Mathematical analysis

Profile decomposition and phase control for circle-valued maps in one dimension

Décomposition en profils et contrôle des phases des applications unimodulaires en dimension un

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A B S T R A C T

When $1 < p < \infty$, maps $f$ in $W^{1,p}(0,1;\mathbb{S}^1)$ have $W^{1,p}$ phases $\psi$, but the $W^{1,p}$-seminorm of $\psi$ is not controlled by the one of $f$. Lack of control is illustrated by "the kink": $f = e^{i\phi}$, where the phase $\phi$ moves quickly from 0 to $2\pi$. A similar situation occurs for maps $f: \mathbb{S}^1 \to \mathbb{S}^1$, with Moebius maps playing the role of kinks. We prove that this is the only loss of control mechanism: each map $f: \mathbb{S}^1 \to \mathbb{S}^1$ satisfying $\int_{\mathbb{S}^1} |f|^p W^{1,p} \leq M$

can be written as $f = e^{i\psi} \prod_{j=1}^K (M_{a_j})^{\pm 1}$, where $M_{a_j}$ is a Moebius map vanishing at $a_j \in \mathbb{D}$,

while the integer $K = K(f)$ and the phase $\psi$ are controlled by $M$. In particular, we have $K \leq c_p M$ for some $c_p$. When $p = 2$, we obtain the sharp value of $c_2$, which is $c_2 = 1/(4\pi^2)$. As an application, we obtain the existence of minimal maps of degree one in $W^{1,p}(\mathbb{S}^1,\mathbb{S}^1)$ with $p \in (2-\varepsilon, 2)$.

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R Ê S U M É

Si $1 < p < \infty$, les applications $f$ appartenant à $W^{1,p}(0,1;\mathbb{S}^1)$ ont des phases $\psi$ dans $W^{1,p}$, mais la seminorme $W^{1,p}$ de $\psi$ n’est pas contrôlée par celle de $f$. L’absence de contrôle est illustrée par «le pli»: $f = e^{i\psi}$, où la phase $\psi$ augmente rapidement de 0 à $2\pi$. Pour des applications $f: \mathbb{S}^1 \to \mathbb{S}^1$, le même phénomène apparaît, avec les transformations de Moebius jouant le rôle des plis. Nous prouvons que cet exemple est essentiellement le seul: toute application $f: \mathbb{S}^1 \to \mathbb{S}^1$ telle que $\int_{\mathbb{S}^1} |f|^p W^{1,p} \leq M$ s’écrit $f = e^{i\psi} \prod_{j=1}^K (M_{a_j})^{\pm 1}$,

où $M_{a_j}$ est une transformation de Moebius s’annulant en $a_j \in \mathbb{D}$, tandis que l’entier $K = K(f)$ et la phase $\psi$ sont contrôlés par $M$. En particulier, nous avons $K \leq c_p M$ pour une constante $c_p$. Pour $p = 2$, nous obtenons la valeur optimale de $c_2$, qui est $c_2 = 1/(4\pi^2)$.

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1. Introduction

Let \(0 < s < 1\), \(1 \leq p < \infty\) and let \(f: (0, 1) \rightarrow S^1\) belong to the space \(W^{s,p}\). Then \(f\) can be written as \(f = e^{i\psi}\), where \(\varphi \in W^{s,p}\) [3]. Once the existence of \(\varphi\) is known, a natural question is whether we can control \(|\varphi|_{W^{s,p}}\) in terms of \(|f|_{W^{s,p}}\). For most of \(s, p\), the answer is positive. The exceptional cases are provided precisely by the spaces \(W^{1/p,p}(0, 1); S^1\), with \(1 < p < \infty\) [3]. In these spaces, lack of control is established via the following explicit example. For \(n \geq 1\), we define \(\varphi_n\) as follows:

\[
\varphi_n(x) := \begin{cases} 
0, & \text{for } 0 < x < 1/2 \\
2\pi \left(n(x - 1/2) + 1/n\right), & \text{for } 1/2 < x < 1/2 + 1/n \\
2\pi, & \text{for } 1/2 + 1/n < x \leq 1
\end{cases}
\]

