# On an elliptic equation of $p$-Kirchhoff type with convection term 

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## A R T I CLE IN F O

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#### Abstract

In this paper, by using Galerkin's approach with a priori estimates, we establish the existence of solutions to a class of elliptic problems given by a system of nonlinear equations of $p$-Kirchhoff type with a convection term.


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## R É S U M É

Dans ce travail, on utilise la méthode de Galerkin avec une estimation a priori pour montrer l'existence de solutions à une classe de problèmes elliptiques, donnée par un système d'équations non linéaires de type $p$-Kirchhoff en présence d'un terme de gradient.
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## 1. Introduction

In this work, we study the existence of a solution for the following problem

$$
\begin{align*}
& -M\left(\int_{\Omega}|\nabla u|^{p} \mathrm{~d} x\right) \Delta_{p} u=f(x, u, \nabla u) \quad \text { in } \Omega \\
& u=0 \quad \text { on } \partial \Omega \tag{1.1}
\end{align*}
$$

where $\Omega \subset \mathbb{R}^{N}$ is a bounded domain with smooth boundary, $1<p<N$. The functional $M$ verifies,
$\left(M_{1}\right) M:(0,+\infty) \rightarrow(0,+\infty)$ continuous and $m_{0}=\inf _{s>0} M(s)>0$.
Because of the term $M\left(\int_{\Omega}|\nabla u|^{p} \mathrm{~d} x\right)$, a problem like (1.1) is nonlocal, which provokes some mathematical difficulties and also makes the study of such a problem particularly interesting. In [9], Kirchhoff proposed a model of equation extending the classical d'Alembert wave equation for free vibrations of elastic strings by considering the effect of a change in the length of the string during the vibration. Many problems of Kirchhoff type have been studied; we just quote some recent references [1-4,6,7,10-12]...

[^0]An other distinguished feature in studying such kind of equations lies in the presence of the gradient term $\nabla u$ (or convection term), which makes the problem (1.1) nonlocal and nonvariational. To overcome the difficulties brought, our approach was motivated by [5] with the use of the method of Galerkin and, as a consequence, by the Brouwer fixed-point theorem.

Assume that

$$
f(x, u, \nabla u)=h(x, u)+g(x, \nabla u)
$$

where $h$ is sublinear function and $g$ is bounded (from above) by a gradient term. For these functions, we set these hypotheses:
$\left(H_{1}\right) h: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is locally Holder continuous, there exist positive constants $a_{1} \in L^{p^{\prime}}\left(\mathbb{R}^{N}\right), b_{1} \in L^{\frac{p}{p-\left(r_{1}+1\right)}}\left(\mathbb{R}^{N}\right)$ and $r_{1} \in(0, p-1)$, such that

$$
0<h(x, t) \leq a_{1}+b_{1}|t|^{r_{1}}, \quad \forall(x, t) \in \Omega \times \mathbb{R}
$$

where $p^{\prime}$ is the conjugate of $p$.
$\left(H_{2}\right) g: \Omega \times \mathbb{R}^{N} \rightarrow \mathbb{R}$ is locally Holder continuous, there exist positive constants $a_{2} \in L^{p^{\prime}}\left(\mathbb{R}^{N}\right), b_{2} \in L^{\frac{p}{p-r_{2}}}\left(\mathbb{R}^{N}\right) \cap L^{\infty}\left(\mathbb{R}^{N}\right)$ and $r_{2} \in(0, p-1)$, such that

$$
0 \leq g(x, \lambda) \leq a_{2}+b_{2}|\lambda|^{r_{2}}, \quad \forall(x, \lambda) \in \Omega \times \mathbb{R}^{N} .
$$

Now, we set the main result of the present paper (Theorem 1.1).
Theorem 1.1. Under the conditions $\left(M_{1}\right),\left(H_{1}\right)-\left(H_{2}\right)$, the problem (1.1) has at least a nontrivial solution.

## 2. Proof of the main result

Let $L^{s}(\Omega)$ be the Lebesgue space equipped with the norm $|u|_{s}=\left(\int_{\Omega}|u|^{s} \mathrm{~d} x\right)^{\frac{1}{s}}, 1 \leq s<\infty$, and let $W_{0}^{1, p}(\Omega)$ be the usual Sobolev space with respect to the norm

$$
\|u\|=\left(\int_{\Omega}|\nabla u|^{p} \mathrm{~d} x\right)^{\frac{1}{p}}
$$

We recall that $u \in W_{0}^{1, p}(\Omega)$ is a weak solution to the problem (1.1) if it verifies

$$
M\left(\|u\|^{p}\right) \int_{\Omega}|\nabla u|^{p-2} \nabla u \nabla v \mathrm{~d} x-\int_{\Omega}(g(x, \nabla u)+h(x, u)) v \mathrm{~d} x=0
$$

for all $v \in W_{0}^{1, p}(\Omega)$.
Lemma 2.1. (See [8].) Let $F: \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ be a continuous function with $\langle F(x), x\rangle \geq 0$, for all $x$ verifying $|x|=\rho>0$, where $\langle$., . $\rangle$ is the usual inner product of $\mathbb{R}^{N}$. Then there exists $\gamma \in B_{\rho}(0)$ such that $F(\gamma)=0$.

