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Faber polynomial coefficient bounds for a subclass of bi-univalent functions

Bornes des coefficients des développements en polynômes de Faber d'une sous-classe de fonctions bi-univalentes

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ABSTRACT

In this work, considering a general subclass of bi-univalent functions and using the Faber polynomials, we obtain coefficient expansions for functions in this class. In certain cases, our estimates improve some of those existing coefficient bounds.

RÉSUMÉ

Dans cet article, on considère une sous-classe de fonctions bi-univalentes ; en utilisant les développements en polynômes de Faber, on obtient les coefficients de ces développements pour les fonctions de la sous-classe considérée. Dans certains cas, les estimations sur les bornes des coefficients améliorent des résultats déjà connus.

1. Introduction

Let A denote the class of functions f that are analytic in the open unit disk $U = \{z : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let S be the subclass of A consisting of functions f that are also univalent in U, and let P be the class of functions

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n$$

that are analytic in U and satisfy the condition $\mathrm{Re}(\varphi(z)) > 0$ in U. By the Caratheodory's lemma (see, e.g., [11]) we have $|\varphi_n| \le 2$.

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Let $f \in A$. We define the differential operator D^p, $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1, 2, \ldots\}$, by (see [24])

$$D^0 f(z) = f(z);$$
$$D^1 f(z) = Df(z) = zf'(z);$$
$$\vdots$$
$$D^p f(z) = D^1\left(D^{p-1} f(z)\right).$$

The Koebe one-quarter theorem [11] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus, every such univalent function has an inverse f^{-1} that satisfies

$$f^{-1}(f(z)) = z \ (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots.$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. For a brief history and interesting examples in the class Σ, see [27].

Lewin [19] studied the class of bi-univalent functions, obtaining the bound 1.51 for the modulus of the second coefficient $|a_2|$. Netanyahu [22] showed that $\max|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Subsequently, Brannan and Clunie [7] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Brannan and Taha [8] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses. Recently, many authors investigated bounds for various subclasses of bi-univalent functions [5,10,13,18,20,21,23,27–29].

The Faber polynomials introduced by Faber [12] play an important role in various areas of mathematical sciences, especially in geometric function theory. Grunsky [14] succeeded in establishing a set of conditions for a given function that are necessary and in their totality sufficient for the univalency of this function, and in these conditions the coefficients of the Faber polynomials play an important role. Schiffer [26] gave differential equations for univalent functions solving certain extremum problems with respect to coefficients of such functions; in this differential equation a polynomial appears again, which is just the derivative of a Faber polynomial (Schaeffer–Spencer [25]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions [6,9,16,15,17]. The coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N}\backslash\{1, 2\};\ \ \mathbb{N} = \{1, 2, 3, \ldots\})$ is still an open problem.

For $f(z)$ and $F(z)$ analytic in U, we say that $f(z)$ is subordinate to $F(z)$, written $f \prec F$, if there exists a Schwarz function

$$u(z) = \sum_{n=1}^{\infty} c_n z^n$$

with $|u(z)| < 1$ in U, such that $f(z) = F(u(z))$. For the Schwarz function $u(z)$, we note that $|c_n| < 1$. (e.g., see Duren [11]).

A function $f \in \Sigma$ is said to be $\mathrm{B}_\Sigma(p, \lambda, \varphi)$, $p \in \mathbb{N}_0$, and $\lambda \geq 0$, if the following subordination holes

$$\frac{(1-\lambda) D^p f(z) + \lambda D^{p+1} f(z)}{z} \prec \varphi(z) \tag{2}$$

and

$$\frac{(1-\lambda) D^p g(w) + \lambda D^{p+1} g(w)}{w} \prec \varphi(w) \tag{3}$$

where $g(w) = f^{-1}(w)$.

In this paper, we use the Faber polynomial expansions to obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions in $\mathrm{B}_\Sigma(p, \lambda, \varphi)$ as well as we provide estimates for the initial coefficients of these functions.

2. Main results

Using the Faber polynomial expansion of functions $f \in A$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as, [3],

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} = & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ & + \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5}\left[a_5 + (-n+2) a_3^2\right] + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6}[a_6 + (-2n+5) a_3 a_4] \\ & + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \tag{4}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $|a_2|, |a_3|, \ldots, |a_n|$ [4]. In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} \frac{1}{2} K_1^{-2} &= -a_2, \\ \frac{1}{3} K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4} K_3^{-4} &= -\left(5a_2^3 - 5a_2 a_3 + a_4\right). \end{aligned} \tag{5}$$

