Differential geometry

Harmonic vector fields on Finsler manifolds

Champs de vecteurs harmoniques sur les variétés finslériennes

Alireza Shahi, Behroz Bidabad

Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Iran

1. Introduction

On a Riemannian manifold a harmonic vector field is a vector field for which the Hodge Laplacian operator of its corresponding 1-form vanishes. Harmonic vector fields are extensively studied by different authors in Riemannian geometry, regarding its applications, see for instance, [5,6,8] and [9]. Akbar–Zadeh has considered a natural horizontal Laplacian operator on SM and generalized the Bochner and Yano’s techniques for Finsler manifolds cf., [1] and [2] page 241. Bao and Lackey in [4] construct a Laplace operator on differential forms and study harmonic forms on the underlying Finsler manifold.

Recently, the present authors have studied harmonic vector fields on Landsberg manifolds and found a necessary and sufficient condition for a vector field to be harmonic. Furthermore the harmonic vector fields on Landsberg manifolds are characterized, cf., [7].

In the present work we characterize the harmonic vector fields on Finsler manifolds through the two scalar functions \( L \) and \( J \) with significant geometric interpretations defined by (9) and (10), respectively. More intuitively, the following theorems are proved.

E-mail addresses: alirezashahi@aut.ac.ir (A. Shahi), bidabad@aut.ac.ir (B. Bidabad).
Theorem 1. Let \((M, F)\) be a compact Finsler manifold without boundary and \(X\) a vector field on \(M\). If \(I \geq 0\) and \(\frac{\partial X_i}{\partial y_j} = 0\), then \(X\) is a harmonic vector field.

Theorem 2. Let \((M, F)\) be a compact Finsler manifold without boundary.

- If \(J = 0\) for a harmonic vector field \(X\), then \(X\) is parallel in Berwald connection.
- If \(J > 0\), then there is no non-zero harmonic vector field.

Theorem 3. Let \((M, F)\) be an \(n\)-dimensional compact Finsler manifold without boundary and \(H^1_{\text{dR}}(SM)\) the first horizontal de Rham cohomology group of \(SM\).

- If \(J = 0\) for all harmonic vector fields, then \(\dim H^1_{\text{dR}}(SM) \leq n\).
- If \(J > 0\), then \(H^1_{\text{dR}}(SM) = 0\).

We show also, if the complete lift of a harmonic vector field \(X\) coincides with its canonical lift on \(SM\), then \(X\) is Killing.

2. Preliminaries

Let \((M, F)\) be a Finsler manifold, \(\pi : TM_0 \to M\) the bundle of non-zero tangent vectors and \(\pi^*TM\) the pullback bundle. We often use notations and terminologies of [2] and sometimes those of [3]. The covariant derivatives of Cartan and Berwald connections are denoted here by \(\nabla\) and \(D\), respectively. Let \(TTM_0 = HTM \oplus VTM\), be the Whitney sum, where \(HTM\) and \(VTM\) are horizontal and vertical bundle respectively and for any \(X \in TTM_0\), \(X = HX + VX\). If \(X\) and \(Y\) are sections of \(\pi^*TM\), then the Cartan and Berwald connections are related by

\[
D_{H\tilde{X}}Y = \nabla_{H\tilde{X}}Y + y^i(\nabla_i T)(X, Y),
\]

\[
D_{V\tilde{X}}Y = V\tilde{X}.Y,
\]

where \(T\) is the Cartan tensor with the components \(T_{kij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k}\) and \(y = y^i \frac{\partial}{\partial x^i} \in T_x M\). The Finsler manifold \((M, F)\) is called Landsberg manifold if the \(h\)-curvature of the Cartan connection vanishes everywhere or equivalently \(\nabla_0 T = 0\). By means of (1), we have \(D_0 g_{ij} = -2\nabla_0 T_{kij}\), where the index 0 denotes the contracted multiplication by \(y\), hence \(D_0 g_{ij} = y^k D_k g_{ij} = -2y^k \nabla_0 T_{kij} = 0\). Equation (2) is written locally \(D_{\tilde{X}}Y^i = \tilde{y}^i\), which leads to the following Ricci identities for the Berwald connection:

\[
D_1 D_k X^i - D_k D_1 X^i = X^T R^l_{jlk} \frac{\partial X^j}{\partial y^l} R^k_{il},
\]

