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Some characterizations of the quasi-sum production models with proportional marginal rate of substitution

Certaines caractérisations des modèles de production quasi-somme avec un taux marginal de substitution proportionnelle

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A R T I C L E I N F O

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This paper is dedicated to Prof. Ieronim Mihăilă on the occasion of his 79th birthday

A B S T R A C T

In this note we classify quasi-sum production functions with constant elasticity of production with respect to any factor of production and with proportional marginal rate of substitution.

R É S U M É

Dans cette note, nous classons les fonctions de production quasi-somme avec élasticité constante de la production par rapport à un facteur de production et avec un taux marginal de substitution proportionnel.

1. Introduction

The notion of *production function* is a key concept in both macroeconomics and microeconomics, being used in the mathematical modeling of the relationship between the output of a firm, an industry, or an entire economy, and the inputs that have been used in obtaining it. Generally, production function is a twice differentiable mapping $f : \mathbb{R}^n_+ \to \mathbb{R}_+$, $f = f(x_1, \dots, x_n)$, where f is the quantity of output, n is the number of the inputs and $x_1, \dots, x_n$ are the factor inputs. A production function f is called *quasi-sum* [3,5] if there are strict monotone functions $G, h_1, \dots, h_n$ with $G' > 0$ such that

$$f(x) = G(h_1(x_1) + \ldots + h_n(x_n)), \tag{1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$. We note that these functions are of great interest because they appear as solutions to the general bisymmetry equation, being related to the problem of consistent aggregation [1].

Among the family of production functions, the most famous is the so-called Cobb–Douglas production function. A generalized Cobb–Douglas production function depending on n-inputs is given by

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$$f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^{n} x_i^{\alpha_i}, \tag{2}$$

where $A, \alpha_1, \dots, \alpha_n > 0$. We recall that a production function of the form $f(x) = G(h(x_1, \dots, x_n))$, where G is a strictly increasing function and h is a homogeneous function of any given degree p, is said to be a *homothetic* production function [7]. It is easy to see that a production function f can be identified with the graph of f, *i.e.* the nonparametric hypersurface of $\mathbb{E}^{n+1}$ defined by

$$L(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n)) \tag{3}$$

and called the *production hypersurface* of f (see [9,11]). Motivated by some recent classification results concerning production hypersurfaces [2,5,7,8,12], in the present work we classify quasi-sum production functions with a proportional marginal rate of substitution and investigate the existence of such production models whose production hypersurfaces have null Gauss–Kronecker curvature or null mean curvature. We recall that, if f is a production function with n inputs $x_1, x_2, \dots, x_n$, $n \geq 2$, the *elasticity of production* with respect to a certain factor of production x_i is defined as

$$E_{x_i} = \frac{x_i}{f} f_{x_i} \tag{4}$$

and the *marginal rate of technical substitution* of input x_j for input x_i is given by

$$\mathrm{MRS}_{ij} = \frac{f_{x_j}}{f_{x_i}}, \tag{5}$$

where the subscripts denote partial derivatives of the function f with respect to the corresponding variables. A production function satisfies *the proportional marginal rate of substitution property* if

$$\mathrm{MRS}_{ij} = \frac{x_i}{x_j}, \text{ for all } 1 \leq i \neq j \leq n. \tag{6}$$

In the last section of the paper we will prove the following theorem that generalizes the results from [10].

Theorem 1.1. *Let f be a quasi-sum production function given by (1). Then:*

i. *The elasticity of production is a constant k_i with respect to a certain factor of production x_i if and only if f reduces to*

$$f(x_1, \dots, x_n) = A \cdot x_i^{k_i} \cdot \exp\left(D \sum_{j \neq i} h_j(x_j)\right), \tag{7}$$

where A and D are positive constants.

ii. *The elasticity of production is a constant k_i with respect to all factors of production x_i, $i = 1, \dots, n$, if and only if f reduces to the generalized Cobb–Douglas production function given by (2).*

iii. *The production function satisfies the proportional marginal rate of substitution property if and only if it reduces to the homothetic generalized Cobb–Douglas production function given by*

$$f(x_1, \dots, x_n) = F\left(\prod_{i=1}^{n} x_i^k\right), \tag{8}$$

where k is a nonzero real number.

iv. *If the production function satisfies the proportional marginal rate of substitution property, then:*

iv_1. *The production hypersurface has vanishing Gauss–Kronecker curvature if and only if, up to a suitable translation, f reduces to the following generalized Cobb–Douglas production function with constant return to scale:*

$$f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^{n} x_i^{\frac{1}{n}}. \tag{9}$$

iv_2. *The production hypersurface cannot be minimal.*

iv_3. *The production hypersurface has vanishing sectional curvature if and only if, up to a suitable translation, f reduces to the following generalized Cobb–Douglas production function:*

$$f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^{n} \sqrt{x_i}. \tag{10}$$

2. Preliminaries on the geometry of hypersurfaces

For general references on the geometry of hypersurfaces, we refer to [4].