Then \(|\varphi_n|_{W^{1/p,p}} \rightarrow \infty\) (since \(\varphi_n \rightarrow \varphi = 2\pi \chi_{(1/2, 1)}\) a.e., and \(\varphi\) does not belong to \(W^{1/p,p}\)). On the other hand, if we extend \(u_n := e^{i\psi_n}\) with the value 1 outside \((0, 1)\) and still denote the extension \(u_n\) then, by scaling,

\[
|u_n|_{W^{1/p,p}(0, 1)} \leq |u_n|_{W^{1/p,p}(\mathbb{R})} = |u_1|_{W^{1/p,p}(\mathbb{R})} < \infty.
\]

Thus \(|u_n|_{W^{1/p,p}(0, 1)} \lesssim 1\) and \(|\varphi_n|_{W^{1/p,p}(0, 1)} \rightarrow \infty\). Finally, we invoke the fact that \(W^{1/p,p}\) phases are unique mod \(2\pi\) [3].

If one considers instead maps \(f: S^1 \rightarrow S^1\), always in the critical case \(f \in W^{1/p,p}, 1 < p < \infty\), then a new phenomenon occurs: \(f\) has a degree \(\deg f\), and does not have a \(W^{1/p,p}\) phase at all when \(\deg f \neq 0\) [11, Remark 10]. However, even if \(\deg f = 0\) (and thus \(f\) has a \(W^{1/p,p}\) phase \(\varphi\)), we have a loss-of-control phenomenon similar to the one on \((0, 1)\). Indeed, let \(M_\alpha(z) := \frac{a - z}{1 - \overline{a}z}, a \in \mathbb{D}, z \in \overline{\mathbb{D}}\), be a Moebius transform (that we identify with its restriction to \(S^1\), \(M_\alpha: S^1 \rightarrow S^1\)). Let \(f_\alpha(z) := z M_\alpha(z)\), so that \(f_\alpha\) is smooth and \(\deg f_\alpha = 0\). One may prove (see below) that \(|M_\alpha|_{W^{1/p,p}} = |\text{Id}|_{W^{1/p,p}}\), and thus \(f_\alpha\) is bounded in \(W^{1/p,p}\). However, if \(a \rightarrow \alpha = e^{i\theta} \in S^1\), then the smooth phase \(\varphi_\alpha\) of \(f_\alpha\) converges a.e. to \(\varphi(\alpha) := \int \frac{\xi - \theta}{2\pi + \xi - \theta}, \quad \text{if } \xi - \pi < \theta < \xi\)

\[
\int \frac{\xi - \theta}{2\pi + \xi - \theta}, \quad \text{if } \xi < \theta < \pi + \xi,
\]

which does not belong to \(W^{1/p,p}\). (Here, uniqueness of the phases and convergence hold mod \(2\pi\).) Thus \(\varphi_\alpha\) is not bounded as \(a \rightarrow \alpha \in S^1\). On the other hand, the plot of \(\varphi_\alpha\) shows that \(\varphi_\alpha\) has a “kink shape”, and thus we have here the analog of the example on \((0, 1)\).

There are evidences that this loss of control mechanism is the only possible one. For example, the phase of the kink is not bounded in \(W^{1/p,p}\), but clearly is in \(W^{1,1}\) (same for \(f_\alpha\), Bourgain and Brézis [4] proved that for every \(f \in W^{1/2,2}(0, 1); S^1\), we may split \(f = e^{i\psi} v\), with \(\psi\) and \(v = e^{i\theta}\) satisfying

\[
|\psi|_{W^{1/2,2}} \lesssim |f|_{W^{1/2,2}} \quad \text{and} \quad |v|_{W^{1,1}} = |v|_{W^{1,1}} \lesssim |f|^2_{W^{1/2,2}}.
\]

Intuitively, one should think at \(v\) as at “the kink part of \(f\)”. The above result was extended by Nguyen [18] to \(1 < p < \infty\): for every \(1 < p < \infty\) and every \(f \in W^{1/p,p}(0, 1); S^1\), we may split \(f = e^{i\psi} v\), with \(\psi\) and \(v = e^{i\theta}\) satisfying

\[
|\psi|_{W^{1/p,p}} \leq C_p |f|_{W^{1/p,p}} \quad \text{and} \quad |v|_{W^{1,1}} = |v|_{W^{1,1}} \leq C_p |f|^p_{W^{1/p,p}}.
\]

Here we present another result in this direction, written for simplicity on the unit circle.

