Topology

On the generators of the polynomial algebra as a module over the Steenrod algebra

Sur les générateurs de l’algèbre polynomiale comme module sur l’algèbre de Steenrod

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ABSTRACT

Let $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ be the polynomial algebra over the prime field of two elements, $\mathbb{F}_2$, in $k$ variables $x_1, x_2, \ldots, x_k$, each of degree 1. We are interested in the Peterson hit problem of finding a minimal set of generators for $P_k$ as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. In this paper, we study the hit problem in degree $(k - 1)(2^d - 1)$, with $d$ a positive integer. Our result implies the one of Mothebe [4,5].

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1. Introduction

Let $P_k$ be the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_k]$, with the degree of each $x_i$ being 1. This algebra arises as the cohomology with coefficients in $\mathbb{F}_2$ of an elementary Abelian 2-group of rank $k$. Then, $P_k$ is a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The action of $\mathcal{A}$ on $P_k$ is determined by the elementary properties of the Steenrod squares $Sq^i$ and subject to the Cartan formula (see Steenrod and Epstein [12]).

An element $g$ in $P_k$ is called hit if it belongs to $\mathcal{A}^+ P_k$, where $\mathcal{A}^+$ is the augmentation ideal of $\mathcal{A}$. This means that $g$ can be written as a finite sum $g = \sum_{a \geq 0} Sq^a(g_a)$ for suitable polynomials $g_a \in P_k$.

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We are interested in the hit problem, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra $P_k$ as a module over the Steenrod algebra. In other words, we want to find a basis of the $\mathbb{F}_2$-vector space $Q P_k := P_k / A^+ P_k = \mathbb{F}_2 \otimes A P_k$.

The hit problem was first studied by Peterson [7], Wood [16], Singer [10], and Priddy [8], who showed its relation to several classical problems respectively in cobordism theory, modular representation theory, the Adams spectral sequence for the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups.

The vector space $Q P_k$ was explicitly calculated by Peterson [7] for $k = 1, 2$, by Kameko [3] for $k = 3$, and recently by the second author [13, 14] for $k = 4$. From the results of Wood [16] and Kameko [3], the hit problem is reduced to the case of degree $n$ of the form

$$n = s(2^d - 1) + 2^d m,$$

where $s, d, m$ are non-negative integers and $1 \leq s < k$ (see [14]). For $s = k - 1$ and $m > 0$, the problem was studied by Crabb and Hubbuck [2], Nam [6], Repka and Selick [9], and the second author [13, 14].

In the present paper, we study the hit problem in degree $n$ of the form (1.1) with $s = k - 1$, $m = 0$ and $d$ an arbitrary positive integer.

Denote by $(Q P_k)_n$ the subspace of $Q P_k$ consisting of the classes represented by the homogeneous polynomials of degree $n$ in $P_k$. From the result of Carlisle and Wood [1] on the boundedness conjecture, one can see that for $d$ big enough, the dimension of $(Q P_k)_n$ does not depend on $d$; it depends only on $k$. In this paper, we prove the following.

**Main Theorem.** Let $n = (k - 1)(2^d - 1)$ with $d$ a positive integer and let $p = \min \{k, d\}$, $q = \min \{k, d - 1\}$. If $k \geq 3$, then

$$\dim (Q P_k)_n \geq c(k, d) := \sum_{t=1}^{p} \binom{k}{t} + (k - 3) \binom{k}{2} \sum_{u=1}^{q} \binom{k}{u},$$

with equality if and only if $k = 3$ or $k = 4$, $d \geq 5$ or $k = 5$, $d \geq 6$.

Note that $c(k, 1) = \binom{k}{1} = k$. If $d > k$, then $c(k, d) = ((k - 3)\binom{k}{2} + 1)(2^k - 1)$. At the end of Section 3, we show that our result implies Mothebe’s result in [4, 5].

In Section 2, we recall the definition of an admissible monomial in $P_k$ and Singer’s criterion on the hit monomials. Our results will be presented in Section 3.

2. Preliminaries

In this section, we recall some needed information from Kameko [3] and Singer [11], which will be used in the next section.

**Notation 2.1.** We denote $N_k = \{1, 2, \ldots, k\}$ and $X_j = X_{j_1, j_2, \ldots, j_{\ell_j}} = \prod_{j \in N_k}^j X_{j_i}^{j_i}, \ J = \{j_1, j_2, \ldots, j_{\ell_j}\} \subset N_k$. In particular, $X_{N_k} = 1, \ X_1 = x_1 x_2 \ldots x_k, \ X_j = x_{j_1} x_{j_2} \ldots x_{j_{\ell_j}}, \ 1 \leq j \leq k$, and $X := X_k \in P_{k-1}$.

