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Algebraic geometry

A symplectic analog of the Quot scheme

Un analogue symplectique du schéma Quot

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ABSTRACT

We construct a symplectic analog of the Quot scheme that parameterizes the torsion quotients of a trivial vector bundle over a compact Riemann surface. Some of its properties are investigated.

RÉSUMÉ

Nous construisons un analogue symplectique du schéma Quot, qui paramètre les modules quotients de torsion d'un fibré vectoriel trivial sur une surface de Riemann compacte, et nous examinons certaines de ses propriétés.

1. Introduction

Let X be a compact connected Riemann surface. For fixed integers n and d, let $\mathcal{Q}(\mathcal{O}_X^{\oplus n}, d)$ denote the Quot scheme that parameterizes the torsion quotients of $\mathcal{O}_X^{\oplus n}$ of degree d (see [8] for construction and properties of a general Quot schemes). This particular Quot scheme $\mathcal{Q}(\mathcal{O}_X^{\oplus n}, d)$ arises in the study of moduli space of vector bundles of rank n on X [4,7,3]. Being a moduli space of vortices, it is also studied in mathematical physics (see [1] and references therein).

Here we consider a symplectic analog of the Quot scheme. Let $\mathcal{O}_X^{\oplus 2r}$ be the trivial vector bundle on X equipped with a symplectic structure given by the standard symplectic form on $\mathbb{C}^{2r}$. Take torsion quotients of it of degree dr that are compatible with the symplectic structure (this is explained in Section 2.1). Fixing X, r and d, let $\mathcal{Q}$ denote the associated symplectic Quot scheme. The projective symplectic group $\mathrm{PSp}(2r, \mathbb{C})$ has a natural action on $\mathcal{Q}$. We show that the connected component, containing the identity element, of the group of all automorphisms of $\mathcal{Q}$ coincides with $\mathrm{PSp}(2r, \mathbb{C})$.

In [4], Bifet, Ghione and Letizia used the usual Quot schemes $\mathcal{Q}(\mathcal{O}_X^{\oplus n}, d)$ associated with X to compute the cohomologies of the moduli spaces of semistable vector bundles of rank n on X. Our hope is to be able to compute the cohomologies of the moduli space of semistable $\mathrm{Sp}(2r, \mathbb{C})$-bundles on X using the symplectic Quot scheme $\mathcal{Q}$.

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2. The symplectic Quot scheme

2.1. Construction of the symplectic Quot scheme

Let

$$\omega' := \sum_{i=1}^{r} (e_i^* \otimes e_{i+r}^* - e_{i+r}^* \otimes e_i^*)$$

be the standard symplectic form on $\mathbb{C}^{2r}$. Let X be a compact connected Riemann surface. The sheaf of holomorphic functions on X will be denoted by $\mathcal{O}_X$. Let

$$E_0 := \mathcal{O}_X^{\oplus 2r}$$

be the trivial holomorphic vector bundle on X of rank $2r$. The above symplectic form ω' defines a symplectic structure on E_0, because the fibers of E_0 are identified with $\mathbb{C}^{2r}$. This symplectic structure on E_0 will be denoted by ω_0.

Fix an integer $d \geq 1$. Let

$$\mathcal{Q} := \mathcal{Q}(\omega_0, d) \tag{1}$$

be the *symplectic Quot scheme* parameterizing all torsion quotients

$$\tau_Q : E_0 \longrightarrow Q$$

of degree dr satisfying the following condition: there is an effective divisor D_Q on X of degree d such that the restricted form $\omega_0|_{\text{kernel}(\tau_Q)}$ factors as

$$\text{kernel}(\tau_Q) \otimes \text{kernel}(\tau_Q) \xrightarrow{\omega_0} \mathcal{O}_X(-D_Q) \hookrightarrow \mathcal{O}_X\,; \tag{2}$$

it should be clarified that D_Q depends on Q. Equivalently, the symplectic Quot scheme $\mathcal{Q}$ parameterizes all coherent analytic subsheaves $F \subset E_0$ of rank $2r$ and degree $-dr$ such that the form $\omega_0|_F$ factors as

$$F \otimes F \xrightarrow{\omega_0} \mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X\,,$$

where D is some effective divisor on X of degree d that depends on F. This description of $\mathcal{Q}$ sends any subsheaf F to the quotient sheaf E_0/F.