**Theorem 1.** Let \(1 < p < \infty\) and \(M > 0\). Then there exist constants \(c_p\) and \(F(M)\) such that: every map \(f \in W^{1/p,p}(S^1; S^1)\) satisfying

\[
|f|^p_{W^{1/p,p}} \leq M \quad \text{can be written as} \quad f = e^{i\psi} \prod_{j=1}^{K} (M_j)^{\epsilon_j}, \quad \text{with} \ \epsilon_j \in \{-1, 1\},
\]

\[
K \leq c_p M,
\]

and

\[
|\psi|^p_{W^{1/p,p}} \leq F(M).
\]

When \(p = 2\), we may take \(c_2 = 1/(4\pi^2)\), and this constant is optimal.

**Corollary 1.** Let \(1 < p < \infty\) and let \(f_n, f \in W^{1/p,p}(S^1; S^1)\) be such that \(f_n \rightharpoonup f\) in \(W^{1/p,p}\). Then, up to a subsequence, there exist \(K \in \mathbb{N}, \epsilon_j \in \{-1, 1\}, a_j, \alpha_j \in S^1, j = 1, \ldots, K, \psi_n \in W^{1/p,p}(S^1; S^1)\), and a constant \(C\), such that:
\[ f_n = e^{\psi_n} \prod_{j=1}^{K} (M_{d_j})^{\psi_j} f; \]
\[ \alpha_j \to \alpha_f \text{ as } n \to \infty; \]
\[ \psi_n \to C \text{ in } W^{1/p,p} \text{ as } n \to \infty. \]

The theorem and the corollary are reminiscent of profile decompositions obtained in different, often geometrical, contexts. We mention, e.g., the work of Sacks and Uhlenbeck [19] on minimal 2-spheres, the analysis of Brézis and Coron [6–8] of constant mean curvature surfaces, or the one of Struwe [20] of equations involving the critical Sobolev exponent. There are also abstract approaches to bubbling as in the work of Lions [16] about concentration-compactness or the characterization of the lack of compactness of critical embeddings in Gérard [12], Jaffard [15] or Bahouri, Cohen and Koch [1].

Let us comment on the connection between (2) and our theorem. First, (2) has the following version for maps on \( S^1 \): we may split \( f = e^{\psi} \nu \), with \( |\nu|_{W^{1/p,p}} \leq C_p |f|_{W^{1/p,p}} \) and \( |\nu|_{W^{1,1}} \leq C_p |f|_{W^{1/p,p}} \). Next, a Moebius map satisfies \( |M_a|_{W^{1,1}} = 2\pi \), and thus

\[
\left| \prod_{j=1}^{K} (M_{d_j})^{\psi_j} \right|_{W^{1,1}} \leq 2\pi K \leq 2\pi c_p M. \tag{5}
\]

Estimate (5) shows that (3) is a refinement of the second part of (2). On the other hand, (4) is weaker than the first part of (2), since \( F(M) \) need not have a linear growth (and actually we do not have any control on \( F \)). This suggests the following conjecture.

**Conjecture.** Let \( 1 < p < \infty \). Then there exist constants \( c_p, d_p \) such that every \( f \in W^{1/p,p}(S^1; S^1) \) satisfying \( |f|_{W^{1/p,p}} \leq M \) can be decomposed as \( f = e^{\psi} \prod_{j=1}^{K} (M_{d_j})^{\psi_j} \), with \( \psi \in [-1,1] \),

\[ K \leq c_p M, \tag{6} \]

and

\[ |\nu|_{W^{1/p,p}} \leq d_p M. \tag{7} \]

In addition, when \( p = 2 \), we may take \( c_2 = 1/(4\pi^2) \).

## 2. Proofs

We start by recalling or establishing few auxiliary results. Given \( 1 \leq p < \infty \), \( f, f_n \) will denote maps in \( W^{1/p,p}(S^1; S^1) \). When \( 1 < p < \infty \), “\( \rightharpoonup \)” refers to weak convergence in \( W^{1/p,p} \).

