Partial differential equations

Applications of Bourgain–Brézis inequalities to fluid mechanics and magnetism

Applications des inégalités de Bourgain–Brézis à la mécanique des fluides et au magnétisme

Sagun Chanillo\textsuperscript{a}, Jean Van Schaftingen\textsuperscript{b}, Po-Lam Yung\textsuperscript{c}

\textsuperscript{a} Department of Mathematics, State University of New Jersey, Rutgers, NJ 08854, USA
\textsuperscript{b} Institut de recherche en mathématique et en physique, Université catholique de Louvain, chemin du Cyclotron 2 bte L7.01.01, 1348 Louvain-la-Neuve, Belgium
\textsuperscript{c} Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong

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A B S T R A C T

As a consequence of inequalities due to Bourgain–Brézis, we obtain local-in-time well-posedness for the two-dimensional Navier–Stokes equation with velocity bounded in spacetime and initial vorticity in bounded variation. We also obtain spacetime estimates for the magnetic field vector through improved Strichartz inequalities.

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R É S U M É

À partir d’inégalités de Bourgain–Brézis, nous démontrons le caractère bien posé localement dans le temps des équations de Navier–Stokes avec vitesse bornée en espace-temps et un tourbillon initial à variation bornée. Nous obtenons également des estimations en espace-temps pour le champ magnétique grâce à des inégalités de Strichartz améliorées.

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1. Incompressible Navier–Stokes flow

Let \( \mathbf{v}(x,t) \in \mathbb{R}^2 \) be the velocity and \( p(x,t) \) be the pressure of a fluid of viscosity \( \nu > 0 \) at position \( x \in \mathbb{R}^2 \) and time \( t \in \mathbb{R} \), governed by the incompressible two-dimensional Navier–Stokes equation:

\[
\begin{align*}
\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} &= \nu \Delta \mathbf{v} - \nabla p, \\
\nabla \cdot \mathbf{v} &= 0,
\end{align*}
\]

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E-mail addresses: chanillo@math.rutgers.edu (S. Chanillo), Jean.VanSchaftingen@uclouvain.be (J. Van Schaftingen), plyung@math.cuhk.edu.hk (P.-L. Yung).

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When the viscosity coefficient \( \nu \) degenerates to zero, (1) becomes the Euler equation. In two spatial dimensions, the vorticity of the flow is a scalar, defined by

\[
\omega = \partial_x v_2 - \partial_y v_1
\]

where we wrote \( \mathbf{v} = (v_1, v_2) \). In the sequel, when we consider the Navier–Stokes equation, without loss of generality we set the viscosity coefficient \( \nu = 1 \).

The vorticity associated with the incompressible Navier–Stokes flow in two dimensions propagates according to the equation

\[
\omega_t - \Delta \omega = - \nabla \cdot (\mathbf{v} \omega).
\]

(2)

This follows from (1) by taking the curl of both sides. We express the velocity \( \mathbf{v} \) in the Navier–Stokes equation in terms of the vorticity through the Biot–Savart relation:

\[
\mathbf{v} = (-\Delta)^{-1}(\partial_x \omega, -\partial_y \omega).
\]

(3)

This follows formally by differentiating \( \omega = \partial_x v_2 - \partial_y v_1 \), and using that \( \nabla \cdot \mathbf{v} = 0 \).

Our theorem states:

**Theorem 1.** Consider the two-dimensional vorticity equation (2) and an initial vorticity \( \omega_0 \in W^{1,1}(\mathbb{R}^2) \) at time \( t = 0 \). If

\[
\|\omega_0\|_{W^{1,1}(\mathbb{R}^2)} \leq A_0,
\]

then there exists a unique solution to the vorticity equation (2) for all time \( t \leq t_0 = C/A_0^2 \) such that

\[
\sup_{t \leq t_0} \|\omega(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \leq C A_0.
\]

Moreover, the solution \( \omega \) depends continuously on the initial data \( \omega_0 \), in the sense that if \( \omega_0^{(i)} \) is a sequence of initial data converging in \( W^{1,1}(\mathbb{R}^2) \) to \( \omega_0 \), then the corresponding solutions \( \omega^{(i)} \) to the vorticity equation (2) satisfy

\[
\sup_{t \leq t_0} \|\omega^{(i)}(\cdot, t) - \omega(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \to 0
\]

as \( i \to \infty \).

