Differential geometry/Mathematical economics

Some characterizations of the quasi-sum production models with proportional marginal rate of substitution

Certaines caractérisations des modèles de production quasi-somme avec un taux marginal de substitution proportionnelle

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\textbf{A R T I C L E   I N F O}

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This paper is dedicated to Prof. Ieronim Mihăilă on the occasion of his 79th birthday

\textbf{A B S T R A C T}

In this note we classify quasi-sum production functions with constant elasticity of production with respect to any factor of production and with proportional marginal rate of substitution.

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\textbf{R É S U M É}

Dans cette note, nous classons les fonctions de production quasi-somme avec élasticité constante de la production par rapport à un facteur de production et avec un taux marginal de substitution proportionnel.

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1. Introduction

The notion of \textit{production function} is a key concept in both macroeconomics and microeconomics, being used in the mathematical modeling of the relationship between the output of a firm, an industry, or an entire economy, and the inputs that have been used in obtaining it. Generally, production function is a twice differentiable mapping \( f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \), \( f = f(x_1, \ldots, x_n) \), where \( f \) is the quantity of output, \( n \) is the number of the inputs and \( x_1, \ldots, x_n \) are the factor inputs. A production function \( f \) is called \textit{quasi-sum} \([3,5]\) if there are strict monotone functions \( G, h_1, \ldots, h_n \) with \( G' > 0 \) such that

\[ f(x) = G(h_1(x_1) + \ldots + h_n(x_n)), \]  

where \( x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n \). We note that these functions are of great interest because they appear as solutions to the general bisymmetry equation, being related to the problem of consistent aggregation \([1]\).

Among the family of production functions, the most famous is the so-called Cobb–Douglas production function. A generalized Cobb–Douglas production function depending on \( n \)-inputs is given by
where $A, \alpha_1, \ldots, \alpha_n > 0$. We recall that a production function of the form $f(x) = G(h(x_1, \ldots, x_n))$, where $G$ is a strictly increasing function and $h$ is a homogeneous function of any given degree $p$, is said to be a homothetic production function \cite{7}. It is easy to see that a production function $f$ can be identified with the graph of $f$, i.e. the nonparametric hypersurface of $\mathbb{E}^{n+1}$ defined by

$$L(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n))$$

and called the production hypersurface of $f$ (see \cite{9,11}). Motivated by some recent classification results concerning production hypersurfaces \cite{2,5,7,8,12}, in the present work we classify quasi-sum production functions with a proportional marginal rate of substitution and investigate the existence of such production models whose production hypersurfaces have null Gauss–Kronecker curvature or null mean curvature. We recall that, if $f$ is a production function with $n$ inputs $x_1, x_2, \ldots, x_n$, $n \geq 2$, the elasticity of production with respect to a certain factor of production $x_i$ is defined as

$$E_{x_i} = \frac{x_i}{f} f_{x_i}$$

and the marginal rate of technical substitution of input $x_j$ for input $x_i$ is given by

$$\operatorname{MRS}_{ij} = \frac{f_{x_j}}{f_{x_i}},$$

where the subscripts denote partial derivatives of the function $f$ with respect to the corresponding variables. A production function satisfies the proportional marginal rate of substitution property if

$$\operatorname{MRS}_{ij} = \frac{x_i}{x_j}, \text{ for all } 1 \leq i \neq j \leq n.$$  

In the last section of the paper we will prove the following theorem that generalizes the results from \cite{10}.

**Theorem 1.1.** Let $f$ be a quasi-sum production function given by \eqref{1}. Then:

i. The elasticity of production is a constant $k_i$ with respect to a certain factor of production $x_i$ if and only if $f$ reduces to

$$f(x_1, \ldots, x_n) = A \cdot \prod_{i=1}^{n} x_i^{k_i}, \quad \text{ (7)}$$

where $A$ and $D$ are positive constants.

ii. The elasticity of production is a constant $k_i$ with respect to all factors of production $x_i$, $i = 1, \ldots, n$, if and only if $f$ reduces to the generalized Cobb–Douglas production function given by \eqref{2}.

iii. The production function satisfies the proportional marginal rate of substitution property if and only if it reduces to the homothetic generalized Cobb–Douglas production function given by

$$f(x_1, \ldots, x_n) = F\left(\prod_{i=1}^{n} x_i^{k}\right), \quad \text{ (8)}$$

where $k$ is a nonzero real number.