Proof of Theorem 1.1. Let $\Gamma=\left\{e_{1}, \ldots, e_{n}, \ldots\right\} \subset W_{0}^{1, p}(\Omega)$ such that

$$
W_{0}^{1, p}(\Omega)=\overline{\operatorname{span}\left\{e_{1}, \ldots, e_{n}\right\}}
$$

since $W_{0}^{1, p}(\Omega)$ is a reflexive and separable Banach space.
Define $V_{n}=\operatorname{span}\left\{e_{1}, \ldots, e_{n}\right\}$. It is known that $V_{n}$ and $\mathbb{R}^{N}$ are isomorphic and for $\eta \in \mathbb{R}^{N}$, we have an unique $v \in V_{n}$ by the identification $\eta \mapsto^{\phi} \sum_{i=1}^{N} \eta_{i} e_{i}=v$.

Define the function $F=\left(F_{1}, \ldots, F_{N}\right): \mathbb{R}^{N} \rightarrow \mathbb{R}$ by

$$
F_{i}(u)=\int_{\Omega} M\left(\int_{\Omega}|\nabla u|^{p} \mathrm{~d} x\right) \int_{\Omega}|\nabla u|^{p-2} \nabla u \nabla e_{i} \mathrm{~d} x-\int_{\Omega}(g(x, \nabla u)+h(x, u)) e_{i} \mathrm{~d} x, u \in V_{i}
$$

Our method consists in considering a class of auxiliary problems,

$$
\begin{equation*}
M\left(\int_{\Omega}\left|\nabla u_{n}\right|^{p} \mathrm{~d} x\right) \int_{\Omega}\left|\nabla u_{n}\right|^{p-2} \nabla u_{n} \nabla e_{i} \mathrm{~d} x=\int_{\Omega}\left(g\left(x, \nabla u_{n}\right)+h\left(x, u_{n}\right)\right) e_{i} \mathrm{~d} x \tag{2.1}
\end{equation*}
$$

we show the existence of weak solutions $u_{n} \in V_{n}$ for the problem (2.1).

For $u \in V_{n}$, we have that

$$
\begin{aligned}
\langle F(u), u\rangle & \geq C_{1}\|u\|^{p}-\int_{\Omega} h(x, u) u \mathrm{~d} x-\int_{\Omega} g(x, \nabla u) u \mathrm{~d} x \\
& \geq C_{1}\|u\|^{p}-\left|a_{1}\right|_{p^{\prime}}\|u\|_{p}-\left|b_{1}\right|_{\frac{p}{p-\left(r_{1}+1\right)}}|u|_{p}^{r_{1}+1}-\left|a_{2}\right|_{p^{\prime}}|u|_{p}-\left(\int_{\Omega} b_{2}^{p^{\prime}}|\nabla u|^{p^{\prime} r_{2}}\right)^{\frac{1}{p^{\prime}}}|u|_{p} \\
& \geq C_{1}\|u\|^{p}-\left|a_{1}\right|_{p^{\prime}}\|u\|_{p}-\left|b_{1}\right|_{\frac{p}{p-\left(r_{1}+1\right)}}|u|_{p}^{r_{1}+1}-\left|a_{2}\right|_{p^{\prime}}|u|_{p}-C\left|b_{2}^{p^{\prime}}\right|_{\frac{p}{p-r_{2}}}|\nabla u|_{p^{\prime}+p}^{r_{2}}|u|_{p}
\end{aligned}
$$

we noticed here that $p p^{\prime}=p+p^{\prime}$.
As we know that

$$
W^{1, p+p^{\prime}}(\Omega) \hookrightarrow W^{1, p}(\Omega)
$$

then there is $C_{5}>0$ such that

$$
\|u\| \leq C_{5}|\nabla u|_{p+p^{\prime}},
$$

thus, there exist $C_{6}, C_{7}>0$ such that

$$
\langle F(u), u\rangle \geq C_{1}\|u\|^{p}-\left|a_{1}\right|_{p^{\prime}}|u|_{p}-C_{6}\left|b_{1}\right|_{\frac{p}{p-\left(r_{1}+1\right)}}\|u\|^{r_{1}+1}-\left|a_{2}\right|_{p^{\prime}}|u|_{p}-C_{7}\left|b_{2}^{p^{\prime}}\right|_{\frac{p}{p-r_{2}}}\|u\|^{r_{2}+1} .
$$

Since $r_{i}+1<p, i=1,2$, there exist positive numbers $\rho$ and $R$ such that

$$
\langle F(u), u\rangle \geq \rho>0 \text { on }\|u\|=R .
$$

$F$ is continuous, so, by Lemma 2.1, the system (2.1) has a solution $u_{n}$ in $V_{n} \subset W_{0}^{1, p}(\Omega)$ with $\left\|u_{n}\right\| \leq R$.
Furthermore, up to a subsequence, we may assume that there exists $u \in W_{0}^{1, p}(\Omega)$ such that

$$
\begin{aligned}
& u_{n} \rightharpoonup u \text { in } W_{0}^{1, p}(\Omega) \\
& u_{n} \rightarrow u \text { a.e } x \in \Omega
\end{aligned}
$$