\[
D_1 D_k X_j - D_k D_1 X_j = -X_l R^l_{jlk} \frac{\partial X^j}{\partial y^l} R^k_{il},
\]

where \(R^l_{jlk}\) is the \(hh\)-curvature tensor of the Berwald connection, cf., [2], page 19. Let \(X = X^i(x) \frac{\partial}{\partial x^i}\) be a vector field on \(M\). We associate with \(X\) the 1-form \(\tilde{X}\) on \(SM\) defined by \(\tilde{X} = X(z) \, dx^i + \hat{X}_i \, dy^j\), where \(\hat{X}_i = \frac{1}{r}(D_0 X_i - y_i D_0 (y^j X_j) F^{-2})\), cf., [2] page 231. The horizontal part of the associated 1-form on \(SM\) is denoted again in this paper by \(X = X_i(z) \, dx^i\), where \(z \in SM\). The differential and co-differential operators of the horizontal 1-form \(X\) are given by

\[
dX = \frac{1}{2} (D_1 X_j - D_j X_1) \, dx^i \wedge dx^j + \frac{\partial X_i}{\partial y^j} \, dx^j \wedge dy^i,
\]

\[
\delta X = -(\nabla^j X_i - X_j \nabla_0 T^i) = -g^{ij} D_i X_j,
\]

respectively, where the co-differential operator \(\delta\) is the formal adjoint of \(d\), in the global scalar product over \(SM\), cf., [2], pages 223 and 239. Let \((M, F)\) be a compact Finsler manifold without boundary, the divergence formula for a horizontal 1-form \(X = X_i(z) \, dx^i\) is given by

\[
\int_{SM} (\delta X) \eta = -\int_{SM} (g^{ij} D_j X_i) \eta = 0,
\]

where \(\eta\) is a volume form on \(SM\), cf., [2], page 66.
3. Harmonic vector fields on Finsler manifolds

Let $(M, F)$ be a Finsler manifold. A vector field $X$ on $M$ is said to be harmonic if its corresponding horizontal 1-form on $SM$ satisfies $\Delta X = d\delta(X) + \delta d(X) = 0$ or $dX = 0$ and $\delta X = 0$, equivalently

$$D_j X_i = D_i X_j, \quad g^{ij} D_i X_j = 0, \quad \frac{\partial X_j}{\partial y^i} = 0. \quad (8)$$

Let us consider the two significant scalar function $I$ and $J$ defined respectively by

$$I = X_i (g^{jk} D_k D_j X^i - (R^i_{jk} + T^i_{jk} R^j_{0k}) X^k) - 2(D_j (X^k V^0 T^i_k) + 2X_j V^0 T^j_k V^0 T^i_k - X_t D^i V^0 T^i_j), \quad (9)$$

$$J = (R_{jk} + T^i_{jk} R^j_{0kt}) X^i X^k + 2X_t (V^0 T^i D^i X_t + X_t D_j V^0 T^i_j). \quad (10)$$

On Landsberg manifolds, for the canonical lift of a harmonic vector field $X$ on $SM$, these scalar functions reduce to $I = g^{jk} X_t D_k D^i X^j$ and $J = R_{jk} X^i X^k$. Therefore on a Landsberg manifold of positive Ricci directional curvature (i.e. $R_{jk} X^i X^k > 0$), we have $J > 0$.

Now we are in a position to prove Theorem 1.

**Proof of Theorem 1.** A straightforward computation using the Ricci identities leads to the following formula. For more details see [7], Eq. (14):

$$D_j (X^k D_k X^j) - D_k (X^k D_j X^j) = (R_{jk} + T^i_{jk} R^j_{0kt}) X^i X^k + D_k X^j D_j X^k - D_k X^k D_j X^j. \quad (11)$$

