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Number theory

Complex Pisot numbers in algebraic number fields



Nombres de Pisot complexes dans des corps de nombres algébriques

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ABSTRACT

Let P(K) be the set of Pisot numbers generating a real algebraic number field K over the field of rationals \mathbb{Q} . Then, a result of Meyer implies that P(K) is relatively dense in the interval $[1, \infty)$ and a theorem of Pisot gives that P(K) contains units, whenever $K \neq \mathbb{Q}$. In the present note, we prove analogous results for the set of complex Pisot numbers generating a non-real number field K' over \mathbb{Q} when K' is neither a quadratic field nor a CM-field.

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RÉSUMÉ

Soient *K* un corps réel de nombres algébriques et P(K) l'ensemble des nombres de Pisot engendrant *K* sur le corps des rationnels \mathbb{Q} . Un résultat, dû à Meyer, montre que P(K) est relativement dense dans l'intervalle $[1, \infty)$, et un théorème de Pisot établit que l'ensemble P(K) contient des unités lorsque $K \neq \mathbb{Q}$. On considère, dans cette note, un corps non réel K' de nombres algébriques, et l'on obtient des résultats similaires aux précédents pour l'ensemble des nombres de Pisot complexes engendrant K' sur \mathbb{Q} lorsque K' n'est, ni un corps quadratique, ni un corps CM.

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1. Introduction

A Pisot number is a real algebraic integer greater than 1 whose other conjugates are of modulus less than 1, and the set of such numbers is usually denoted by S. Pisot, who was the first to investigate the family

$$P(K) := \{ \pm \theta \mid \theta \in \mathbb{S} \cap K, \ K = \mathbb{Q}(\theta) \},\$$

where *K* is an algebraic number field (or simply a number field), which is contained in \mathbb{R} and he showed, in particular, that P(K) contains units (see for instance [2,4,13]), whenever $K \neq \mathbb{Q}$. Fan and Schmeling [6] have proved that the set P(K) is relatively dense in \mathbb{R} . Using some theorems, due to Meyer, on harmonious sets [11,12,16], the second author pointed out, in [17], that P(K) is a real Meyer set.

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Recall, generally, that a subset *P* of the Euclidean space \mathbb{R}^d is said to be a Delone set if there are two positive numbers *r* and *R* such that every ball, in \mathbb{R}^d , with radius *r*, contains at most one element of *P* (i.e., *P* is uniformly discrete), and every ball, in \mathbb{R}^d , with radius *R* contains at least one element of *P* (i.e., *P* is relatively dense). A Meyer set, in \mathbb{R}^d , is a Delone set *P* in \mathbb{R}^d satisfying the relation $P - P \subset P + F$, for some finite set $F \subset \mathbb{R}^d$ [10]. Given a full-rank lattice *L* of \mathbb{R}^d , we obtain a model set (or a cut-and-project set [10]) by projecting onto $x_a \in \mathbb{R}^a$ those points $(x_a, x_b) \in L$, where a + b = d and x_b belongs to an open bounded subset *W* of \mathbb{R}^b . Notice also that a result of Meyer yields that a model set is a Meyer set [10,11].

The aim of this note is to extend the two above-mentioned results on Pisot numbers to complex Pisot numbers. A complex Pisot number is a non-real algebraic integer with modulus greater than 1, whose other conjugates, except its complex conjugate, are of modulus less than 1. Although certain families of complex Pisot numbers have been considered in various questions (see for instance [8,14] and [15]), the set, say S_c , of complex Pisot numbers has been little studied. In fact, the only three papers we know that discuss complex Pisot numbers explicitly are due to Kelly [9], generalizing Salem's beautiful result that the set of Pisot numbers is closed to prove that $S \cup S_c$ is also closed and to Chamfy [5] and Garth [7], giving some results on small complex Pisot numbers. To state our results, set

$$P_{c}(K) := \{ \theta \mid \theta \in \mathbb{S}_{c} \cap K, \ K = \mathbb{Q}(\theta) \},\$$

where *K* is a non-real number field. As usual, Ω_K denotes the group of roots of unity belonging to *K* and Z_K the ring of integers of *K*. Recall also that *K* is said to be a CM-field if it is a totally non-real quadratic extension of a totally real number field (which is unique), say R_K .

Theorem 1.1. Let *K* be a non-real number field. Then, $P_c(K)$ is a complex Meyer set. Moreover, if *K* is neither quadratic, nor a CM-field with $\Omega_K = \{\pm 1\}$, then there are units in $P_c(K)$.

