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Harmonic analysis

Rough fractional integrals and its commutators on variable Morrey spaces

Intégrales fractionnaires rugueuses et ses commutateurs sur les espaces de Morrey avec exposant variable

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ABSTRACT

In this paper, the authors obtain the boundedness of fractional integrals with rough kernel on variable Morrey spaces. The corresponding boundedness for commutators generalized by the fractional integral and BMO function is also considered.

RÉSUMÉ

Dans cet article, les auteurs obtiennent la bornitude des intégrales fractionnaires avec un noyau singulier dans des espaces de Morrey (avec exposant variable). De plus, la bornitude des commutateurs généralisés entre ces opérateurs et la multiplication par une fonction BMO est aussi considérée.

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1. Introduction

In this paper, we are concerned with variable Morrey spaces $\mathcal{M}_{p(\cdot),u}$ introduced in [8], which are equipped with a Morrey weight function u and a variable exponent function $p(\cdot)$. The main results of this paper consist of the boundedness of rough fractional integrals and its commutators with BMO functions together with the boundedness of the rough fractional maximal operators on variable Morrey spaces.

Let $\mathbb{S}^{n-1}$ denote the unit sphere in Euclidean space $\mathbb{R}^n$ and $\Omega \in L^s(\mathbb{S}^{n-1})$ $(s \geq 1)$ be homogeneous of degree zero on $\mathbb{R}^n$. For $0 < \alpha < n$, the rough fractional integral is defined by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n-\alpha}} f(x-y)\,\mathrm{d}y$$

and a related fractional maximal operator $M_{\Omega,\alpha}$ is defined by

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$$M_{\Omega,\alpha} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|\le r} |\Omega(y') f(x-y)|\,\mathrm{d}y,$$

where $y' = \frac{y}{|y|}$ for any $y \neq 0$.

It is easy to see that $T_{\Omega,\alpha}$ is just the Riesz potential I_α when $\Omega \equiv 1$. The Hardy–Littlewood–Sobolev theorem (see [6,14]) states that I^α is a bounded operator from Lebesgue spaces $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $0 < \alpha < n$, $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. The boundedness of $T_{\Omega,\alpha}$ on Lebesgue spaces was first introduced by Muckenhoupt and Wheeden [11]. Recently, Wang studied the weighted norm inequalities for $T_{\Omega,\alpha}$ on Morrey spaces in [17]. It extends Spanne's result on the boundedness of the fractional integrals in [13]. On the other hand, moving in another direction, variable exponent function spaces theory has attracted much attention, due to its application to partial differential equations and the calculus of variations (see [3,4,12,18]). In many applications, a crucial step has been to show that the classical operators of harmonic analysis, such as maximal operators, singular integrals and fractional integrals, are bounded on variable exponent function spaces. In 2007, Capone et al. [1] established the boundedness of I_α on variable Lebesgue spaces. In 2013, the Hardy–Littlewood–Sobolev theorem was generalized in [8] by Ho to the case of variable Morrey spaces on unbounded domains. In 2015, one of the authors and Liu in [15] showed that the Hardy–Littlewood–Sobolev theorem still held for homogeneous fractional integrals on variable Lebesgue spaces as well as on variable Hardy spaces. These results leave open the question of the Morrey spaces estimates for $T_{\Omega,\alpha}$ and related operators in the variable exponent setting. Before stating our results, we need some notations and definitions on variable exponent analysis.

For a measurable subset $E \subset \mathbb{R}^n$, we denote $p^-(E) = \inf_{x\in E} p(x)$ and $p^+(E) = \sup_{x\in E} p(x)$. Especially, we denote $p^- = p^-(\mathbb{R}^n)$ and $p^+ = p^+(\mathbb{R}^n)$. Let $p(\cdot)$: $\mathbb{R}^n \to (0,\infty)$ be a measurable function with $0 < p^- \le p^+ < \infty$ and $\mathcal{P}$ be the set of all measurable functions $p(\cdot): \mathbb{R}^n \to [1,\infty)$ such that $1 < p^- \le p^+ < \infty$. Let $\mathcal{B}$ be the set of $p(\cdot) \in \mathcal{P}$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}$.