The first and second terms in the left-hand side are written respectively:

$$D_j (X^k D_k X^j) = D_j [X^k g^{ij} D_k D_j X^i] = D_j [X^k g^{ij} D_k D_k X^i + X^k D^i X^j D_k g^{ij}]$$

$$= g^{ij} D_j (X^k D_k X^i) + D_j g^{ij} X^k D_k X^i + D_j (X^k D^i X^j D_k g^{ij})$$

$$= g^{ij} D_j (X^k D_k X^i) - 2(V^0 T^i_j) (X^k D_k X^i) - 2D_j (X^k X^i V^0 T^i_k), \quad (12)$$

and

$$D_k (X^k D_j X^j) = D_k (g^{kt} X_t D_j X^j) = g^{kt} D_k (X_t D_j X^j) + X_t D_j X^j (D_k g^{kt})$$

$$= g^{kt} D_k (X_t D_j X^j) - 2X_t D_j X^j V^0 T^i. \quad (13)$$

Plugging (12) and (13) into (11) yields

$$D_j (X^k D_k X^j) - D_k (X^k D_j X^j) = g^{ij} D_j (X^k D_k X^i) - g^{kt} D_k (X_t D_j X^j)$$

$$- 2(V^0 T^i_j) (X^k D_k X^i) - 2D_j (X^k X^i V^0 T^i_k) + 2X_t D_j X^j V^0 T^i. \quad (14)$$

Both first two terms on the right-hand side of (14) are divergence. In fact, considering $\omega_k = X^k D_k X^j$ as a 1-form, $D^i \omega_k$ is a divergence function. Integrating (11) over $SM$, using (14) and the divergence formula (7) we obtain:

$$\int_{SM} [(R_{jk} + T^i_{jk} R^j_{0kt}) X^i X^k + D_k X^j D_j X^k - D_k X^k D_j X^j$$

$$+ 2(V^0 T^i_j) (X^k D_k X^i) + D_j (X^k X^i V^0 T^i_k) - X_t D_j X^j V^0 T^i)] \eta = 0. \quad (15)$$

Consider the scalar function $\phi = X_t X^i$ on $SM$. Contracting its second Berwald covariant derivative yields

$$g^{jk} D_k D_j \phi = g^{jk} [D_k X_t D_j X^i + X_t D_k D_j X^i + D_k X^i D_j X_t + X^i D_k D_j X_t]. \quad (16)$$

Note that

$$g^{jk} X_t D_k D_j X^i = g^{jk} X_t [D_k [g^{jt} D_j X_t + X_t D_j g^{jt}]$$

$$= g^{jk} X_t [g^{jt} D_k D_j X_t + (D_k g^{jt}) (D_j X_t) + D_k (X_t D_j g^{jt})]$$

$$= g^{jk} X^i D_k D_j X_t + g^{jk} [X_t (D_k g^{jt}) (D_j X_t) + D_k (X_t D_j g^{jt}) - (D_k X_t) X_t D_j g^{jt}]$$

$$= g^{jk} X^i D_k D_j X_t + X_t (D^i g^{jt}) (D_j X_t) + g^{jk} D_k (X_t X_t D_j g^{jt}) - X_t (D_k X_t) D^j g^{jt}$$

$$= g^{jk} X^i D_k D_j X_t + g^{jk} D_k (X_t X_t D_j g^{jt}). \quad (17)$$

On the other hand,
\[ g^{jk} D_k X_j D^i = D_k X_i D^k X^i = g^{jk} D_j X_i D_k X^i. \] (18)

Therefore plugging (17) and (18) in (16) we get
\[ g^{jk} D_k D_j \phi = 2 X_i g^{jk} D_k D_j X^i + 2 D^j X^i D_j X_i - g^{jk} D_k (X_i X_j D^i g^{ij}). \] (19)

A moment’s thought shows that in the equation (19) the left-hand side and the last term of the right-hand side are divergence. Hence, by integration over SM and using the divergence formula (7), we obtain
\[ \int_{SM} \left[ X_i g^{jk} D_k D_j X^i + D^j X^k D^i D_k X^i \right] \eta = 0. \] (20)

Therefore by means of equations (15) and (20), we obtain
\[ \int_{SM} \left[ X_i \left[ g^{jk} D_k D_j X^i - (R^i_k + T^i_r R^{kr}_{Okt}) X^k - (D^j X^k D_j X_k - D_k X^j D_k X^i) \right. \right. \\
+ D_k X^k D_j X^i - 2 \left( X^k D_k X_i (\nabla_0 T^i) + D_j (X^i X_j \nabla_0 T^i) - X_i D_j X^i \nabla_0 T^i \right) \left. \right] \eta = 0. \] (21)