1. Recall that, up to a multiplicative constant \( \alpha \in S^1 \), the Moebius transforms give all the conformal representations \( u : \mathbb{D} \to \mathbb{D} \). In particular, \( M_a : S^1 \to S^1 \) is a smooth orientation preserving diffeomorphism, and thus \( \text{deg} M_a = 1 \). Consequence: if \( g : S^1 \to S^1 \) is continuous, then \( \text{deg} g \circ M_a = \text{deg} g \).

2. If \( 1 \leq p < \infty \) and \( a \in \mathbb{D} \), then \( |f|_{W^{1/p,p}} = |f|_{W^{1/p,p}} (M_a) \). (Here, we let \( |f|_{W^{1,1}} := \int_{S^1} |f| = \int_{0}^{2\pi} |df(\varphi)|/d\varphi |d\varphi \) and, for \( 1 < p < \infty \), \( |f|_{W^{1/p,p}} := \int_{S^1} \int_{S^1} |f(x) - f(y)|^p/|x-y|^2 \text{d}x \text{d}y \).) In order to prove the desired equality when \( p = 1 \), we write \( M_a = e^{\psi(\varphi)} \), \( 0 \leq \varphi \leq 2\pi \), with \( \varphi \) smooth and increasing. Then

\[ |f|_{W^{1,1}} = \int_{S^1} \frac{d}{d\varphi}|f(e^{\psi(\varphi)})| d\varphi = \int_{0}^{\psi^{-1}(2\pi)} \frac{d}{d\varphi}|f(e^{\psi(\varphi)})| d\varphi \]

When \( 1 < p < \infty \), we rely on the following identity, valid for measurable functions \( F : S^1 \times S^1 \to [0, \infty] \):

\[ \int_{S^1} \int_{S^1} \frac{F(M_a(x), M_a(y))}{|x-y|^2} \text{d}x \text{d}y = \int_{S^1} \int_{S^1} \frac{F(x, y)}{|x-y|^2} \text{d}x \text{d}y. \tag{8} \]

Proof of (8): We have \( |M_a|^{-1} = M_a \) and thus, after change of variables, (8) amounts to

\[ |x-y|^2 |M_a(x)| |M_a(y)| = |M_a(x) - M_a(y)|^2, \quad \forall x, y \in S^1. \tag{9} \]

In turn, (9) follows immediately from the straightforward equality \( |M_a(x)| = \frac{1-|a|^2}{|1-\overline{a}x|^2} \).

3. If \( 1 \leq p < \infty \) and \( a \in \mathbb{D} \), then \( \text{deg}[f \circ M_a] = \text{deg} f \). Indeed, to start with, such \( f \) has a degree, since \( W^{1/p,p} \hookrightarrow \text{VMO} \) and \( \text{VMO} \) maps gave a degree stable with respect to BMO convergence [11]. By item 1, the desired equality holds true
for smooth $f$. The general case follows by density of $C(\mathbb{S}; \mathbb{S})$ into $W^{1/p,p}(\mathbb{S}; \mathbb{S})$ [11, Lemmas A11 and A12] and by stability of the VMO degree.

4. If $1 \leq p < \infty$ and the degree of $f$ is $d$, then we may write $f(z) = e^{\psi(z)} z^d$, with $\psi \in W^{1/p,p}(\mathbb{S}; \mathbb{R})$. This follows easily from the fact that maps $f \in W^{1/p,p}(0,1; \mathbb{S})$ lift within $W^{1/p,p}$ [3].

5. Let $1 < p < \infty$. For $f \in W^{1/p,p}(\mathbb{S}; \mathbb{S})$, let $u = u(f)$ be its harmonic extension. Set $c'_p := \inf \{ |f|^p_{W^{1/p,p}} : u(0) = 0 \}$. Clearly, $c'_p$ is achieved, and therefore $c'_p > 0$.