Finally, the velocity vector \( \mathbf{v} \) defined by the Biot–Savart relation (3) solves the 2-dimensional incompressible Navier–Stokes equation (1), and satisfies

\[
\sup_{t \leq t_0} \|\mathbf{v}(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} + \sup_{t \leq t_0} \|\nabla \mathbf{v}(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq C A_0.
\]

Via the Gagliardo–Nirenberg inequality, we can conclude from our theorem that

\[
\sup_{0 \leq t \leq t_0} \|\omega(\cdot, t)\|_{L^p(\mathbb{R}^2)} \leq C, \quad 1 \leq p \leq 2.
\]

In particular, this is enough to apply Theorem II of Kato [8] to express the velocity vector in the Navier–Stokes equation (1) in terms of the vorticity via the Biot–Savart relation displayed above.

In [7,8], it was proved that under the hypothesis that the initial vorticity is a measure, there is a global solution that is well-posed to the vorticity and Navier–Stokes equation; see also an alternative approach in Ben-Artzi [1], and a stronger uniqueness result in Brézis [4]. The velocity constructed then satisfies the estimate [8, (0.5)]:

\[
\|\mathbf{v}(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \leq C t^{-1/2}, \quad t \to 0.
\]

(4)

In contrast, in Theorem 1 we have \( \mathbf{v} \in L^\infty L_\chi^\infty, x \in \mathbb{R}^2 \), though we are assuming that the initial vorticity has bounded variation, that is, its gradient is a measure.

The estimate (4) is indeed sharp as can be seen by the famous example of the Lamb–Oseen vortex [9], which consists of an initial vorticity \( \omega_0 = \alpha_0 \delta_0 \), a Dirac mass at the origin of \( \mathbb{R}^2 \) with strength \( \alpha_0 \). The constant \( \alpha_0 \) is called the total circulation of the vortex. A unique solution to the vorticity equation (2) can be obtained by setting

\[
\omega(x, t) = \frac{\alpha_0}{4\pi t} e^{-\frac{\|x\|^2}{4t}}, \quad \mathbf{v}(x, t) = \frac{\alpha_0}{2\pi} \frac{(-x_2, x_1)}{|x|^2} \left( 1 - e^{-\frac{\|x\|^2}{4t}} \right).
\]

It can be seen from the identities above that

\[
\|\omega(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \sim \|\mathbf{v}(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \sim C t^{-1/2}, \quad t \to 0.
\]
Hence the assumption that the initial vorticity is a measure cannot yield an estimate like in Theorem 1. Thus to get uniform-in-time, $L^\infty$ space bounds all the way to $t = 0$, we need a stronger hypothesis and one such is vorticity in $BV$ (bounded variation).

It is also helpful to further compare our result with that of Kato [8], who establishes in (0.4) of his paper that given that the initial vorticity is a measure, one has for the vorticity at further time
\[
\|\nabla \omega(\cdot, t)\|_{L^q(\mathbb{R}^2)} \leq C t^{\frac{3}{4} - \frac{2}{q}}, \quad 1 < q \leq \infty.
\]
In contrast, we obtain uniform-in-time bounds for $q = 1$, as opposed to singular bounds for $q > 1$ when $t \to 0$.

It is an open question whether there is a global version of Theorem 1 of our paper.

In order to prove Theorem 1, we rely on a basic proposition that follows from the work of Bourgain and Brézis [2,3]. A part of this proposition also holds in three dimensions. Recall that if $v(x, t) \in \mathbb{R}^3$ is the velocity of a fluid at a point $x \in \mathbb{R}^3$ at time $t$, then the vorticity of $v$ is defined by
\[
\omega = \nabla \times v.
\]
Under the assumption that the flow is incompressible, the Biot–Savart relation reads
\[
v = (-\Delta)^{-1}(\nabla \times \omega).
\]

**Proposition 2.**

(a) Consider the velocity $v$ in three spatial dimensions. Assume that $v$ satisfies the Biot–Savart relation (5). Then at any fixed time $t$,
\[
\|v(\cdot, t)\|_{L^3(\mathbb{R}^3)} + \|\nabla v(\cdot, t)\|_{L^2(\mathbb{R}^3)} \leq C\|\nabla \times \omega(\cdot, t)\|_{L^1(\mathbb{R}^3)}
\]
where $C$ is a constant independent of $t$, $v$, and $\omega$.

(b) Consider the velocity $v$ in two spatial dimensions. Assume that $v$ satisfies the Biot–Savart relation (3). Then at any fixed time $t$,
\[
\|v(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla v(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla \omega(\cdot, t)\|_{L^1(\mathbb{R}^2)}
\]
where $C$ is a constant independent of $t$, $v$ and $\omega$.

