptiques, donnée par un système d'équations non linéaires de type *p*-Kirchhoff en présence d'un terme de gradient. © 2015 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

In this work, we study the existence of a solution for the following problem

$$-M\left(\int_{\Omega} |\nabla u|^{p} dx\right) \Delta_{p} u = f(x, u, \nabla u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega, \qquad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, 1 . The functional <math>M verifies, $(M_1) \ M : (0, +\infty) \to (0, +\infty)$ continuous and $m_0 = \inf_{s>0} M(s) > 0$.

Because of the term $M(\int_{\Omega} |\nabla u|^p dx)$, a problem like (1.1) is nonlocal, which provokes some mathematical difficulties and also makes the study of such a problem particularly interesting. In [9], Kirchhoff proposed a model of equation extending the classical d'Alembert wave equation for free vibrations of elastic strings by considering the effect of a change in the length of the string during the vibration. Many problems of Kirchhoff type have been studied; we just quote some recent references [1–4,6,7,10–12]...

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An other distinguished feature in studying such kind of equations lies in the presence of the gradient term ∇u (or convection term), which makes the problem (1.1) nonlocal and nonvariational. To overcome the difficulties brought, our approach was motivated by [5] with the use of the method of Galerkin and, as a consequence, by the Brouwer fixed-point theorem.

Assume that

$$f(x, u, \nabla u) = h(x, u) + g(x, \nabla u)$$

where h is sublinear function and g is bounded (from above) by a gradient term. For these functions, we set these hypotheses.

 (H_1) $h: \Omega \times \mathbb{R} \to \mathbb{R}$ is locally Holder continuous, there exist positive constants $a_1 \in L^{p'}(\mathbb{R}^N), b_1 \in L^{\frac{p}{p-(r_1+1)}}(\mathbb{R}^N)$ and $r_1 \in (0, p-1)$, such that

 $0 < h(x,t) \le a_1 + b_1 |t|^{r_1}, \ \forall (x,t) \in \Omega \times \mathbb{R},$

where p' is the conjugate of p.

 (H_2) $g: \Omega \times \mathbb{R}^N \to \mathbb{R}$ is locally Holder continuous, there exist positive constants $a_2 \in L^{p'}(\mathbb{R}^N), b_2 \in L^{\frac{p}{p-r_2}}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ and $r_2 \in (0, p-1)$, such that

$$0 \le g(x,\lambda) \le a_2 + b_2 |\lambda|^{r_2}, \ \forall (x,\lambda) \in \Omega \times \mathbb{R}^N$$

Now, we set the main result of the present paper (Theorem 1.1).

Theorem 1.1. Under the conditions (M_1) , (H_1) – (H_2) , the problem (1.1) has at least a nontrivial solution.

2. Proof of the main result

Let $L^{s}(\Omega)$ be the Lebesgue space equipped with the norm $|u|_{s} = \left(\int_{\Omega} |u|^{s} dx\right)^{\frac{1}{s}}, 1 \le s < \infty$, and let $W_{0}^{1,p}(\Omega)$ be the usual Sobolev space with respect to the norm

$$||u|| = \left(\int_{\Omega} |\nabla u|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

We recall that $u \in W_0^{1,p}(\Omega)$ is a weak solution to the problem (1.1) if it verifies

$$M\left(\|u\|^{p}\right)\int_{\Omega}|\nabla u|^{p-2}\nabla u\nabla v\,\mathrm{d}x-\int_{\Omega}\left(g(x,\nabla u)+h(x,u)\right)v\,\mathrm{d}x=0,$$

for all $v \in W_0^{1,p}(\Omega)$.

Lemma 2.1. (See [8].) Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function with $(F(x), x) \ge 0$, for all x verifying $|x| = \rho > 0$, where $\langle ., . \rangle$ is the usual inner product of \mathbb{R}^N . Then there exists $\gamma \in B_\rho(0)$ such that $F(\gamma) = 0$.

Proof of Theorem 1.1. Let $\Gamma = \{e_1, \ldots, e_n, \ldots\} \subset W_0^{1,p}(\Omega)$ such that

$$W_0^{1,p}(\Omega) = \overline{span\{e_1,\ldots,e_n\}},$$

since $W_0^{1,p}(\Omega)$ is a reflexive and separable Banach space. Define $V_n = span\{e_1, \dots, e_n\}$. It is known that V_n and \mathbb{R}^N are isomorphic and for $\eta \in \mathbb{R}^N$, we have an unique $v \in V_n$ by

the identification $\eta \mapsto^{\phi} \sum_{i=1}^{N} \eta_i e_i = v$.

Define the function $F = (F_1, \ldots, F_N) : \mathbb{R}^N \to \mathbb{R}$ by

$$F_i(u) = \int_{\Omega} M\left(\int_{\Omega} |\nabla u|^p \, \mathrm{d}x\right) \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla e_i \, \mathrm{d}x - \int_{\Omega} \left(g(x, \nabla u) + h(x, u)\right) e_i \, \mathrm{d}x, u \in V_i.$$

Our method consists in considering a class of auxiliary problems,

$$M\left(\int_{\Omega} |\nabla u_n|^p \,\mathrm{d}x\right) \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla e_i \,\mathrm{d}x = \int_{\Omega} \left(g(x, \nabla u_n) + h(x, u_n)\right) e_i \,\mathrm{d}x,\tag{2.1}$$

we show the existence of weak solutions $u_n \in V_n$ for the problem (2.1).