The above pairing

$$F \otimes F \xrightarrow{\omega_0} \mathcal{O}_X(-D)$$

produces an injective homomorphism

$$\mu : F \longrightarrow \mathcal{O}_X(-D) \otimes F^* \tag{3}$$

between coherent analytic sheaves of rank $2r$; the homomorphism μ is injective because it is injective over the complement of the support of E_0/F. Since

$$\text{degree}(F) = -dr = \text{degree}(\mathcal{O}_X(-D) \otimes F^*)\,,$$

the homomorphism μ in (3) is an isomorphism. This means that the pairing $F_x \otimes F_x \xrightarrow{\omega_0(x)} \mathcal{O}_X(-D)_x$ is nondegenerate for every $x \in X$. The divisor D is uniquely determined by F because μ is an isomorphism. More precisely, consider the homomorphism of coherent analytic sheaves

$$\widetilde{\mu} : F \longrightarrow F^*$$

given by the restriction $\omega_0|_F$. The divisor D is the scheme theoretic support of the quotient sheaf $F^*/\widetilde{\mu}(F)$.

The group of all permutations of $\{1, \cdots, d\}$ will be denoted by S_d. The quotient

$$X^d/S_d \tag{4}$$

of X^d for the natural action of S_d is the symmetric product $\text{Sym}^d(X)$. Let

$$\varphi : \mathcal{Q} \longrightarrow \text{Sym}^d(X) \tag{5}$$

be the morphism that sends any quotient Q to the divisor D_Q (see (2)).

Let

$$\widetilde{\mathcal{Q}} := \mathcal{Q}(\mathcal{O}_X^{\oplus 2r}, rd) \tag{6}$$

be the Quot scheme that parameterizes all torsion quotients of $\mathcal{O}_X^{\oplus 2r} = E_0$ of degree rd. It is an irreducible smooth complex projective variety of dimension $2r^2d$. For any subsheaf F of E_0 with $E_0/F \in \widetilde{\mathcal{Q}}$, we have

$$T_{E_0/F}\widetilde{\mathcal{Q}} = H^0(X,\ (E_0/F) \otimes F^*)\,. \tag{7}$$

Now assume that $E_0/F \in \mathcal{Q}$. First consider the homomorphism $F \otimes E_0 \xrightarrow{\omega_0} \mathcal{O}_X$ obtained by restricting ω_0. Note that $\omega_0(F \otimes F) \subset \mathcal{O}_X(-\varphi(E_0/F))$, where φ is the morphism in (5) (see (2)). Therefore, ω_0 produces a homomorphism

$$\theta : F \otimes (E_0/F) \longrightarrow \mathcal{O}_X/(\mathcal{O}_X(-\varphi(E_0/F)))\,.$$

The subspace

$$T_{E_0/F}\mathcal{Q} \subset T_{E_0/F}\widetilde{\mathcal{Q}}$$

consists of all homomorphisms $\alpha : F \longrightarrow E_0/F$ (see (7)) such that

$$\theta(v \otimes \alpha(w)) = \theta(w \otimes \alpha(v))\,.$$

Clearly $\mathcal{Q}$ is a closed subscheme of $\widetilde{\mathcal{Q}}$.

Lemma 1. *The scheme $\mathcal{Q}$ is an irreducible projective variety of dimension $d(r^2 + r + 2)/2$.*

Proof. The morphism φ in (5) is clearly surjective. Also, each fiber of φ is irreducible. Since $\mathrm{Sym}^d(X)$ is irreducible, we conclude that $\mathcal{Q}$ is also irreducible.