An important subset of $\mathcal{B}$ is the class of globally log-Hölder continuous functions $p \in LH$, with $1 < p^- \le p^+ < \infty$. Recall that $p(\cdot) \in LH(\mathbb{R}^n)$, if $p(\cdot)$ satisfies

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)}, \quad |x-y| \le 1/2, \quad |p(x) - p(y)| \le \frac{C}{\log|x| + e}, \quad |y| \ge |x|.$$

Definition 1.1. (See [8].) Let $q \in \mathcal{P}$. A Lebesgue measurable function $u(x,r): \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ is said to be a Morrey weight function $\mathbb{W}_q$ for $L^{p(\cdot)}$ if there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r \ge 0$, u fulfills

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{q(\cdot)}}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{q(\cdot)}}} u(x, 2^{j+1} r) < C\, u(x,r).$$

Definition 1.2. (See [2,5].) The variable Lebesgue space $L^{p(\cdot)}$ is defined as the set of all measurable functions f for which the quantity $\int_{\mathbb{R}^n} |\varepsilon f(x)|^{p(x)} \mathrm{d}x$ is finite for some $\varepsilon > 0$ and

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \le 1\right\}.$$

As a special case of the theory of Nakano and Luxemberg, we see that $L^{p(\cdot)}$ is a quasi-normed space. Especially, when $p^- \ge 1$, $L^{p(\cdot)}$ is a Banach space.

Definition 1.3. (See [7].) The variable Morrey space $\mathcal{M}_{p(\cdot),u}$ is the collection of all Lebesgue measurable functions f satisfying

$$\|f\|_{\mathcal{M}_{p(\cdot),u}} = \sup_{z\in\mathbb{R}^n, R>0} \frac{1}{u(z,R)} \|\chi_{B(z,R)} f\|_{L^{p(\cdot)}} < \infty,$$

for $p(x) \in \mathcal{P}$ and $u(x,r): \mathbb{R}^n \times (0.\infty) \to (0,\infty)$.

Throughout this paper, C or c denotes a positive constant that may vary at each occurrence, but is independent of the main parameter, and $A \sim B$ means that there are constants $C_1 > 0$ and $C_2 > 0$ independent of the main parameter such that $C_1 B \le A \le C_2 B$. Given a measurable set $S \subset \mathbb{R}^n$, $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function.

Now, let us formulate our main results.

Theorem 1.1. *Suppose that $\Omega \in L^s(\mathcal{S}^{n-1})$ with $1 < s \le \infty$. Let $\alpha > 0$, $p(\cdot), q(\cdot) \in \mathcal{B}$, $\frac{q(\cdot)(n-\alpha)}{n} \in \mathcal{B}$ and $u \in \mathbb{W}_{q(\cdot)}$. Suppose that $1 \le s' < p_- \le p_+ < \frac{n}{\alpha}$ and $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$, a.e. on $\mathbb{R}^n$, then*

$$\|T_{\Omega,\alpha} f\|_{\mathcal{M}_{q(\cdot),u}} \le C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|f\|_{\mathcal{M}_{p(\cdot),u}}$$

and

$$\|M_{\Omega,\alpha} f\|_{\mathcal{M}_{q(\cdot),u}} \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{\mathcal{M}_{p(\cdot),u}}.$$

Furthermore, if $b \in BMO$ *and*

$$\sum_{j=0}^{\infty}(j+1)\frac{\|\chi_{B(x,r)}\|_{L^{q(\cdot)}}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{q(\cdot)}}}u(x,2^{j+1}r) < Cu(x,r),$$

then

$$\|[b, T_{\Omega,\alpha}]f\|_{\mathcal{M}_{q(\cdot),u}} \le C\|b\|_*\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{\mathcal{M}_{p(\cdot),u}}.$$

2. Proof of the main results

Before we prove the main results, we need some more facts on variable exponent function spaces. First we state results concerning the boundedness of the rough fractional integrals on variable Lebesgue spaces given in [15].