If M is a hypersurface of the Euclidean space $\mathbb{E}^{n+1}$, then it is known that the *Gauss map* $\nu : M \to S^n$ maps M to the unit hypersphere S^n of $\mathbb{E}^{n+1}$. With the help of the differential $d\nu$ of ν it can be defined a linear operator on the tangent space T_pM, denoted by S_p and known as the *shape operator*, by $g(S_p v, w) = g(d\nu(v), w)$, for $v, w \in T_pM$, where g is the metric tensor on M induced from the Euclidean metric on $\mathbb{E}^{n+1}$. The eigenvalues of the shape operator are called *principal curvatures*. The determinant of the shape operator S_p, denoted by $K(p)$, is called the *Gauss–Kronecker curvature*. When $n = 2$, the Gauss–Kronecker curvature is simply called the *Gauss curvature*, which is intrinsic due to famous Gauss's Theorem Egregium. The trace of the shape operator S_p is called the *mean curvature* of the hypersurfaces. In contrast to the Gauss–Kronecker curvature, the mean curvature is extrinsic, which depends on the immersion of the hypersurface. A hypersurface is said to be *minimal* if its mean curvature vanishes identically. We recall now the following lemma which will be used in the proof of Theorem 1.1.

Lemma 2.1. *(See [4].) For the production hypersurface defined by (3) and* $w = \sqrt{1 + \sum_{i=1}^{n} f_i^2}$, *we have:*

i. *The Gauss–Kronecker curvature K is given by*

$$K = \frac{\det(f_{x_i x_j})}{w^{n+2}}. \tag{11}$$

ii. *The mean curvature H is given by*

$$H = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{f_{x_i}}{w} \right). \tag{12}$$

iii. *The sectional curvature K_{ij} of the plane section spanned by $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial x_j}$ is*

$$K_{ij} = \frac{f_{x_i x_i} f_{x_j x_j} - f_{x_i x_j}^2}{w^2 \left(1 + f_{x_i}^2 + f_{x_j}^2\right)}. \tag{13}$$

3. Proof of Theorem 1.1

Let f be a quasi-sum production function given by (1). Then we have

$$f_{x_i}(x) = G'(u) h_i'(x_i) \tag{14}$$

with $u = h_1(x_1) + \ldots + h_n(x_n)$ and from (14) we derive

$$f_{x_i x_i} = G''(h_i')^2 + G'h_i'', \ i = 1, \ldots, n, \tag{15}$$

$$f_{x_i x_j} = G'' h_i' h_j', \ i \neq j. \tag{16}$$

i. We first prove the left-to-right implication. If the elasticity of production is a constant k_i with respect to a certain factor of production x_i, then from (4) we obtain

$$f_{x_i} = k_i \frac{f}{x_i}. \tag{17}$$

Using now (1) and (14) in (17) we get

$$\frac{G'}{G} = k_i \frac{1}{x_i h_i'}. \tag{18}$$

By taking the partial derivative of (18) with respect to x_j, $j \neq i$, we obtain

$$h_j' \frac{G''G - (G')^2}{G^2} = 0.$$

Now, taking into account that h_j is a strict monotone function, we find

$$G(u) = C \cdot e^{Du}, \tag{19}$$

for some positive constants C and D. Hence from (18) and (19) we obtain

$$h_i(x_i) = \frac{k_i}{D} \ln x_i + A_i, \tag{20}$$

where A_i is a real constant. Finally, combining (1), (19) and (20) we get a function of the form (7), where $A = Ce^{D \cdot A_i}$. The converse can be verified easily by direct computation.

ii. The assertion is an immediate consequence of i.