Take a point $\widehat{x} := \{x_1 \cdots, x_d\} \in \mathrm{Sym}^d(X)$ such that all $x_i \in X$, $1 \le i \le d$, are distinct. Let

$$\mathbb{L} \subset \mathrm{Gr}(\mathbb{C}^{2r}, r) \tag{8}$$

be the variety parameterizing all Lagrangian subspaces of $\mathbb{C}^{2r}$ for the standard symplectic form ω'. We have a map $\mathbb{L}^d \longrightarrow \varphi^{-1}(\widehat{x})$ that sends any $(V_1, \cdots, V_d) \in \mathbb{L}^d$ to the composition

$$\mathcal{O}_X^{2r} \longrightarrow \mathcal{O}_X^{2r}|_{x_1\cup\cdots\cup x_d} = \bigoplus_{i=1}^{d} \mathbb{C}^{2r}_{x_i} \longrightarrow \bigoplus_{i=1}^{d} \mathbb{C}^{2r}_{x_i}/V_i\,,$$

where $\mathbb{C}^{2r}_{x_i}$ is the sheaf supported at x_i with stalk $\mathbb{C}^{2r}$; note that this composition is surjective and hence it defines an element of $\varphi^{-1}(\widehat{x})$. This map $\mathbb{L}^d \longrightarrow \varphi^{-1}(\widehat{x})$ is clearly an isomorphism. Since $\dim \mathbb{L} = r(r+1)/2$, it follows that $\dim \mathcal{Q} = d(r^2 + r + 2)/2$. $\square$

2.2. Vortex equation and stability

Fix a Hermitian metric ω_X on X; note that ω_X is Kähler.

Take a subsheaf $F \subset \mathcal{O}_X^{\oplus 2r}$ such that the quotient $\mathcal{O}_X^{\oplus 2r}/F$ is a torsion sheaf of degree dr, so $F \in \widetilde{\mathcal{Q}}$ (see (6)). Let

$$\mathcal{O}_X^{\oplus 2r} \hookrightarrow F^* \tag{9}$$

be the dual of the inclusion of F in $\mathcal{O}_X^{\oplus 2r}$. Note that the homomorphism in (9) defines a $2r$-pair of rank $2r$ [2, p. 535 (3.1)]. Therefore, each element of $\mathcal{Q}$ defines a $2r$-pair of rank $2r$.

Take any real number τ, and take an element $z_F \in \mathcal{Q}$. Let $F \subset \mathcal{O}_X^{\oplus 2r}$ be the subsheaf represented by z_F. We note that the subsheaf F is τ-stable if $rd < \tau$ (see [2, p. 535, Definition 3.3] for the definition of τ-stability).

We assume that $\tau > rd$. Therefore, all elements of $\mathcal{Q}$ are τ-stable. Hence every $F \subset \mathcal{O}_X^{\oplus 2r}$ lying in $\mathcal{Q}$ admits a unique Hermitian structure that satisfies the $2r$–τ-vortex equation [2, p. 536, Theorem 3.5].

Take any $F \subset \mathcal{O}_X^{\oplus 2r}$ represented by an element of $\mathcal{Q}$. Let h_F denote the unique Hermitian structure on F that satisfies the $2r$–τ-vortex equation. The isomorphism μ in (3) takes h_F to the unique Hermitian structure on $\mathcal{O}_X(-D) \otimes F^*$ that satisfies the $2r$–τ-vortex equation for $\mathcal{O}_X(-D) \otimes F^* \in \mathcal{Q}$. Indeed, this follows immediately from the uniqueness of the Hermitian structure satisfying the $2r$–τ-vortex equation.

3. Automorphisms of the symplectic Quot scheme

In this section we assume that $\mathrm{genus}(X) \ge 2$.