We remark that in two dimensions, by the Poincaré inequality, it follows from $\|\nabla v\|_{L^2(\mathbb{R}^2)} < \infty$, that $v$ lies in $VMO(\mathbb{R}^2)$, i.e. has vanishing mean oscillation.

**Proof of Proposition 2.** Note that
\[
\nabla \cdot (\nabla \times \omega) = 0.
\]
Thus we can immediately apply the result of Bourgain–Brézis [3] (see also [2,5,10]) to the Biot–Savart formula (5) and get the desired conclusions in part (a).

To consider the 2-dimensional flow, note that $(-\partial_y \omega, \partial_x \omega)$ is a vector field in $\mathbb{R}^2$ with vanishing divergence. In view of the two-dimensional Biot–Savart relation (3), we can then use the two-dimensional Bourgain–Brézis result [3], and we obtain (b).

We note further that the proposition applies to both the Euler (inviscid) or the Navier–Stokes (viscous) flow.

**Proof of Theorem 1.** Now set $K_t$ for the heat kernel in two dimensions, i.e.
\[
K_t(x) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}.
\]
Rewriting (2) as an integral equation for $\omega$ using Duhamel’s theorem, where $\omega_0$ is the initial vorticity, we have
\[
\omega(x, t) = K_t \ast \omega_0(x) + \int_0^t \partial_s K_{t-s} \ast [v \omega(x, s)] \, ds
\]
where $v$ is given by (3).
We apply a Banach fixed point argument to the operator $T$ given by
\[
T \omega(x, t) = K_t \ast \omega_0(x) + \int_0^t \partial_s K_{t-s} \ast [v \omega(x, s)] \, ds,
\]
where again $\mathbf{v}$ is given by (3). Let us set
\[
E = \left\{ g \mid \sup_{0<t_0} \|g(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \leq A \right\}.
\]
We will first show that $T$ maps $E$ into itself, for $t_0$ chosen as in the theorem.
Differentiating (7) in the space variable once, we get
\[
(T\omega(x, t))_x = K_t \cdot f_0(x) + \int_0^t \partial_x K_{t-s} \cdot (\mathbf{v}_x \omega) ds + \int_0^t \partial_x K_{t-s} \cdot (\mathbf{w}_x \omega) ds.
\]
Here we denote by $f_0$ the spatial derivative of the initial vorticity $\omega_0$. Using Young’s convolution inequality, we have
\[
\|T\omega(\cdot, t)\|_{L^1(\mathbb{R}^2)} \leq \|f_0\|_{L^1(\mathbb{R}^2)} + C \int_0^t (t-s)^{-1/2}(\|\mathbf{v}_x \omega\|_{L^1(\mathbb{R}^2)} + \|\mathbf{w}_x \omega\|_{L^1(\mathbb{R}^2)}) ds.
\]
Now we apply Proposition 2(b) to each of the terms on the right. For the first term, we have, by Cauchy–Schwarz,
\[
\|\mathbf{v}_x \omega\|_{L^1(\mathbb{R}^2)} \leq C \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)} \|\omega\|_{L^2(\mathbb{R}^2)}.
\]
The Gagliardo–Nirenberg inequality applies as $\omega \in E$ and so $\omega(\cdot, t) \in L^1(\mathbb{R}^2)$ and so,
\[
\|\omega\|_{L^2(\mathbb{R}^2)} \leq C \|\nabla \omega\|_{L^1(\mathbb{R}^2)},
\]
and to $\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}$ we apply Proposition 2(b). Similarly, for the second term,
\[
\|\mathbf{w}_x \omega\|_{L^1(\mathbb{R}^2)} \leq \|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)} \|\omega\|_{L^1(\mathbb{R}^2)}.
\]
Again we apply Proposition 2(b) to $\|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)}$. Hence in all we have,
\[
\|T\omega(\cdot, t)\|_{L^1(\mathbb{R}^2)} \leq \|f_0\|_{L^1(\mathbb{R}^2)} + C \int_0^t (t-s)^{-1/2}(\|\nabla \omega\|_{L^2(\mathbb{R}^2)}) ds.
\]
Thus setting $\|f_0\|_{L^1(\mathbb{R}^2)} = \|\omega_0\|_{W^{1,1}(\mathbb{R}^2)} \leq A_0$, we get for $t \leq t_0$ and since $\omega \in E$,
\[
\|\nabla (T\omega)(\cdot, t)\|_{L^1(\mathbb{R}^2)} \leq A_0 + C t_0^{1/2} A^2.
\]
Next from Young’s convolution inequality it follows from (7) that
\[
\|T\omega(\cdot, t)\|_{L^1(\mathbb{R}^2)} \leq A_0 + \int_0^t (t-s)^{-1/2}(\|\mathbf{w}_x \omega(\cdot, s)\|_{L^1(\mathbb{R}^2)}) ds.
\]
But by Proposition 2(b) again,
\[
\|\mathbf{w} \omega\|_{L^1(\mathbb{R}^2)} \leq \|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)} \|\omega\|_{L^1(\mathbb{R}^2)} \leq c A^2.
\]
Thus
\[
\|T\omega(\cdot, t)\|_{L^1(\mathbb{R}^2)} \leq A_0 + c t_0^{1/2} A^2.
\]
So, adding the estimates for $T\omega$ and $\nabla (T\omega)$, we have:
\[
\sup_{t \leq t_0} \|T\omega(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \leq 2 A_0 + c t_0^{1/2} A^2.
\]
By choosing $A$ so that $A_0 = A/8$ and $t < t_0 = C/A_0^2$, we can assure that if $\omega \in E$, then
\[
\sup_{t \leq t_0} \|\nabla (T\omega)(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \leq \frac{A}{2}.
\]
Thus $T\omega \in E$, if $\omega \in E$. If we establish that $T$ is a contraction, then we are done.
Next we observe that the estimates in Proposition 2(b) are linear estimates. That is
\[
\|\mathbf{v}_1 - \mathbf{v}_2\|_{L^\infty} + \|\nabla \mathbf{v}_1 - \nabla \mathbf{v}_2\|_{L^2} \leq C \|\omega_1 - \omega_2\|_{W^{1,1}(\mathbb{R}^2)}.
\]
We easily can see from the computations above, that we have
\[
\sup_{t \leq t_0} \| T \omega_1 - T \omega_2 \|_{W^{1,1}(\mathbb{R}^3)} \leq C a_0^{1/2} \sup_{t \leq t_0} \| \omega_1 - \omega_2 \|_{W^{1,1}(\mathbb{R}^3)}.
\]