For $u \in V_n$, we have that

$$\langle F(u), u \rangle \ge C_1 ||u||^p - \int_{\Omega} h(x, u) u \, dx - \int_{\Omega} g(x, \nabla u) u \, dx$$

$$\ge C_1 ||u||^p - |a_1|_{p'} ||u||_p - |b_1|_{\frac{p}{p-(r_1+1)}} |u|_p^{r_1+1} - |a_2|_{p'} |u|_p - \left(\int_{\Omega} b_2^{p'} |\nabla u|^{p'r_2}\right)^{\frac{1}{p'}} |u|_p$$

$$\ge C_1 ||u||^p - |a_1|_{p'} ||u||_p - |b_1|_{\frac{p}{p-(r_1+1)}} |u|_p^{r_1+1} - |a_2|_{p'} |u|_p - C|b_2^{p'}|_{\frac{p}{p-r_2}} |\nabla u|_{p'+p}^{r_2} |u|_p,$$

we noticed here that pp' = p + p'. As we know that

$$W^{1,p+p'}(\Omega) \hookrightarrow W^{1,p}(\Omega),$$

then there is $C_5 > 0$ such that

$$\|u\| \leq C_5 |\nabla u|_{p+p'},$$

thus, there exist C_6 , $C_7 > 0$ such that

$$\langle F(u), u \rangle \ge C_1 \|u\|^p - |a_1|_{p'} |u|_p - C_6 |b_1|_{\frac{p}{p-(r_1+1)}} \|u\|^{r_1+1} - |a_2|_{p'} |u|_p - C_7 |b_2^{p'}|_{\frac{p}{p-r_2}} \|u\|^{r_2+1}.$$

Since $r_i + 1 < p$, i = 1, 2, there exist positive numbers ρ and R such that

$$\langle F(u), u \rangle \ge \rho > 0$$
 on $||u|| = R$.

F is continuous, so, by Lemma 2.1, the system (2.1) has a solution u_n in $V_n \subset W_0^{1,p}(\Omega)$ with $||u_n|| \le R$. Furthermore, up to a subsequence, we may assume that there exists $u \in W_0^{1,p}(\Omega)$ such that

$$u_n \rightarrow u \text{ in } W_0^{1,p}(\Omega),$$

$$u_n \to u$$
 a.e $x \in \Omega$.

By using the Dominated Convergence Theorem, we get

$$\int_{\Omega} h(x, u_n) \omega \, \mathrm{d}x \to \int_{\Omega} h(x, u) \omega \, \mathrm{d}x, \text{ for } \omega \in V_k.$$

Now, we set $A_n(x) = g(x, \nabla u_n(x))$.

In view of condition (H_2) , we have

$$|A_n(x)|_{\frac{p}{r_2}} \le |a_2|_{\frac{p}{r_2}} + \left| |b_2|^{\frac{p}{r_2}} \right|_{\infty} |\nabla u_n|_p^{r_2} \le c_1 + c_2 R^{r_2}.$$
(2.2)

From the reflexivity of $L^{\frac{p}{r_2}}(\Omega)$, passing to a subsequence if necessary; there is $A \in L^{\frac{p}{r_2}}(\Omega)$ such that

$$\int_{\Omega} A_n \varphi \, \mathrm{d} x \to \int_{\Omega} A \varphi \, \mathrm{d} x, \ \forall \varphi \in L^q(\Omega)$$

with

 $\frac{r_2}{p} + \frac{1}{q} = 1.$

Therefore, $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, because $q < p^*$, it yields

$$M\left(\int_{\Omega} |\nabla u_n|^p \, \mathrm{d}x\right) \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \, \mathrm{d}x = \int_{\Omega} \left(g(x, \nabla u_n) + h(x, u_n)\right) \varphi \, \mathrm{d}x, \ \forall \varphi \in W_0^{1, p}(\Omega)$$

On the other hand we have

 $u_n \rightarrow u$ (weakly),

so when $n \to \infty$,

$$\int_{\Omega} \left(h(x, u_n) - h(x, u) \right) (u_n - u) \, \mathrm{d}x + \int_{\Omega} \left(A_n(x) - A(x) \right) (u_n - u) \, \mathrm{d}x \to 0,$$

then

$$m_0 \int_{\Omega} \left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) (\nabla u_n - \nabla u) \, \mathrm{d} x \to 0.$$

Using the following inequality $\forall x, y \in \mathbb{R}^N$

$$\begin{aligned} |x - y|^{\gamma} &\leq 2^{\gamma} (|x|^{\gamma - 2} x - |y|^{\gamma - 2} y).(x - y) \text{ if } \gamma \geq 2, \\ |x - y|^2 &\leq \frac{1}{\gamma - 1} (|x| + |y|)^{2 - \gamma} (|x|^{\gamma - 2} x - |y|^{\gamma - 2} y).(x - y) \text{ if } 1 < \gamma < 2, \end{aligned}$$

where *x*.*y* is the inner product in \mathbb{R}^N , we get

$$\int_{\Omega} |\nabla u_n - \nabla u|^p \mathrm{d}x \leq \int_{\Omega} \left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) (\nabla u_n - \nabla u) \,\mathrm{d}x.$$

Hence,

$$|| u_n - u || \to 0,$$

and then

$$u_n \to u$$
 in $W_0^{1,p}(\Omega)$.

Since *M* is continuous,

$$M\left(\int_{\Omega} |\nabla u_n|^p \,\mathrm{d}x\right) \to \int_{\Omega} M\left(\int_{\Omega} |\nabla u|^p\right) \mathrm{d}x.$$

So we obtain that *u* is a weak solution to the problem (1.1) and from (*H*₁), $u \neq 0$. \Box

Remark 2.2. By a standard argument and straightforward computations, there exists C > 0 such that

$$\|u\| \leq C,$$

so we can conclude that the solution u is a decay solution of our problem, i.e

$$\lim_{|x|\to\infty}u(x)=0.$$

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