The group of all holomorphic automorphisms of $\mathcal{Q}$ will be denoted by $\mathrm{Aut}(\mathcal{Q})$. Let

$$\mathrm{Aut}(\mathcal{Q})^0 \subset \mathrm{Aut}(\mathcal{Q})$$

be the connected component of it containing the identity element. The group of linear automorphisms of the symplectic vector space $(\mathbb{C}^{2r}, \omega')$ is $\mathrm{Sp}(2r, \mathbb{C})$. The standard action of $\mathrm{Sp}(2r, \mathbb{C})$ on $\mathbb{C}^{2r}$ produces an action of $\mathrm{Sp}(2r, \mathbb{C})$ on $\mathcal{Q}$. The center $\mathbb{Z}/2\mathbb{Z}$ of $\mathrm{Sp}(2r, \mathbb{C})$ acts trivially on $\mathcal{Q}$. Therefore, we get a homomorphism

$$\rho : \mathrm{PSp}(2r, \mathbb{C}) = \mathrm{Sp}(2r, \mathbb{C})/(\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathrm{Aut}(\mathcal{Q})^0\,.$$

Theorem 2. *The above homomorphism ρ is an isomorphism.*

Proof. The standard action of $\mathrm{Sp}(2r, \mathbb{C})$ on $\mathbb{C}^{2r}$ produces an action of $\mathrm{PSp}(2r, \mathbb{C})$ on the variety $\mathbb{L}$ in (8). This action on $\mathbb{L}$ is effective. From this it follows immediately that the homomorphism ρ is injective (recall that the general fiber of φ is $\mathbb{L}^d$).

For each $1 \leq i \leq d$, let $p_i : X^d \longrightarrow X$ be the projection to the i-th factor of the Cartesian product. For each pair $1 \leq i < j \leq d$, let

$$\Delta_{i,j} \subset X^d$$

be the divisor over which the two maps p_i and p_j coincide. Let

$$\widetilde{U} := X^d \setminus \Big(\bigcup_{1 \leq i < j \leq d} \Delta_{i,j} \Big)$$

be the complement. Consider all point $\{x_1 \cdots, x_d\} \in \mathrm{Sym}^d(X)$ such that $x_k \neq x_\ell$ for all $k \neq \ell$. The complement in $\mathrm{Sym}^d(X)$ of the subset defined by such points will be denoted by U. The quotient map $X^d \longrightarrow X^d/S_d$ (see (4)) sends $\widetilde{U}$ to U. The quotient map

$$f : \widetilde{U} := X^d \setminus \Big(\bigcup_{1 \leq i < j \leq d} \Delta_{i,j} \Big) \longrightarrow U \tag{10}$$

is an étale Galois covering with Galois group S_d.

The inverse image $\varphi^{-1}(U) \subset \mathcal{Q}$ will be denoted by $\mathcal{V}$, where φ is the projection in (5). Let

$$\varphi' := \varphi|_{\mathcal{V}} : \mathcal{V} \longrightarrow U \tag{11}$$

be the restriction of φ. We note that $\mathcal{V}$ is a fiber bundle over U with fibers isomorphic to $\mathbb{L}^d$ (see (8) for $\mathbb{L}$). In particular, $\mathcal{V}$ is contained in the smooth locus of $\mathcal{Q}$.

The Lie algebra of $\mathrm{Sp}(2r, \mathbb{C})$ will be denoted by $\mathrm{sp}(2r, \mathbb{C})$. The Lie algebra of $\mathrm{Aut}(\mathcal{Q})^0$ is contained in the space of algebraic vector fields $H^0(\mathcal{V}, T\mathcal{V})$ equipped with the Lie bracket operation of vector fields. Let

$$d\rho : \mathrm{sp}(2r, \mathbb{C}) \longrightarrow H^0(\mathcal{V}, T\mathcal{V}) \tag{12}$$

be the homomorphism of Lie algebras associated with the homomorphism ρ of Lie groups. To prove that ρ is surjective, it suffices to show that $d\rho$ is surjective. We note that $d\rho$ is injective because ρ is injective.