In general, for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of K_{n-1}^{p} is as, [3],

$$K_{n-1}^{p} = pa_n + \frac{p(p-1)}{2} E_{n-1}^2 + \frac{p!}{(p-3)!3!} E_{n-1}^3 + \ldots + \frac{p!}{(p-n+1)!(n-1)!} E_{n-1}^{n-1}, \tag{6}$$

where $E_{n-1}^{p} = E_{n-1}^{p}(a_2, a_3, \ldots)$ and by [1],

$$E_{n-1}^{m}(a_2, \ldots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \ldots (a_n)^{\mu_{n-1}}}{\mu_1! \ldots \mu_{n-1}!}, \quad \text{for } m \leq n$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \ \ldots \ + \mu_{n-1} &= m, \\ \mu_1 + 2\mu_2 + \ \ldots \ + (n-1)\mu_{n-1} &= n-1. \end{aligned}$$

Evidently, $E_{n-1}^{n-1}(a_2, \ldots, a_n) = a_2^{n-1}$, [2]; or equivalently,

$$E_n^m(a_1, a_2, \ldots, a_n) = \sum_{n=1}^{\infty} \frac{m!(a_1)^{\mu_1} \ldots (a_n)^{\mu_n}}{\mu_1! \ldots \mu_n!}, \quad \text{for } m \leq n$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \ \ldots \ + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \ \ldots \ + n\mu_n &= n. \end{aligned}$$

It is clear that $E_n^n(a_1, a_2, \ldots, a_n) = a_1^n$. The first and the last polynomials are:

$$E_n^1 = a_n, \qquad E_n^n = a_1^n.$$

Theorem 1. *For $0 \leq \beta < 1$, $\lambda \geq 1$ and $p \in \mathbb{N}_0$, let $f \in \mathrm{B}_\Sigma(p, \lambda, \varphi)$. If $a_m = 0$; $2 \leq m \leq n-1$, then*

$$|a_n| \leq \frac{2}{n^p[1 + (n-1)\lambda]}; \quad n \geq 4. \tag{7}$$

Proof. Let f be given by (1). We have

$$\frac{(1-\lambda)\,D^{p}f\,(z)\ +\ \lambda D^{p+1}f\,(z)}{z}=1+\sum_{n=2}^{\infty}n^{p}\left[1+(n-1)\,\lambda\right]a_{n}z^{n-1},\tag{8}$$

and for its inverse map, $g=f^{-1}$, we have

$$\begin{aligned}\frac{(1-\lambda)\,D^{p}g\,(w)\ +\ \lambda D^{p+1}g\,(w)}{w}&=1+\sum_{n=2}^{\infty}n^{p}\left[1+(n-1)\,\lambda\right]\frac{1}{n}K_{n-1}^{-n}\left(a_{2},a_{3},\ldots,a_{n}\right)w^{n-1}\\&=1+\sum_{n=2}^{\infty}n^{p}\left[1+(n-1)\,\lambda\right]b_{n}w^{n-1}.\end{aligned}\tag{9}$$

On the other hand, for $f\in B_{\Sigma}\left(p,\lambda,\varphi\right)$ and $\varphi\in P$ there are two Schwarz functions

$$u\,(z)=\sum_{n=1}^{\infty}c_{n}z^{n}$$

and

$$v\,(w)=\sum_{n=1}^{\infty}d_{n}w^{n}$$

such that

$$\frac{(1-\lambda)\,D^{p}f\,(z)\ +\ \lambda D^{p+1}f\,(z)}{z}=\varphi(u(z))\tag{10}$$

and

$$\frac{(1-\lambda)\,D^{p}g\,(w)\ +\ \lambda D^{p+1}g\,(w)}{w}=\varphi(v(w))\tag{11}$$

where

$$\varphi(u(z))=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\varphi_{k}E_{n}^{k}\left(c_{1},c_{2},\ldots,c_{n}\right)z^{n}\tag{12}$$

and

$$\varphi(v(w))=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\varphi_{k}E_{n}^{k}\left(d_{1},d_{2},\ldots,d_{n}\right)w^{n}.\tag{13}$$

Comparing the corresponding coefficients of (10) and (12) yields

$$n^{p}\left[1+(n-1)\,\lambda\right]a_{n}=\sum_{k=1}^{n-1}\varphi_{k}E_{n-1}^{k}\left(c_{1},c_{2},\ldots,c_{n-1}\right),\ n\geq 2\tag{14}$$

and similarly, from (11) and (13), we obtain

$$n^{p}\left[1+(n-1)\,\lambda\right]b_{n}=\sum_{k=1}^{n-1}\varphi_{k}E_{n-1}^{k}\left(d_{1},d_{2},\ldots,d_{n-1}\right),\ \ n\geq 2.\tag{15}$$

Note that for $a_{m}=0;\ 2\leq m\leq n-1$ we have $b_{n}=-a_{n}$ and so

$$\begin{aligned}n^{p}\left[1+(n-1)\,\lambda\right]a_{n}&=\varphi_{1}c_{n-1},\\-n^{p}\left[1+(n-1)\,\lambda\right]a_{n}&=\varphi_{1}d_{n-1}.\end{aligned}\tag{16}$$