iv. If the production function satisfies the proportional marginal rate of substitution property, then:

iv1. The production hypersurface has vanishing Gauss–Kronecker curvature if and only if, up to a suitable translation, $f$ reduces to the following generalized Cobb–Douglas production function with constant return to scale:

$$f(x_1, \ldots, x_n) = A \cdot \prod_{i=1}^{n} x_i^{\frac{1}{2}}. \quad \text{ (9)}$$

iv2. The production hypersurface cannot be minimal.

iv3. The production hypersurface has vanishing sectional curvature if and only if, up to a suitable translation, $f$ reduces to the following generalized Cobb–Douglas production function:

$$f(x_1, \ldots, x_n) = A \cdot \prod_{i=1}^{n} \sqrt{x_i}. \quad \text{ (10)}$$
2. Preliminaries on the geometry of hypersurfaces

For general references on the geometry of hypersurfaces, we refer to [4].

If $M$ is a hypersurface of the Euclidean space $\mathbb{R}^{n+1}$, then it is known that the Gauss map $\nu : M \rightarrow S^n$ maps $M$ to the unit hypersphere $S^n$ of $\mathbb{R}^{n+1}$. With the help of the differential $d\nu$ of $\nu$ it can be defined a linear operator on the tangent space $T_p M$, denoted by $S_p$ and known as the shape operator, by $g(S_p v, w) = g(d\nu(v), w)$, for $v, w \in T_p M$, where $g$ is the metric tensor on $M$ induced from the Euclidean metric on $\mathbb{R}^{n+1}$. The eigenvalues of the shape operator are called principal curvatures. The determinant of the shape operator $S_p$, denoted by $K(p)$, is called the Gauss–Kronecker curvature. When $n = 2$, the Gauss–Kronecker curvature is simply called the Gauss curvature, which is intrinsic due to famous Gauss's Theorem Egregium. The trace of the shape operator $S_p$ is called the mean curvature of the hypersurfaces. In contrast to the Gauss–Kronecker curvature, the mean curvature is extrinsic, which depends on the immersion of the hypersurface. A hypersurface is said to be minimal if its mean curvature vanishes identically. We recall now the following lemma which will be used in the proof of Theorem 1.1.

**Lemma 2.1.** (See [4].) For the production hypersurfaces defined by (3) and $w = \sqrt{1 + \sum_{i=1}^{n} f_i^2}$, we have:

i. The Gauss–Kronecker curvature $K$ is given by

$$K = \frac{\text{det}(f_{x_i x_j})}{w^{n+2}}.$$  \hfill (11)

ii. The mean curvature $H$ is given by

$$H = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{f_{x_i}}{w} \right).$$ \hfill (12)

iii. The sectional curvature $K_{ij}$ of the plane section spanned by $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$ is

$$K_{ij} = \frac{f_{x_i x_j} f_{x_j x_i} - f_{x_i}^2}{w^2 \left( 1 + f_i^2 + f_j^2 \right)}.$$ \hfill (13)

3. Proof of Theorem 1.1

Let $f$ be a quasi-sum production function given by (1). Then we have

$$f_{x_i}(x) = G'(u) h'_i(x_i)$$  \hfill (14)

with $u = h_1(x_1) + \ldots + h_n(x_n)$ and from (14) we derive

$$f_{x_i x_j} = G''(h'_i)^2 + G' h''_i, \quad i = 1, \ldots, n,$$

$$f_{x_i x_j} = G'' h'_i h'_j, \quad i \neq j.$$  \hfill (16)

i. We first prove the left-to-right implication. If the elasticity of production is a constant $k_i$ with respect to a certain factor of production $x_i$, then from (4) we obtain

$$f_{x_i} = k_i \frac{f}{x_i}.$$  \hfill (17)

Using now (1) and (14) in (17) we get

$$\frac{G'}{G} = k_i \frac{1}{x_i h'_i}.$$ \hfill (18)

By taking the partial derivative of (18) with respect to $x_j, j \neq i$, we obtain

$$\frac{h'_j}{G} \frac{G''}{G} - \left( \frac{G'}{G} \right)^2 = 0.$$  \hfill (19)

Now, taking into account that $h_j$ is a strict monotone function, we find

$$G(u) = C \cdot e^{Du},$$  \hfill (20)
for some positive constants C and D. Hence from (18) and (19) we obtain

\[ h_i(x_i) = \frac{k_i}{D} \ln x_i + A_i, \]

(20)

where \( A_i \) is a real constant. Finally, combining (1), (19) and (20) we get a function of the form (7), where \( A = Ce^{D-A_i} \). The converse can be verified easily by direct computation.

ii. The assertion is an immediate consequence of i.