Take any algebraic vector field

$$\gamma \in H^0(\mathcal{V}, T\mathcal{V}).$$

Let

$$d\varphi' : T\mathcal{V} \longrightarrow \varphi'^* TU$$

be the differential of the projection φ' in (11). As noted before, the fibers of φ' are isomorphic to $\mathbb{L}^d$, in particular, they are connected smooth projective varieties, so any holomorphic function on a fiber of φ' is a constant function. This implies that the section $d\varphi'(\gamma)$ descends to U. In other words, there is a holomorphic vector field γ' on U such that

$$d\varphi'(\gamma) = \varphi'^* \gamma'. \tag{13}$$

Let

$$\gamma'' := f^* \gamma' \in H^0(\widetilde{U}, T\widetilde{U}) \tag{14}$$

be the pullback, where f is the projection in (10). Since the vector field γ is algebraic, the above vector field γ'' is meromorphic on X^d, meaning

$$\gamma'' \in H^0\Big(X^d, (TX^d) \otimes \mathcal{O}_{X^d}\Big(\sum_{1 \leq i < j \leq d} m \cdot \Delta_{i,j} \Big)\Big)$$

for some integer m. It is known that there are no such nonzero sections [5, Proposition 2.3]. Therefore, we have $\gamma'' = 0$, and hence from (13) and (14) it follows that

$$d\varphi'(\gamma) = 0. \tag{15}$$

The standard action of $\mathrm{Sp}(2r, \mathbb{C})$ on $\mathbb{C}^{2r}$ produces an action of $\mathrm{Sp}(2r, \mathbb{C})$ on $\mathbb{L}$ defined in (8). Let

$$\mathrm{sp}(2r, \mathbb{C}) \longrightarrow H^0(\mathbb{L}, T\mathbb{L})$$

be the corresponding homomorphism of Lie algebras. It is known that the above homomorphism is an isomorphism. Let

$$T_{\rm rel} \longrightarrow \widetilde{U} \times_U \mathcal{V} \longrightarrow \widetilde{U}$$

be the relative algebraic tangent bundle for the projection $\widetilde{U} \times_U \mathcal{V} \longrightarrow \widetilde{U}$. Since

$$\widetilde{U} \times_U \mathcal{V} = \widetilde{U} \times \mathbb{L}^d ,$$

and $H^0(\widetilde{U}, \mathcal{O}_{\widetilde{U}}) = \mathbb{C}$ [5, Lemma 2.2], we have

$$H^0(\widetilde{U} \times_U \mathcal{V}, T_{\rm rel}) = H^0(\mathbb{L}, T\mathbb{L})^{\oplus d} = {\rm sp}(2r, \mathbb{C})^{\oplus d} . \tag{16}$$

Let

$$T\mathcal{V} \supset T_{\varphi'} \longrightarrow \mathcal{V}$$

be the relative algebraic tangent bundle for the projection φ'. Since f in (10) is a Galois étale covering with Galois group S_d, from (16) we have

$$H^0(\mathcal{V}, T_{\varphi'}) = H^0(\widetilde{U} \times_U \mathcal{V}, T_{\rm rel})^{S_d} = ({\rm sp}(2r, \mathbb{C})^{\oplus d})^{S_d} = {\rm sp}(2r, \mathbb{C}) .$$

Combining this with (15) we conclude that

$$H^0(\mathcal{V}, T\mathcal{V}) = {\rm sp}(2r, \mathbb{C}) .$$

In particular the homomorphism $d\rho$ in (12) is surjective. □

Corollary 3. *The isomorphism class of the variety $\mathcal{Q}$ uniquely determines the isomorphism class of X, except when $d = 2 = \text{genus}(X)$.*

Proof. This follows from Theorem 2 and [6]. The argument is similar to the proof of Theorem 5.1 in [5]. □

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