On the other hand \((R^i_k + T^i_r R^{kr}_{Okt}) X^k = (R^i_k + T^i_r R^{kr}_{Okt}) X_i X^k, \) and \(D_j (X^i X_j \nabla_0 T^i) = X_i D_j (X^i \nabla_0 T^i) + X_i \nabla_0 T^i (D_j X_i). \) Direct computations show that
\[ D^j X^k D_j X_k - D_k X^j D_j X^i = \frac{1}{2} (D^j X^k - D^k X^j) (D_j X_k - D_k X_j) + 2 X_i \nabla_0 T^k D_i X^k, \]
and
\[ \nabla_0 T^k D_i X^k = D_i X_j \nabla_0 T^j - 2 X_j \nabla_0 T^i D_k X^k. \]

Replacing these terms in (21) leads to
\[ \int_{SM} \left[ X_i \left[ g^{jk} D_k D_j X^i - (R^i_k + T^i_r R^{kr}_{Okt}) X^k - 2 D^j (X^i \nabla_0 T^i_k) - 4 X_i \nabla_0 T^j_k \nabla_0 T^i_k \right. \right. \\
- 2 \left( \nabla_0 T^i (D^j X_i) + 2 D^j X^i \nabla_0 T^i \right) + \frac{1}{2} (D^j X^k - D^k X^j) (D_j X_k - D_k X_j) + D_k X^k D_j X^i \right] \eta = 0. \] (22)

Consider the following norm on SM
\[ \| (D_j X_k - D_k X_j) \|^2 := g^{ij} g^{lk} (D_l X_i - D_i X_l) (D_l X_k - D_k X_l) \]
\[ = (g^{ij} [D_l X_i - X_i (D_l g^{ij})] - g^{lk} [D_l X_l - X_l (D_l g^{ij})]) [D_l X_k - D_k X_l] \]
\[ = [D^k X^l - X_l D^k X^l - X_i D^k X^l] [D_l X_k - D_k X_l] \]
\[ = [D^k X^l - D^k X^l + 2 X_t \nabla_0 T^{jk} - 2 X_j \nabla_0 T^{kl}] [D_l X_k - D_k X_l] \]
\[ = [D^k X^l - D^k X^l] [D_l X_k - D_k X_l]. \] (23)

and
\[ D_k X^k D_j X^i = (g^{jk} D_k X_i + X_i D_k g^{ik})(g^{jl} D_j X_l + X_l D_j g^{jl}) \]
\[ = (g^{jk} D_j X_i)^2 - 2 X_i \nabla_0 T^i g^{jk} D_k X_i - 2 X_i \nabla_0 T^i g^{jl} D_j X_l + 4 X_i X_l \nabla_0 T^i \nabla_0 T^l \]
\[ = (g^{jk} D_j X_i)^2 - 4 X_i \nabla_0 T^i g^{jk} D_k X_i + 4 X_i X_l \nabla_0 T^i \nabla_0 T^l \]
\[ = (g^{jk} D_j X_i)^2 - 4 X_i \nabla_0 T^i (D_k X^k - X_i D_k g^{ik}) + 4 X_i X_l \nabla_0 T^i \nabla_0 T^l \]
\[ = (g^{jk} D_j X_i)^2 - 4 X_i \nabla_0 T^i (D_k X^k) - 8 X_i X_l \nabla_0 T^i \nabla_0 T^l + 4 X_i X_l \nabla_0 T^i \nabla_0 T^l \]
\[ = (g^{jk} D_j X_i)^2 - 4 X_i \nabla_0 T^i (D_k X^k) - 4 X_i X_l \nabla_0 T^i \nabla_0 T^l. \] (24)

Replacing (23) and (24) in (22) yields
\[ \int_{SM} \left[ X_i \left[ g^{jk} D_k D_j X^i - (R^i_k + T^i_r R^{kr}_{Okt}) X^k - 2 D^j (X^i \nabla_0 T^i_k) + 2 X_j \nabla_0 T^i_k \nabla_0 T^i_k + (\nabla_0 T^i (D^j X_i) + D_j X^i \nabla_0 T^i \right. \right. \\
+ 2 X_j \nabla_0 T^i \nabla_0 T^i \right] + \frac{1}{2} \| (D_j X_k - D_k X_j) \|^2 + (g^{jk} D_j X_i)^2 \right] \eta = 0. \] (25)
Finally note that
\[ D_j X^j \nabla_0 T^i = g^{jk} D_j X_k \nabla_0 T^i - 2X_k \nabla_0 T^k \nabla_0 T^i. \]  
(26)