Proposition 2.1. *Let* $p(\cdot), q(\cdot) \in \mathcal{P}$, $0 < \alpha < n$, $1 < p^- \le p^+ < \frac{n}{\alpha}$ *and* $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ *for any* $x \in \mathbb{R}^n$. *If* $\frac{q(\cdot)(n-\alpha)}{n} \in \mathcal{B}$, $\Omega \in L^s$ *and* $1 \le s' < p^-$, *then*

$$\|T^\alpha_\Omega f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Similarly, we can obtain the boundedness of the commutators of rough fractional integral on variable Lebesgue spaces. For the sake of brevity, here we state the result without proof.

Proposition 2.2. *Let* $b \in BMO$, $p(\cdot), q(\cdot) \in \mathcal{P}$, $0 < \alpha < n$, $1 < p^- \le p^+ < \frac{n}{\alpha}$ *and* $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ *for any* $x \in \mathbb{R}^n$. *If* $\frac{q(\cdot)(n-\alpha)}{n} \in \mathcal{B}$, $\Omega \in L^s(\mathbb{S}^{n-1})$ *and* $1 \le s' < p^-$, *then*

$$\|[b, T]^\alpha_\Omega f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

The next basic property satisfied by $L^{p(\cdot)}$ is given in [8].

Proposition 2.3. *Let* $p(\cdot) \in \mathcal{B}$, *then we have a constant* $C > 0$ *so that for any balls or cubes* B,

$$\|\chi_B\|_{L^{p(\cdot)}}\|\chi_B\|_{L^{p'(\cdot)}} \sim |B|.$$

Proof of Theorem 1.1. Let $f \in \mathcal{M}_{p(\cdot),u}$. For any $z \in \mathbb{R}^n$, $r > 0$ and decompose $f = f_0 + \sum_{j=1}^{\infty} f_j$, where $f_0 = \chi_{B(z,2r)}f$ and $f_j = \chi_{B(z,2^{j+1}r)\backslash B(z,2^j r)}f$, $j \in \mathbb{N}\backslash\{0\}$.
Since $T_{\Omega,\alpha}$ is a linear operator, then we can write

$$\frac{1}{u(z,r)}\|\chi_{B(z,r)}T_{\Omega,\alpha}f\|_{L^{q(\cdot)}} \tag{1}$$

$$\le \frac{1}{u(z,r)}\|\chi_{B(z,r)}T_{\Omega,\alpha}f_0\|_{L^{q(\cdot)}} + \frac{1}{u(z,r)}\sum_{j=1}^{\infty}\|\chi_{B(z,r)}T_{\Omega,\alpha}f_j\|_{L^{q(\cdot)}} = I + II. \tag{2}$$

Notice that $p(\cdot), q(\cdot) \in \mathcal{P}$, $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. Proposition 2.1 shows that $\|T_{\Omega,\alpha}f\|_{L^{q(\cdot)}} \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p(\cdot)}}$. Thus, we can obtain that

$$I \le \frac{1}{u(z,r)}\|T_{\Omega,\alpha}f_0\|_{L^{q(\cdot)}} \le \frac{C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}}{u(z,2r)}\|f\chi_{B(z,2r)}\|_{L^{p(\cdot)}} \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{\mathcal{M}_{p(\cdot),u}}. \tag{3}$$

We now turn to deal with the term II. By Hölder's inequality,

$$|T_{\Omega,\alpha}f_j(x)| \le \left(\int_{R_j}|\Omega(x-y)|^s\mathrm{d}y\right)^{1/s}\left(\int_{R_j}\frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}\mathrm{d}y\right)^{1/s'},$$

where $R_j = B(z,2^{j+1}r)\backslash B(z,2^j r)$. When $x \in B(z,r)$ and $y \in R_j$, it easy to see that $|x-y| \sim |y-z| \sim 2^j r$. Therefore, $R_j \subset \{y : |x-y| \le C2^j r\}$. Then we can get

$$\left(\int_{R_j} |\Omega((x-y)')|^s \mathrm{d}y\right)^{1/s} \leq \left(\int_{|x-y|\leq C2^jr} |\Omega((x-y)')|^s \mathrm{d}y\right)^{1/s}$$
$$\leq \left(\int_{\mathbb{S}^{n-1}} \int_0^{C2^jr} R^{n-1}|\Omega((x-y)')|^s \,\mathrm{d}R\,\mathrm{d}\sigma(y')\right)^{1/s} \leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}(2^jr)^{n/s}.$$