iii. Assume first that f satisfies the proportional marginal rate of substitution property. Then from (5), (6) and (14) we derive $x_i h_i' = x_j h_j'$, $\forall i \neq j$. Hence we conclude that there exists a nonzero real number k such that: $x_i h_i' = k$, $i = 1, \ldots, n$, and therefore we obtain

$$h_i(x_i) = k \ln x_i + C_i, \; i = 1, \ldots, n, \tag{21}$$

for some real constants $C_1, \ldots, C_n$. Now, from (1) and (21) we derive

$$f(x) = G\left(k \sum_{i=1}^{n} \ln x_i + \overline{A}\right),$$

where $\overline{A} = \sum_{i=1}^{n} C_i$ and hence we find

$$f(x) = (G \circ \ln)\left(A \cdot \prod_{i=1}^{n} x_i^k\right), \tag{22}$$

where $A = e^{\overline{A}}$. Therefore we get a production function of the form (8), where $F(u) = (G \circ \ln)(A \cdot u)$.

The converse is easy to verify.

iv$_1$. We first prove the left-to-right implication. If the production hypersurface has null Gauss–Kronecker curvature, then from (11) we get

$$\det(f_{x_i x_j}) = 0. \tag{23}$$

On the other hand, the determinant of the Hessian matrix of f is given by [6]

$$\det(f_{x_i x_j}) = (G')^n \prod_{i=1}^{n} h_i'' + (G')^{n-1} G'' \sum_{i=1}^{n} h_1'' \cdot \ldots \cdot h_{i-1}''(h_i')^2 h_{i+1}'' \cdot \ldots \cdot h_n''. \tag{24}$$

By using (21), (23) and (24), we obtain

$$(-1)^n (G')^{n-1} k^n (G' - knG'') = 0.$$

But $G' > 0$ and $k \neq 0$ and hence we derive

$$\frac{G''}{G'} = \frac{1}{kn}. \tag{25}$$

After solving (25) we find

$$G(u) = Cnke^{\frac{u}{nk}} + D \tag{26}$$

for some constants C, D with $C > 0$. Combining (22) and (26), after a suitable translation, we conclude that the function f reduces to the form (9). The converse follows easily by direct computation.

iv$_2$. Let us assume that the production hypersurface is minimal. Then we have $H = 0$ and from (12) we derive

$$\sum_{i=1}^{n} f_{x_i x_i}\left(1 + \sum_{i=1}^{n} f_{x_i}^2\right) - \sum_{i,j=1}^{n} f_{x_i} f_{x_j} f_{x_i x_j} = 0$$

which reduces to

$$\sum_{i=1}^{n} f_{x_i x_i} + \sum_{i \neq j}\left(f_{x_i}^2 f_{x_j x_j} - f_{x_i} f_{x_j} f_{x_i x_j}\right) = 0. \tag{27}$$

By introducing (14), (15) and (16) in (27), we get

$$G'' \sum_{i=1}^{n} (h_i')^2 + G' \sum_{i=1}^{n} h_i'' + (G')^3 \sum_{i \neq j} (h_i')^2 h_j'' = 0. \tag{28}$$

By using now (21) in (28) and taking into account that $k \neq 0$, we obtain

$$(kG'' - G') \sum_{i=1}^{n} \frac{1}{x_i^2} - k^2 (G')^3 \sum_{i \neq j} \frac{1}{x_i^2 x_j^2} = 0. \tag{29}$$

But the only solution to the equation (29) is $G(u) = \textit{constant}$, which is a contradiction because $G' > 0$. Hence the production hypersurface cannot be minimal.

iv$_3$. Assume first that the production hypersurface has $K_{ij} = 0$. Then from (13) we get

$$f_{x_i x_i} f_{x_j x_j} - f_{x_i x_j}^2 = 0. \tag{30}$$

By introducing (14), (15) and (16) into (30), since $G' \neq 0$, we obtain

$$[(h_i')^2 h_j'' + (h_j')^2 h_i''] G'' + h_i'' h_j'' G' = 0. \tag{31}$$

By using now (21) in (31) and taking into account that $k \neq 0$, we obtain

$$\frac{G''}{G'} = \frac{1}{2k}. \tag{32}$$

After solving (32) we get

$$G(u) = 2kCe^{\frac{u}{2k}} + D \tag{33}$$

for some constants C, D with $C > 0$. Finally, combining (22) and (33), after a suitable translation, we conclude that the function f reduces to the Cobb–Douglas production function given by (10). The converse is easy to verify by direct computation.

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