iii. Assume first that \( f \) satisfies the proportional marginal rate of substitution property. Then from (5), (6) and (14) we derive \( x_i h_i = x_j h_j, \forall i \neq j \). Hence we conclude that there exists a nonzero real number \( k \) such that: \( x_i h_i = k, i = 1, \ldots, n \), and therefore we obtain

\[ h_i(x_i) = k \ln x_i + C_i, \quad i = 1, \ldots, n, \]

(21)

for some real constants \( C_1, \ldots, C_n \). Now, from (1) and (21) we derive

\[ f(x) = G \left( k \sum_{i=1}^{n} \ln x_i + \bar{A} \right), \]

where \( \bar{A} = \sum_{i=1}^{n} C_i \) and hence we find

\[ f(x) = (G \circ \ln) \left( A \cdot \prod_{i=1}^{n} x_i \right), \]

(22)

where \( A = e^{\bar{A}} \). Therefore we get a production function of the form (8), where \( F(u) = (G \circ \ln)(A \cdot u) \).

The converse is easy to verify.

iv1. We first prove the left-to-right implication. If the production hypersurface has null Gauss–Kronecker curvature, then from (11) we get

\[ \det(f_{x_i x_j}) = 0. \]

(23)

On the other hand, the determinant of the Hessian matrix of \( f \) is given by [6]

\[ \det(f_{x_i x_j}) = (G')^n \prod_{i=1}^{n} h_i'' + (G')^{n-1} G'' \sum_{i=1}^{n} h_i'' \cdot \ldots \cdot h_i' \cdot \ldots \cdot h_i'' \cdot h_i' \cdot \ldots \cdot h_i''. \]

(24)

By using (21), (23) and (24), we obtain

\[ (-1)^n (G')^{n-1} k^n (G' - k n G'') = 0. \]

But \( G' > 0 \) and \( k \neq 0 \) and hence we derive

\[ \frac{G''}{G'} = \frac{1}{k n} \]

(25)

After solving (25) we find

\[ G(u) = C n k e^{\frac{u}{k n}} + D \]

(26)

for some constants \( C, D \) with \( C > 0 \). Combining (22) and (26), after a suitable translation, we conclude that the function \( f \) reduces to the form (9). The converse follows easily by direct computation.

iv2. Let us assume that the production hypersurface is minimal. Then we have \( H = 0 \) and from (12) we derive

\[ \sum_{i=1}^{n} f_{x_i x_i} \left( 1 + \sum_{i=1}^{n} f_{x_i}^2 \right) - \sum_{i,j=1}^{n} f_{x_i x_j} f_{x_i x_j} = 0 \]

which reduces to

\[ \sum_{i=1}^{n} f_{x_i x_i} + \sum_{i \neq j} \left( f_{x_i} f_{x_j} x_j - f_{x_i} f_{x_j} x_j \right) = 0. \]

(27)

By introducing (14), (15) and (16) in (27), we get
\[ G'' \sum_{i=1}^{n} (h_i')^2 + G' \sum_{i=1}^{n} h_i'' + (G')^3 \sum_{i \neq j} (h_i')^2 h_j'' = 0. \]  

(28)

By using now (21) in (28) and taking into account that \( k \neq 0 \), we obtain

\[ (kG'' - G') \sum_{i=1}^{n} \frac{1}{x_i^2} - k^2(G')^3 \sum_{i \neq j} \frac{1}{x_i^2 x_j^2} = 0. \]

(29)

But the only solution to the equation (29) is \( G(u) = \text{constant} \), which is a contradiction because \( G' > 0 \). Hence the production hypersurface cannot be minimal.

iv3. Assume first that the production hypersurface has \( K_{ij} = 0 \). Then from (13) we get

\[ f_{x_i x_j} x_{x_j} - f_{x_i}^2 = 0. \]

(30)

By introducing (14), (15) and (16) into (30), since \( G' \neq 0 \), we obtain

\[ [(h_i')^2 h_j'' + (h_j')^2 h_i''] G'' + h_i'h_j'' G' = 0. \]

(31)

By using now (21) in (31) and taking into account that \( k \neq 0 \), we obtain

\[ \frac{G''}{G'} = \frac{1}{2k}. \]

(32)

After solving (32) we get

\[ G(u) = 2k C e^{\frac{u}{2k}} + D \]

(33)

for some constants \( C, D \) with \( C > 0 \). Finally, combining (22) and (33), after a suitable translation, we conclude that the function \( f \) reduces to the Cobb–Douglas production function given by (10). The converse is easy to verify by direct computation.

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