By similar computation, we get
\[ X_i \nabla_0 T^i D^j X_j + X_i \nabla_0 T^j D^k X_k = g^{jk} D_k (\nabla_0 T^i X_i X_j) - X_i X_j D^i (\nabla_0 T^i). \]  
(27)

The first term on the right-hand side of (27) is divergence. Plugging (26) and (27) into (25), we have
\[
\begin{align*}
\int_{SM} & [X_i (g^{jk} D_k D_j X^i - (R^j_i + T^j_i R^r_{ijk}) X^k - 2(D_j (X^k \nabla_0 T^j_k) + 2X_j \nabla_0 T_t^j \nabla_0 T^j_k - X_i D^i (\nabla_0 T^i))] \\
& + \frac{1}{2} \| (D_j X_k - D_k X_j) \| ^2 + (g^{ij} D_j X_i)^2 \| \eta = 0. \\
\end{align*}
\]
(28)

By the assumption \( I \geq 0 \) and the fact that the last two terms are non-negative, we conclude the last two terms are zero. Therefore, we obtain \( D_j X_k - D_k X_j = 0 \) and \( g^{ij} D_j X_i = 0 \). Thus \( X \) is a harmonic vector field on \( M \). This completes the proof.  \( \square \)

**Proposition 1.** Let \((M, F)\) be a Finsler manifold and \( X \) a harmonic vector field on \( M \). If the complete lift of \( X \) coincides with its canonical lift on \( SM \), then \( X \) is Killing.

**Proof.** Let \( X = X^j(x) \frac{\partial}{\partial x^j} \) be a harmonic vector field on \((M, F)\) and \( X^j(x, y) = X^j \circ \pi(x, y) \) the canonical lift of \( X \) on \( SM \), where \( \pi : SM \rightarrow M \) is the projection map. The corresponding horizontal 1-form related to \( X \) on \( SM \) is given by \( X = X_i(x, y) dx^i := g_{ij}(X^j(x, y)) dx^i \). By definition of harmonic vector fields we have \( \frac{\partial X_j}{\partial x^j} = 0 \). The vertical derivation leads to \( T^j_i X^i = 0 \), and by contraction we have \( \nabla_0 (T^j_i X^i) = 0 \). Let \( \tilde{X} = X_i(z) dx^i + \tilde{X}_i dy^i \), where \( \tilde{X}_i = \frac{1}{T} (D_0 X_i - y_i D_0(y_i X_j) F^{-2}) \), be the 1-form corresponding to the complete lift of \( X \) on \( SM \). If it coincides with the corresponding 1-form associated with the canonical lift \( X = X_i(x, y) dx^i \), then, by a proposition in [2] (page 242), we conclude that \( X \) is a conformal vector field. Therefore, we have \( L_{\tilde{X}} g_{ij} = 2 \phi g_{ij} \), where \( \phi \) is the potential function on \( M \) and \( L_{\tilde{X}} g_{ij} = D_i \tilde{X}_j + D_j \tilde{X}_i + D_0 \frac{\partial X_j}{\partial y^j} \) is the Lie derivative, cf., [2], page 227. Contracting with \( g^{ij} \) and using \( \frac{\partial X_i}{\partial y^j} = 0 \), yields \( \phi = 0 \), and hence \( X \) is Killing. This completes the proof.  \( \square \)

**Remark.** In Theorem 1, on a Landsberg manifold a vector field \( X \) on \( M \) with canonical lift, is harmonic if and only if \( g^{jk} D_j D_k X^i = (R^i_j) X^j \) and \( \frac{\partial X_i}{\partial y^j} = 0 \), cf. [7].