6. When $p = 2$, we have the following straightforward calculations: if $f = \sum_{n \in \mathbb{N}} a_n e^{in\theta}$, then $|f|^2_{W^{1/2,2}} = 4\pi^2 \sum_{n \in \mathbb{N}} |a_n|^2$ [10, Chapter 13], and $\deg f = \sum_{n \in \mathbb{N}} n |a_n|^2$ [11, eq. (25)]. This leads to $4\pi^2 \deg f \leq |f|^2_{W^{1/2,2}}$, with equality e.g. when $f(z) := z^d$. On the other hand, if $u(0) = 0$, then $a_0 = 0$ and thus

$$|f|^2_{W^{1/2,2}} = 4\pi^2 \sum_{n \neq 0} |a_n|^2 \geq 4\pi^2 \sum_{n \neq 0} |a_n|^2 = 4\pi^2 \sum_{n \in \mathbb{N}} |a_n|^2 = 2\pi \|f\|_{L^2}^2 = 4\pi^2.$$ 

Thus $c'_2 \geq 4\pi^2$, and the example $f(z) := z$ shows that $c'_2 = 4\pi^2$.

7. For $1 < p < \infty$, there exists some constant $c'_p$ such that $c'_p \| \deg f \| \leq |f|^p_{W^{1/p,p}}, \forall f \in W^{1/p,p}(\mathbb{S}, \mathbb{S})$ [5, Corollary 0.5]. We let $c'_p$ be the best constant such that this estimate holds, and set $c'_p := \min\{c'_p, c'_p\}$. We also set $c_p := 1/c'_p$. By item 6, for $p = 2$ we have $c'_2 = c'_2 = 4\pi^2$, and $c_2 = 1/(4\pi^2)$.

8. Let $1 < p < \infty$. Let $\delta > 0$ and assume that $|u(0)| \geq \delta$ in $\mathbb{D}$. Then there exists some $C = C(\delta, p)$ such that

$$f = e^{\psi'}, \quad \psi' \in W^{1/p,p}(\mathbb{S}; \mathbb{R}) \quad \text{and} \quad |\psi'|_{W^{1/p,p}} \leq C |f|_{W^{1/p,p}}. \quad (10)$$

Indeed, set $v := u/|u|$, and write $v = e^{\psi'}$, with smooth $\psi$. By standard properties of the functional calculus and of trace theory, and by the lifting estimates in [3], we have $\psi \in W^{2/p,p}(\mathbb{D}; \mathbb{R})$, and then $\psi := \text{tr} \psi \in W^{1/p,p}(\mathbb{S}; \mathbb{R})$ satisfies

$$|\psi|_{W^{1/p,p}} \leq C(p) |\psi|_{W^{2/p,p}} \leq C(p) \|v\|_{W^{2/p,p}} \leq C(\delta, p) \|f\|_{W^{1/p,p}}.$$ 

9. Let $1 < p < \infty$ and $c < c'_p$. If $|f|^p_{W^{1/p,p}} \leq c$, then there exists some $\delta > 0$ such that $|u(0)| \geq \delta$ in $\mathbb{D}$. Proof by contradiction: assume that $|f|^p_{W^{1/p,p}} \leq c$, $f_n \to g$ and $|\psi_n|_{W^{1/p,p}} \leq 1/n$. Since $u(g \circ M_a) = u(g) \circ M_a$, we may assume (by item 2) that $a_n = 0$. We find that $u(0) = 0$ and $|f|^p_{W^{1/p,p}} < c'_p$, which is impossible.

10. Let $1 < p < \infty$. Assume that $f_n \to f$ and $f_n \to f$ a.e. Then $|f|^p_{W^{1/p,p}} = |f|^p_{W^{1/p,p}} + |f_n|^p_{W^{1/p,p}} + o(1)$. Indeed, if we set $g_n := f_n / f$, then this follows from the Brézis–Lieb lemma [9] and the identity

$$\frac{\overline{\psi_n}}{\overline{\psi_n}}(x) = f_n(x) - f_n(y) = f(x) - f(y) + \frac{\overline{\psi_n}}{\overline{\psi_n}}(x) f(y) [g_n(x) - g_n(y)].$$

Proof of Theorem 1. The proof is by complete induction on the integer part $L := I(c_p, M) = I(\text{M}/c_p^*)$ of $c_p M$. The case where $L = 0$ follows from items 8 and 9. Let $L > 0$ and let $M$ be such that $I(M/c_p^*) = L$. Assume, by contradiction, that the theorem does not hold for $M$. We may thus find a sequence $(f_n)$ with the following properties:

(a) $|f_n|^p_{W^{1/p,p}} \leq M;
(b) \quad \text{for any} \ K \leq L \text{ and any choice of} a_1, \ldots, a_K \in \mathbb{D} \text{ and of signs} \ e_j = \pm 1 \text{ such that} \sum \limits_{j=1}^K e_j = \deg f_n, \text{if we write} \ f_n = e^{\psi_n} \prod \limits_{j=1}^K (M_{a_j})^{e_j}, \text{then we have} \ |\psi_n|_{W^{1/p,p}} \to \infty. \text{(It is always possible to take} K, a_j, e_j \text{ and} \ \psi_n \text{ as above: it suffices to let} \ K := \deg f \leq I(M/c_p^*) \leq I(M/c_p^*) = L, \ e_j := \pm \deg f_n, \text{and} \ a_j = 0.)
(by item 8 and property (b), there exist points $a_\eta \in \mathbb{D}$ such that $u(f_n)(a_\eta) \to 0$. By item 2, we may assume in addition that $a_\eta = 0$. Thus, in addition to (a) and (b), we may assume:

(c) $f_n \to f$ and $f_n \to f$ a.e., for some $f$ with $u(f)(0) = 0$.

Set $g_n := f_n / f$. By item 10 and the definition of $c'_p$, we have $|f_n|^p_{W^{1/p,p}} \geq c'_p \geq c'_p$, and $|g_n|^p_{W^{1/p,p}} = M - |f|^p_{W^{1/p,p}} + o(1)$. Let $N > M - |f|^p_{W^{1/p,p}}$ be such that $I(N/c'_p) = I(M - |f|^p_{W^{1/p,p}}/c'_p) \leq L - 1$. For large $n$, we have $|g_n|^p_{W^{1/p,p}} \leq N$. By the induction hypothesis, we may write (up to a subsequence) $g_n = e^{\eta_n} \prod \limits_{j=1}^N (M_{a_j})^{e_j}$, with $|\eta_n|_{W^{1/p,p}} \leq F(N)$ and $R \leq N/c'_p$. On the other hand, if $d := \deg f, b_j := 0$ and $e_j := \pm \deg n$, then we may write $e^{\eta_n} \prod \limits_{j=b+1}^{R+|d|} (M_{a_j})^{e_j}$, with $\eta \in W^{1/p,p}$ (item 4). In addition, we have $|d| \leq |f|^p_{W^{1/p,p}}/c'_p$ (item 7). Finally, with $\psi_n := \psi_n + \eta$ and $K := R + |d| \leq M/c_p^*$, we have $f_n = e^{\psi_n} \prod \limits_{j=1}^N (M_{a_j})^{e_j}$, and $(\psi_n)$ is bounded in $W^{1/p,p}$. This contradiction completes the proof of the first part of the theorem.

Optimality of (3) when $p = 2$ follows from the fact that, by item 6, $f(z) := z^d, d > 0$, satisfies $|f|^2_{W^{1/2,2}} = c_2 d$ and requires at least $d$ Möbius maps in its decomposition. □

Proof of Corollary 1. By replacing $f_n$ with $f_n / f$, we may assume that $f_n \to 1$. Up to a subsequence, we may write $f_n = e^{i\eta_n} \prod \limits_{j=1}^N (M_{a_j})^{e_j}$, with $a_j \to a_j \in \mathbb{D}, j = 1, \ldots, P$, and $\eta_n \to \eta$. With no loss of generality, we assume that $a_1, \ldots, a_K \in \mathbb{S}$ and $a_{K+1}, \ldots, a_P \in \mathbb{D}$. Since (clearly) $M_{a_j} \to a_j, j = 1, \ldots, K$, we find that $1 = e^{i(\eta - C)} \prod \limits_{j=K+1}^P (M_{a_j})^{e_j}$ for some appropriate $C$. Thus, with $\zeta_n := \eta_n - \eta$, we have
\( f_n = e^{i\theta_n + \xi_n} \prod_{j=1}^K (M_{\alpha_{jn}})^{\xi_j} \prod_{j=K+1}^P (M_{\beta_{jn}} M_{\alpha_j})^{\xi_j} = e^{i\psi_n} \prod_{j=1}^K (M_{\alpha_{jn}})^{\xi_j}, \)

for some \( \psi_n \) such that \( \psi_n - \xi_n \to C \) in \( W^{1/p,p} \), and thus \( \psi_n \to C \). □