Moreover,

$$\left(\int_{R_j} \frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}\mathrm{d}y\right)^{1/s'} \leq 2^{-j(n-\alpha)}r^{-n+\alpha}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{s'}}.$$

The generalized Hölder inequality given in [10] ensures that

$$\|f\chi_{B(z,2^{j+1}r)}\|_{L^{s'}} \leq C(2^jr)^{-\frac{n}{s}}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}.$$

In fact, we can choose $\tilde{p}(\cdot) > 0$ such that $\frac{1}{s'} = \frac{1}{p(x)} + \frac{1}{\tilde{p}(x)}$ due to $1 \leq s' < p^-$, we have

$$\|f\chi_{B(z,2^{j+1}r)}\|_{L^{s'}} \leq C\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{\tilde{p}(\cdot)}}.$$

Moreover, observing that $\frac{1}{\tilde{p}(\cdot)} = \frac{1}{s'} - \frac{1}{p(\cdot)} = \frac{1}{p'(\cdot)} - \frac{1}{s}$, by Lemma 2.4 in [16], we have

$$\|\chi_{B(z,2^{j+1}r)}\|_{L^{\tilde{p}(\cdot)}} \sim (2^jr)^{-\frac{n}{s}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}.$$

Therefore, we get that

$$|T_{\Omega,\alpha}f_j(x)\chi_{B(z,r)}(x)| \leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\chi_{B(z,r)}(x)(2^jr)^{\alpha-n}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}$$

and the rest of the proof is the same as the proof of Theorem 2.1 in [8]. Thus, we can prove that

$$\|T_{\Omega,\alpha}f\|_{\mathcal{M}_{q(\cdot),u}} \leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{\mathcal{M}_{p(\cdot),u}}.$$

It is not hard to check that $M_{\Omega,\alpha}f(x) \leq CT_{|\Omega|,\alpha}(|f|)(x)$, so the proof of rough fractional maximal operators follows from the boundedness of $T_{\Omega,\alpha}f$.

Next, we prove that $\|[b, T_{\Omega,\alpha}]f\|_{\mathcal{M}_{q(\cdot),u}} \leq C\|b\|_*\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{\mathcal{M}_{p(\cdot),u}}$.

$$\frac{1}{u(z,r)}\|\chi_{B(z,r)}[b, T_{\Omega,\alpha}]f\|_{L^{q(\cdot)}}$$
$$\leq \frac{1}{u(z,r)}\|\chi_{B(z,r)}[b, T_{\Omega,\alpha}]f_0\|_{L^{q(\cdot)}} + \frac{1}{u(z,r)}\sum_{j=1}^{\infty}\|\chi_{B(z,r)}[b, T_{\Omega,\alpha}]f_j\|_{L^{q(\cdot)}} = J_1 + J_2.$$

Applying Proposition 2.2, we have $J_1 \leq C\|b\|_*\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{\mathcal{M}_{q(\cdot),u}}$.