**Proof of Theorem 2.** Let \( X \) be a harmonic vector field on \( M \), then
\[
(D_k X^j)(D_j X^k) = (g^{kl} D_k X_l + (D_k g^{jl}) X_l)(g^{kt} D_t X_t + (D_t g^{kt}) X_t) \\
= \| D_k X_l \|^2 + 2g^{kl} X_l (D_l g^{kt}) + X_l X_t (D_k g^{jl})(D_j g^{kt}) \\
= \| D_k X_l \|^2 - 4X_l (D_k X_l)(\nabla_0 T^l_k) + 4X_l X_t (\nabla_0 T^l_k)(\nabla_0 T^t_j) \\
= \| D_k X_l \|^2 - 4X_l (D_k X_l)(\nabla_0 T^l_k) + 4 \| X_t \nabla_0 T^t_j \|^2. \\
\]  
(29)

On the other hand, we compute the following terms in (15):
\[
(D_k X^k)(D_j X^j) = (g^{kt} D_k X_t + X_t D_k g^{kt})(g^{jl} D_j X_l + X_l D_j g^{jl}) \\
= (X_t D_k g^{kt})(X_t D_j g^{jl}) = 4X_t X_l (\nabla_0 T^t)(\nabla_0 T^l). \\
\]  
(30)

We also have
\[
X_i D_j X^j \nabla_0 T^i = (X_t X_l D_j g^{jl})(\nabla_0 T^l) = -2X_t X_l \nabla_0 T^l \nabla_0 T^i. \\
\]  
(31)

Moreover, we have
\[
D_j (X^k X_l \nabla_0 T^j_k) = D_j (X_t X_l \nabla_0 T^j_l) = 2X_t \nabla_0 T^j_l (D_j X_l) + X_t X_l D_j (\nabla_0 T^j_l). \\
\]  
(32)

And finally
\[
(\nabla_0 T^l)(X^k D_k X_l) = X_t (\nabla_0 T^l)(D^l X_t). \\
\]  
(33)

Substituting (29), (30), (31), (32) and (33) into (15) yields
\[
\int_{SM} \left[ \left( R_{jk} + T^r_{ij} R^{T}_{oklr} \right) X^j X^k + 2X_l \left( (\nabla_0 T^i) (D^j X_l) + X_l D_j (\nabla_0 T^{i,j}) \right) + \parallel D_k X_l \parallel^2 + 4 \parallel X_l \nabla_0 T^j_{ij} \parallel^2 \right] \eta = 0.
\]

By definition of \( J \) or (10), the above equation is written
\[
\int_{SM} \left[ J + \parallel D_k X_l \parallel^2 + 4 \parallel X_l \nabla_0 T^j_{ij} \parallel^2 \right] \eta = 0.
\]

If the vector field \( X \) satisfies \( J = 0 \), by means of (35) we get, \( D_k X_l = 0 \). By the definition of the harmonic vector fields, we have \( D^*_i X_j = 0 \), thus \( X \) is horizontally and vertically parallel in Berwald connection. This completes the proof of the first assertion.

If \( X \) satisfies \( J > 0 \), then it contradicts (35) and therefore there is not any non-zero harmonic vector field satisfying \( J > 0 \). This completes the proof of the second assertion. \( \square \)

**Remark 2.** In Theorem 2, on a Landsberg manifold \((M, F)\), the assumptions \( J = 0 \) and \( J > 0 \) for a harmonic vector field \( X \) with canonical lift on \( SM \), reduce to \( R_{jk} X^j X^k = 0 \) and \( R_{jk} X^j X^k > 0 \), respectively, cf. [7].

**Proof of Theorem 3.** Let \( X \) be a harmonic vector field on \((M, F)\) endowed with a Berwald connection that satisfies \( J = 0 \). By the definition of the harmonic vector fields and Theorem 2, we have \( D_i X_j = D_i^* X_j = 0 \). Therefore, the 1-form \( X = X_i(z) dx^i \) is parallel with respect to the Berwald connection on \( SM \). It is well known that a parallel 1-form on a manifold is determined by its value at a point on the underlying manifold. Hence the dimension of the vector space of parallel horizontal 1-forms is at most equal to \( n \), the dimension of its restriction to the horizontal 1-forms on the cotangent space \( T^*_x(SM) \). Therefore, for the restriction of the first de Rham cohomology group \( H^1_{dR}(SM) \) of these 1-forms, we have \( \dim H^1_{dR}(SM) \leq n \). This completes the proof of the first assertion.

Next, by means of Theorem 2, there is no non-zero harmonic vector field \( X \) that satisfies \( J > 0 \). Hence \( SM \) has no non-trivial harmonic 1-form. Thus \( H^1_{dR}(SM) = 0 \) and this completes the proof of the second assertion. \( \square \)

**References**