Now taking the absolute values of either of the above two equations in (16) and using the facts that $|\varphi_{1}|\leq 2$, $|c_{n-1}|\leq 1$ and $|d_{n-1}|\leq 1$, we obtain

$$|a_{n}|\leq\frac{|\varphi_{1}c_{n-1}|}{|n^{p}\left[1+(n-1)\,\lambda\right]|}=\frac{|\varphi_{1}d_{n-1}|}{|n^{p}\left[1+(n-1)\,\lambda\right]|}\leq\frac{2}{n^{p}\left[1+(n-1)\,\lambda\right]}.\qquad\square\tag{17}$$

Theorem 2. *Let* $f \in B_{\Sigma}(p,\lambda,\varphi)$, $\lambda \geq 1$. *Then*

(i) $$|a_2| \leq \min\left\{\frac{1}{2^{p-1}(1+\lambda)}, \frac{2}{\sqrt{3^p(1+2\lambda)}}\right\} = \frac{1}{2^{p-1}(1+\lambda)},$$

(ii) $$|a_3| \leq \min\left\{\frac{1}{2^{2p-2}(1+\lambda)^2} + \frac{2}{3^p(1+2\lambda)}, \frac{2}{3^{p-1}(1+2\lambda)}\right\}$$

$$= \frac{1}{2^{2p-2}(1+\lambda)^2} + \frac{2}{3^p(1+2\lambda)},$$

(iii) $$\left|a_3 - \eta a_2^2\right| \leq \tfrac{2\eta}{3^p(1+2\lambda)}; \quad \eta = 1, 2. \tag{18}$$

Proof. Replacing n by 2 and 3 in (14) and (15), respectively, we find that

$$2^p(1+\lambda)a_2 = \varphi_1 c_1, \tag{19}$$

$$3^p(1+2\lambda)a_3 = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{20}$$

$$-2^p(1+\lambda)a_2 = \varphi_1 d_1, \tag{21}$$

$$3^p(1+2\lambda)(2a_2^2 - a_3) = \varphi_1 d_2 + \varphi_2 d_1^2 \tag{22}$$

From (19) or (21) we obtain

$$|a_2| \leq \frac{|\varphi_1 c_1|}{2^p(1+\lambda)} = \frac{|\varphi_1 d_1|}{2^p(1+\lambda)} \leq \frac{1}{2^{p-1}(1+\lambda)}. \tag{23}$$

Adding (20) to (22) implies

$$2.3^p(1+2\lambda)a_2^2 = \varphi_1(c_2+d_2) + \varphi_2\left(c_1^2 + d_1^2\right)$$

or, equivalently,

$$|a_2| \leq \frac{2}{\sqrt{3^p(1+2\lambda)}}. \tag{24}$$

From (20)

$$|a_3| = \frac{\left|\varphi_1 c_2 + \varphi_2 c_1^2\right|}{3^p(1+2\lambda)} \leq \frac{4}{3^p(1+2\lambda)}.$$

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (22) from (20). We thus get

$$2.3^p(1+2\lambda)\left(a_3 - a_2^2\right) = \varphi_1(c_2 - d_2) + \varphi_2\left(c_1^2 - d_1^2\right) \tag{25}$$

or

$$|a_3| = |a_2|^2 + \frac{|\varphi_1(c_2 - d_2)|}{3^p(1+2\lambda)} \leq |a_2|^2 + \frac{2}{3^p(1+2\lambda)}. \tag{26}$$

Upon substituting the value of a_2^2 from (23) and (24) into (26), it follows that

$$|a_3| \leq \frac{1}{2^{2p-2}(1+\lambda)^2} + \frac{2}{3^p(1+2\lambda)}$$

and

$$|a_3| \leq \frac{2}{3^{p-1}(1+2\lambda)}.$$

Finally, we rewrite (22) as

$$3^p(1+2\lambda)(a_3 - 2a_2^2) = -\left(\varphi_1 d_2 + \varphi_2 d_1^2\right)$$

and therefore

$$\left|a_3 - 2a_2^2\right| = \frac{\left|\varphi_1 d_2 + \varphi_2 d_1^2\right|}{3^p(1+2\lambda)} \leq \frac{4}{3^p(1+2\lambda)}.$$

Solving the equation (25) for $\left(a_3 - a_2^2\right)$, we obtain

$$\left|a_3 - a_2^2\right| = \frac{\left|\varphi_1\left(c_2 - d_2\right) + \varphi_2\left(c_1^2 - d_1^2\right)\right|}{3^p\left(1+2\lambda\right)} \le \frac{2}{3^p\left(1+2\lambda\right)}. \qquad \square$$

Remark 3. If we put $p = 0$ in Theorem 2, we obtain that the bounds on $|a_2|$ and $|a_3|$ are improvement of the estimates given in Frasin and Aouf [13].

Remark 4. If we put $p = 0,\ \lambda = 1$ in Theorem 2, we obtain that the bounds on $|a_2|$ and $|a_3|$ are improvement of the estimates given in Srivastava et al. [27].

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