Denote that $f_{B(z,r)} = \frac{1}{|B(z,r)|}\int_{B(z,r)} f(y)\,\mathrm{d}y$. Now we consider J_2.

$$|[b, T_{\Omega,\alpha}]f_j(x)\chi_{B(z,r)}(x)|$$
$$\leq \chi_{B(z,r)}(x)2^{(j+1)(\alpha-n)}r^{\alpha-n}\int_{B(z,2^{j+1}r)} |b(x) - b_{B(z,r)} + b_{B(z,r)} - b(y)||\Omega(x-y)||f(y)|\,\mathrm{d}y$$
$$\leq \chi_{B(z,r)}(x)2^{(j+1)(\alpha-n)}r^{\alpha-n}|b(x) - b_{B(z,r)}|\int_{B(z,2^{j+1}r)} |\Omega(x-y)||f(y)|\,\mathrm{d}y$$
$$+ \chi_{B(z,r)}(x)2^{(j+1)(\alpha-n)}r^{\alpha-n}\int_{B(z,2^{j+1}r)} |b_{B(z,r)} - b(y)||\Omega(x-y)||f(y)|\,\mathrm{d}y = J_3 + J_4.$$

It is easy to check that

$$J_3 \leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\chi_{B(z,r)}(x)(2^jr)^{\alpha-n}|b(x) - b_{B(z,r)}|\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}.$$

In the other hand, we estimate J_4.

$$J_4 \le (2^j r)^{\alpha-n}\chi_{B(z,r)}(x)\int\limits_{B(z,2^{j+1}r)}|\Omega(x-y)||b(y)-b_{B(z,2^{j+1}r)}||f(y)|\,\mathrm{d}y$$
$$+(2^j r)^{\alpha-n}\chi_{B(z,r)}(x)|b_{B(z,2^{j+1}r)}-b_{B(z,r)}|\int\limits_{B(z,2^{j+1}r)}|\Omega(x-y)|f(y)|\,\mathrm{d}y = K_1+K_2.$$

As an application of Hölder's inequality yields

$$\begin{aligned} K_1 &\le C(2^j r)^{\alpha-n+n/s}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\chi_{B(z,r)}(x)\|(b-b_{B(z,2^{j+1}r)})f\chi_{B(z,2^{j+1}r)}\|_{L^{s'}} \\ &\le C(2^j r)^{\alpha-n}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\chi_{B(z,r)}(x)\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\|(b-b_{B(z,2^{j+1}r)})\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}} \\ &\le C(2^j r)^{\alpha-n}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|b\|_*\chi_{B(z,r)}(x)\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}, \end{aligned}$$

where the last inequality follows from the fact that $\|b\|_* \sim \sup_Q \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}}\|(b-b_Q)\chi_Q\|_{L^{p(\cdot)}}$ holds for all $b \in BMO$ (see Lemma 3 in [9]).

On the other hand, in view of $|b_{B(z,2^{j+1}r)}-b_{B(z,r)}| \le Cj\|b\|_*$, we can get that

$$K_2 \le C(2^j r)^{\alpha-n}\chi_{B(z,r)}(x)j\|b\|_*\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}.$$

Therefore, by Proposition 2.3, we can obtain that

$$\begin{aligned} &\|[b,T_{\Omega,\alpha}]f_j(x)\chi_{B(z,r)(x)}\|_{L^{q(\cdot)}} \\ &\le C(j+1)(2^j r)^{\alpha-n}\|b\|_*\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B(z,r)}\|_{L^{q(\cdot)}}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}} \\ &\le C\|b\|_*\|\Omega\|_{L^s(\mathbb{S}^{n-1})}(j+1)\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}}}u(z,2^{j+1}r)\sup_{y\in\mathbb{R}^n,R>0}\frac{1}{u(y,R)}\|f\chi_{B(y,R)}\|_{L^{p(\cdot)}} \\ &\le C\|b\|_*\|\Omega\|_{L^s(\mathbb{S}^{n-1})}(j+1)\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}}}u(z,2^{j+1}r)\|f\|_{\mathcal{M}_{p(\cdot),u}}. \end{aligned}$$

Notice that $\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}}}\frac{u(z,2^{j+1}r)}{u(z,r)} \le C$.

Thus,

$$J_2 \le C\|b\|_*\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}}}\frac{u(z,2^{j+1}r)}{u(z,r)}\|f\|_{\mathcal{M}_{p(\cdot),u}} \le C\|b\|_*\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{\mathcal{M}_{p(\cdot),u}}.$$

So we have completed the proof of Theorem 1.1. □

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