It is clear when $K = \mathbb{Q}(\sqrt{D})$ and D is a square-free negative rational integer (i.e., when K is a non-real quadratic field), that the ring \mathbb{Z}_K is a complex Meyer set, the set Ω_K is finite $(\Omega_{\mathbb{Q}(\sqrt{-1})} = \{\pm 1, \pm \sqrt{-1}\}, \Omega_{\mathbb{Q}(\sqrt{-3})} = \{\pm 1, \pm (1 \pm \sqrt{-3})/2\}$ and $\Omega_{\mathbb{Q}(\sqrt{D})} = \{\pm 1\}$ for $D \notin \{-1, -3\}$), and $P_c(K) = \mathbb{Z}_K \setminus (\Omega_K \cup \{0\})$; thus there is no unit in $P_c(K)$, since there is no unit in K others than roots of unity. Next, we characterize complex Pisot units generating a CM-field K in terms of the elements of $P(R_K)$.

Theorem 1.2. The following assertions hold for a CM-field K.

- (1) Any element of $P_c(K)$ is of the form $\omega\sqrt{\alpha}$, where $\omega \neq \pm 1$, $\omega^2 \in \Omega_K$ and $\alpha \in P(R_K)$ is a totally positive unit.
- (2) If $\beta \in P(R_K)$ and $\zeta \in \Omega_K \setminus \{-1, 1\}$, then $\zeta \beta \in P_c(K)$.
- (3) Suppose $\Omega_K = \pm 1$. Then $P_c(K)$ contains units only when $K = \mathbb{Q}(\sqrt{-\alpha})$, $\alpha \in P(R_K)$ is a totally positive unit and $\sqrt{\alpha} \notin R_K$. Moreover, in this last case, the elements of $P_c(K)$ are the numbers of the form $r\sqrt{-\alpha}$, where $r \in R_K$ and $r^2\alpha \in P(R_K)$ is a totally positive unit.

Using Pari calculator [1], we obtain, for instance, that the quartic field $K := \mathbb{Q}(\sqrt{2}, \sqrt{-5})$ is a CM-field such that $\Omega_K = \{\pm 1\}$ and $R_K = \mathbb{Q}(\sqrt{2})$. Since the Pisot number $1 + \sqrt{2}$ is the unique fundamental unit, greater than 1 in R_K , we have that every totally positive unit belonging to $P(R_K)$ is a square of an element of R_K ($1 + \sqrt{2}$ has norm -1). It follows by Theorem 1.2(3) that there is no unit in $P_c(K)$. In fact, Theorem 1.2(3) has been recently used, in [3], to characterize number fields without unit primitive element.

The first proposition in Theorem 1.1 follows easily from the above-mentioned result of Meyer. To show Theorem 1.2 and the second part of Theorem 1.1, we use, respectively, Kronecker's classical theorem and Dirichlet's units theorem. Notice also, that we may define, similarly as in [6] and in [16], for any $0 < \varepsilon \le 1$ the set, say $\mathbb{S}_{c,\varepsilon}$, of ε -complex Pisot numbers, which consists of those non-real algebraic integers whose other conjugates, except their complex conjugates, are of modulus less than ε , and in a same way as in the proof below, we can show that the set { $\theta \in \mathbb{S}_{c,\varepsilon} \cap K \mid K = \mathbb{Q}(\theta)$ }, where K is a non-real number field, is also a complex Meyer set.

2. The proofs

2.1. Proof of Theorem 1.1

Proof. Let $\sigma_1, \ldots, \sigma_{2s+r}$ be the distinct embeddings of the non-real number field K into \mathbb{C} , where $\sigma_1, \ldots, \sigma_{2s}$ are non-real, σ_1 is the identity of K, $\sigma_{2s+1}, \ldots, \sigma_{2s+r}$ are real, and $\sigma_{j+s}(\alpha) = \overline{\sigma_j(\alpha)}$, $\forall j \in \{1, \ldots, s\}$, $\forall \alpha \in K$. Set d = 2s + r, $W = \{z \in \mathbb{C} \mid |z| < 1\}^{s-1} \times \{x \in \mathbb{R} \mid |x| < 1\}^r$ and L the image of the ring \mathbb{Z}_K in \mathbb{R}^d by the standard embedding. Then, L is a full-rank lattice of \mathbb{R}^d . By identifying \mathbb{C} as \mathbb{R}^2 and projecting onto $x_2 \in \mathbb{R}^2$ those points $(x_2, x_{d-2}) \in L$, where $x_{d-2} \in W$, we obtain that the set

$$M = \left\{ \theta \in \mathbb{Z}_K \mid \left| \sigma_j(\theta) \right| < 1, \forall j \in \{2, \dots, d\} \setminus \{1 + s\} \right\}$$

is a complex model set. Hence, M is a complex Meyer set, and so is $P_c(K)$, since

$$M \setminus P_c(K) \subset (-\mathbb{S}) \cup \mathbb{S} \cup \{0\} \subset \mathbb{R}.$$