**Remark.** The corollary implies the “bubbling-off of circles along a sequence of graphs”. More specifically, the behavior of weakly converging sequences of manifold-valued maps can be investigated within the theory of Cartesian currents of Giaquinta, Modica and Souček [13]; see also [14,17] for the specific case of \( W^{1/2,2}(S^1; S^1) \). When \( p = 2 \), it is possible to define (as a current) the graph \( \mathcal{G}_f \) of \( f \in W^{1/2,2}(S^1; S^1) \). With the notation in the corollary, if \( p = 2 \) and \( f_n \to f \), “bubbling-off” reads

\[
\mathcal{G}_{f_n} \to \mathcal{G}_f + \sum_{j=1}^K \varepsilon_j \delta_{\alpha_j} \times [S^1] \text{ in } D_1(S^1 \times S^1).
\]

This can be obtained directly from (1) [17, Proposition 3.1], but also as an immediate consequence of the corollary. Details are left to the reader.

3. Applications

We start with an immediate consequence of **Theorem 1**.

**Corollary 2.** Let \( d \) be a non-negative integer and \( \delta > 0 \). Then there exists a constant \( F(d, \delta) \) such that: every map \( f \in W^{1/2,2}(S^1; S^1) \) satisfying \( \deg f = \delta \) and \( |f|_{W^{1/2,2}}^2 \leq 4\pi^2 (d+1) - \delta \) can be written as \( f = e^{i\psi} \prod_{j=1}^d M_{\alpha_j} \), with \( |\psi|_{W^{1/2,2}} \leq F(d, \delta) \).

**Corollary 2** with \( d = 1 \), as well as a weak version of the corollary when \( d \geq 2 \) were obtained in [2, Theorem 4.4, Theorem 4.8]. As an application of **Corollary 2**, we obtain the following theorem.

**Theorem 2.** There exists some \( \varepsilon > 0 \) such that, for \( p \in (2-\varepsilon, 2] \),

\[
m_p := \min|f|_{W^{1/p,p}}^p, \quad \deg f = 1
\]

is achieved.

**Proof.** When \( p = 2 \), it follows from item 6 that \( m_2 \) is achieved by multiples of Möbius maps.

When \( 1 < p < 2 \), consider a minimizing sequence for \( m_p \). Since \( m_p \leq |\text{id}|_{W^{1/p,p}} \), we may assume that

\[
|f_n|_{W^{1/p,p}} \leq I_p \to I_2 = 4\pi^2 \quad \text{as } p \to 2.
\]

On the other hand, when \( f : S^1 \to S^1 \) we have \( |f|^2_{W^{1/2,2}} \leq 2^{2-p} |f|^p_{W^{1/p,p}} \). Thus

\[
|f_n|^2_{W^{1/2,2}} \leq I_p := 2^{2-p} I_p \to 4\pi^2 \quad \text{as } p \to 2.
\]

For \( p \) sufficiently close to 2 and fixed \( \delta > 0 \), we have \( I_p \leq 8\pi^2 - \delta \). We next apply **Corollary 2** to \( f_n \) and write \( f_n = e^{i\phi_n} M_{\alpha_n} \), with \( |\psi_n|_{W^{1/2,2}} \leq F(1, \delta) \). Set \( g_n := f_n \circ M_{\alpha_n} \). By item 2, \( (g_n) \) is a minimizing sequence for \( m_p \). On the other hand, we have \( g_n = e^{i\phi_n} \text{id} \), with \( \phi_n := \psi_n \circ M_{\alpha_n} \) bounded in \( W^{1/2,2}(S^1; \mathbb{R}) \) by (8). Therefore, up to a subsequence \( g_n \to g \) in \( W^{1/2,2} \), and thus \( g_n \to g := e^{i\phi} \text{id} \) in \( W^{1/2,2} \). We find that \( \deg g = 1 \). Since \( (g_n) \) is bounded in \( W^{1/p,p} \), we obtain that \( g_n \to g \) in \( W^{1/p,p} \). By a standard argument, \( g \) achieves \( m_p \). □

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