By using the Dominated Convergence Theorem, we get

$$
\int_{\Omega} h\left(x, u_{n}\right) \omega \mathrm{d} x \rightarrow \int_{\Omega} h(x, u) \omega \mathrm{d} x, \text { for } \omega \in V_{k}
$$

Now, we set $A_{n}(x)=g\left(x, \nabla u_{n}(x)\right)$.
In view of condition $\left(\mathrm{H}_{2}\right)$, we have

$$
\begin{align*}
\left|A_{n}(x)\right| \frac{p}{r_{2}} & \leq\left|a_{2}\right|_{\frac{p}{r_{2}}}+\left|\left|b_{2}\right|^{\frac{p}{r_{2}}}\right|_{\infty}\left|\nabla u_{n}\right|_{p}^{r_{2}} \\
& \leq c_{1}+c_{2} R^{r_{2}} \tag{2.2}
\end{align*}
$$

From the reflexivity of $L^{\frac{p}{r_{2}}}(\Omega)$, passing to a subsequence if necessary; there is $A \in L^{\frac{p}{r_{2}}}(\Omega)$ such that

$$
\int_{\Omega} A_{n} \varphi \mathrm{~d} x \rightarrow \int_{\Omega} A \varphi \mathrm{~d} x, \forall \varphi \in L^{q}(\Omega)
$$

with

$$
\frac{r_{2}}{p}+\frac{1}{q}=1
$$

Therefore, $W_{0}^{1, p}(\Omega) \hookrightarrow L^{q}(\Omega)$, because $q<p^{*}$, it yields

$$
M\left(\int_{\Omega}\left|\nabla u_{n}\right|^{p} \mathrm{~d} x\right) \int_{\Omega}\left|\nabla u_{n}\right|^{p-2} \nabla u_{n} \nabla \varphi \mathrm{~d} x=\int_{\Omega}\left(g\left(x, \nabla u_{n}\right)+h\left(x, u_{n}\right)\right) \varphi \mathrm{d} x, \quad \forall \varphi \in W_{0}^{1, p}(\Omega) .
$$

On the other hand we have

$$
u_{n} \rightharpoonup u \text { (weakly) }
$$

so when $n \rightarrow \infty$,

$$
\int_{\Omega}\left(h\left(x, u_{n}\right)-h(x, u)\right)\left(u_{n}-u\right) \mathrm{d} x+\int_{\Omega}\left(A_{n}(x)-A(x)\right)\left(u_{n}-u\right) \mathrm{d} x \rightarrow 0
$$

then

$$
m_{0} \int_{\Omega}\left(\left|\nabla u_{n}\right|^{p-2} \nabla u_{n}-|\nabla u|^{p-2} \nabla u\right)\left(\nabla u_{n}-\nabla u\right) \mathrm{d} x \rightarrow 0
$$

Using the following inequality
$\forall x, y \in \mathbb{R}^{N}$

$$
\begin{aligned}
& |x-y|^{\gamma} \leq 2^{\gamma}\left(|x|^{\gamma-2} x-|y|^{\gamma-2} y\right) \cdot(x-y) \text { if } \gamma \geq 2, \\
& |x-y|^{2} \leq \frac{1}{\gamma-1}(|x|+|y|)^{2-\gamma}\left(|x|^{\gamma-2} x-|y|^{\gamma-2} y\right) \cdot(x-y) \text { if } 1<\gamma<2,
\end{aligned}
$$

where $x . y$ is the inner product in $\mathbb{R}^{N}$, we get

$$
\int_{\Omega}\left|\nabla u_{n}-\nabla u\right|^{p} \mathrm{~d} x \leq \int_{\Omega}\left(\left|\nabla u_{n}\right|^{p-2} \nabla u_{n}-|\nabla u|^{p-2} \nabla u\right)\left(\nabla u_{n}-\nabla u\right) \mathrm{d} x
$$

Hence,

$$
\left\|u_{n}-u\right\| \rightarrow 0
$$

and then

$$
u_{n} \rightarrow u \text { in } W_{0}^{1, p}(\Omega)
$$

Since $M$ is continuous,

$$
M\left(\int_{\Omega}\left|\nabla u_{n}\right|^{p} \mathrm{~d} x\right) \rightarrow \int_{\Omega} M\left(\int_{\Omega}|\nabla u|^{p}\right) \mathrm{d} x
$$

So we obtain that $u$ is a weak solution to the problem (1.1) and from $\left(H_{1}\right), u \neq 0$.
Remark 2.2. By a standard argument and straightforward computations, there exists $C>0$ such that

$$
\|u\| \leq C,
$$

so we can conclude that the solution $u$ is a decay solution of our problem, i.e

$$
\lim _{|x| \rightarrow \infty} u(x)=0
$$

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