To prove the second assertion in Theorem 1.1, consider the restriction of the logarithmic representation l of K into \mathbb{R}^{s+r} to the group of units U_K of K sending any $u \in U_K$ to

$$l(u) = (\log |\sigma_1(u)|^2, \dots, \log |\sigma_s(u)|^2, \log |\sigma_{2s+1}(u)|, \dots, \log |\sigma_{2s+r}(u)|).$$

Then, the lattice $l(U_K)$ is contained in the hyperplane H: $x_1 + x_2 + \cdots + x_{s+r} = 0$ of \mathbb{R}^{r+s} . Since $(r + s - 1, -1, -1, \dots, -1) \in H$, the set H has a non-empty intersection with the quadrant Q: $x_1 > 0, x_2 < 0, x_3 < 0, \dots, x_{s+r} < 0$, and because $l(U_K)$ has, by Dirichlet's theorem, full dimension r + s - 1 in H, there are points of $l(U_K)$ in Q. Hence, there is an element, say θ belonging to $P(K) \cap U_K$ or to $P(R_K) \cap U_K$ and we conclude by Theorem 1.2(2) when $\theta \notin P(K)$. \Box

2.2. Proof of Theorem 1.2

Proof. Let *K* be a CM-field and let $\theta = \theta_1, \ldots, \theta_d$ be the conjugates of an element $\theta \in P_c(K)$. Because *K* is closed for the complex conjugation and the complex conjugation commutes with each embedding of *K* into \mathbb{C} , we have that the corresponding conjugates of $\theta\overline{\theta}$ (resp. of $\theta + \overline{\theta}$) are the numbers $\theta_j\overline{\theta_j}$, (resp. the numbers $\theta_j + \overline{\theta_j}$), where $j \in \{1, \ldots, d\}$. Hence, $\alpha := |\theta|^2 \in K$ is a totally positive Pisot unit, and the conjugates of the totally real algebraic integer $(\theta + \overline{\theta})/\sqrt{\theta\overline{\theta}}$ lie to the interval [-2, 2]. It follows by Kronecker's theorem that the number $(\theta + \overline{\theta})/|\theta|$ is of the form $\omega + \overline{\omega}$, where ω is a root of unity, and so the minimal polynomial of θ over R_K is

$$(x-\theta)(x-\overline{\theta}) = x^2 - (\theta+\overline{\theta})x + \theta\overline{\theta} = x^2 - (\omega+\overline{\omega})|\theta|x+|\theta|^2 = (x-\omega|\theta|)(x-\overline{\omega}|\theta|).$$

Hence, $\theta = \omega \sqrt{\alpha}$ (by replacing if necessary ω by $\overline{\omega}$), $\omega \neq \pm 1$, since otherwise $\theta = \pm \sqrt{\alpha} \in \mathbb{R}$, $\alpha \in P(R_K)$, $\omega^2 = \theta^2/\alpha \in K$ and so Theorem 1.2(1) holds.

It is also clear when $\beta \in P(R_K)$ and $\zeta \in \Omega_K \setminus \{-1, 1\}$ that $R_K \subsetneq R_K(\zeta) \subseteq K$, $R_K(\zeta) = K = \mathbb{Q}(\beta\zeta)$ and $\beta\zeta$ is a complex Pisot unit; thus $\zeta\beta \in P_c(K)$ and Theorem 1.2(2) is true.

Now suppose $\Omega_K = \{\pm 1\}$. Then the equality $\omega^2 = \pm 1$ yields $\omega \in \{i, -i\}$, say $\omega = i$ and so $\theta = i |\theta| = \sqrt{-\alpha}$. Hence, if $\theta' \in P_c(K)$, then $\theta' = \sqrt{-\alpha'}$ for some totally positive unit $\alpha' \in P(R_K)$, $r := \theta'/\theta \in R_K$ is a unit, $r^2\alpha = \alpha'$, and $\sqrt{\alpha} \notin K$, since otherwise $i = \theta/\sqrt{\alpha} \in \Omega_K$ (and $i^2 = -1$). To end the proof of Theorem 1.2(3), notice that, when $K' := \mathbb{Q}(\sqrt{-\gamma})$ for some totally positive Pisot unit γ , $\gamma \in K'$, $K' = \mathbb{Q}(\gamma)(\sqrt{-\gamma})$ is a totally non-real quadratic extension of the totally real field $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\beta) = R_{K'}$. \Box

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