By the choice of \( t_0 \), it is seen that \( T \) is a contraction. Thus using the Banach fixed-point theorem on \( E \), we obtain our operator \( T \) has a fixed point and so the integral equation (6) has a solution in \( E \). The remaining part of our theorem follows easily from Proposition 2(b).

2. Magnetism

We next turn to our results on magnetism. We denote by \( B(x, t) \) and \( E(x, t) \) the magnetic and electric field vectors at \((x, t) \in \mathbb{R}^3 \times \mathbb{R}\). Let \( j(x, t) \) denote the current density vector. The Maxwell equations imply

\[
\begin{align*}
\nabla \cdot B &= 0, \\
\partial_t B + \nabla \times E &= 0, \\
\partial_t E - \nabla \times B &= -j.
\end{align*}
\]

Differentiating (10) in \( t \) and using (11), together with the vector identity \( \nabla \times (\nabla \times B) = \nabla(\nabla \cdot B) - \Delta B \) and (9), one obtains an inhomogeneous wave equation for \( B \):

\[
B_{tt} - \Delta B = \nabla \times j.
\]

The right side of (12) satisfies the vanishing divergence condition

\[
\nabla \cdot (\nabla \times j) = 0
\]

for any fixed time \( t \). Thus an improved Strichartz estimate, namely Theorem 1 in \([6]\), applies. We point out that the Bourgain–Brézis inequalities play a key role in the proof of Theorem 1 in \([6]\). We conclude easily:

**Theorem 3.** Let \( B \) satisfy (12) and let \( B(x, 0) = B_0, \partial_t B(x, 0) = B_1 \) denote the initial data at time \( t = 0 \). Let \( s, k \in \mathbb{R} \). Assume \( 2 \leq q \leq \infty \), \( 2 < q \leq \infty \) and \( 2 \leq r < \infty \). Let \((q, r)\) satisfy the wave compatibility condition

\[
\frac{1}{q} + \frac{1}{r} \leq \frac{1}{2},
\]

and assume the following scale invariance condition is verified:

\[
\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - s = \frac{1}{q'} + 1 - k.
\]

Then, for \( \frac{1}{q} + \frac{1}{q'} = 1 \), we have

\[
\|B\|_{L_t^q L_x^r} + \|B\|_{C_t^q H_x^s} + \|\partial_t B\|_{C_t^q H_x^{s-1}} \leq C(\|B_0\|_{H^s} + \|B_1\|_{H_t^{s-1}} + \|(-\Delta)^{k/2}(\nabla j)\|_{L^q_t L^r_x}).
\]

The main point in the theorem above is that we have \( L^1 \) norm in space on the right side.

**References**