Let $a_0(a)$ denote the $i$-th coefficient in dyadic expansion of a non-negative integer $a$. That means $a = a_0(a)2^0 + a_1(a)2^1 + a_2(a)2^2 + \ldots$ for $a_0(a) = 0$ or $1$ with $i \geq 0$. Set $\alpha(a) = \sum_{i \geq 0} a_i(a)$.

Let $x = x_1^{a_1(x)} x_2^{a_2(x)} \ldots x_k^{a_k(x)} \in P_k$. Denote $v_j(x) = a_j, \ 1 \leq j \leq k$. Set $J_i(x) = \{j \in N_k : \alpha_i(v_j(x)) = 0\}, \ for \ i \geq 0$. Then, we have $x = \prod_{i \geq 0} X_{J_i(x)}^{a_i(x)}$.

**Definition 2.2.** For a monomial $x$ in $P_k$, define two sequences associated with $x$ by

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots), \ \sigma(x) = (v_1(x), v_2(x), \ldots, v_k(x)),$$

where $\omega_i(x) = \sum_{1 \leq j \leq \ell_j} a_i-1(v_j(x)) \deg X_{J_i-1(x)}^{a_i-1(x)}, \ i \geq 1$. The sequence $\omega(x)$ is called the weight vector of $x$.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of non-negative integers. The sequence $\omega$ is called the weight vector if $\omega_1 = 0$ for $i > 0$.

The sets of the weight vectors and the sigma vectors are given the left lexicographical order.

For a weight vector $\omega$, we define $deg \omega = \sum_{i \geq 0} 2^i - 1 \omega_i$. If there are $i_0 = 0, i_1, i_2, \ldots, i_r > 0$ such that $i_1 + i_2 + \ldots + i_r = m$, $\omega_{i_1+i_2+\ldots+i_r} = b_r, \ 1 \leq r \leq r, \ 1 \leq s \leq r$, and $\omega_0 = 0$ for all $i > m$, then we write $\omega = (b_1^{(i_1)}, b_2^{(i_2)}, \ldots, b_r^{(i_r)}).$ Denote $b_u^{(i)} = b_u$.

For example, $\omega = (3, 3, 2, 1, 1, 0, \ldots) = (3^{(2)}, 2, 1^{(3)})$.

Denote by $P_k(\omega)$ the subspace of $P_k$ spanned by monomials $y$ such that $\deg y = \deg \omega, \ \omega(y) \leq \omega$, and by $P_k^-(\omega)$ the subspace of $P_k$ spanned by monomials $y \in P_k(\omega)$ such that $\omega(y) < \omega$.

**Definition 2.3.** Let $\omega$ be a weight vector and $f, g$ two polynomials of the same degree in $P_k$. 

i) \( f \equiv g \) if and only if \( f - g \in \mathcal{A}^+ P_k \). If \( f \equiv 0 \) then \( f \) is called hit.

ii) \( f \equiv_\omega g \) if and only if \( f - g \in \mathcal{A}^+ P_k + P_k^- (\omega) \).

Obviously, the relations \( \equiv \) and \( \equiv_\omega \) are equivalence ones. Denote by \( QP_k(\omega) \) the quotient of \( P_k(\omega) \) by the equivalence relation \( \equiv_\omega \). Then, we have \( QP_k(\omega) = P_k(\omega)/(\mathcal{A}^+ P_k \cap P_k(\omega)) \) and \( (QP_k)_n \equiv \bigoplus_{\deg \omega = n} QP_k(\omega) \) (see Walker and Wood [15]).

We note that the weight vector of a monomial is invariant under the permutation of the generators \( x_i \), hence \( QP_k(\omega) \) has an action of the symmetric group \( \Sigma_k \).

For a polynomial \( f \in P_k(\omega) \), we denote by \( |f|_\omega \) the class in \( QP_k(\omega) \) represented by \( f \). Denote by \( |S| \) the cardinal of a set \( S \).

**Definition 2.4.** Let \( x, y \) be monomials of the same degree in \( P_k \). We say that \( x < y \) if and only if one of the following holds:

i) \( \omega(x) < \omega(y) \);

ii) \( \omega(x) = \omega(y) \) and \( \sigma(x) < \sigma(y) \).

**Definition 2.5.** A monomial \( x \) is said to be inadmissible if there exist monomials \( y_1, y_2, \ldots, y_m \) such that \( y_t < x \) for \( t = 1, 2, \ldots, m \) and \( x - \sum_{t=1}^m y_t \in \mathcal{A}^+ P_k \).

A monomial \( x \) is said to be admissible if it is not inadmissible.

Obviously, the set of the admissible monomials of degree \( n \) in \( P_k \) is a minimal set of \( \mathcal{A} \)-generators for \( P_k \) in degree \( n \). Now, we recall a result of Singer [11] on the hit monomials in \( P_k \).

**Definition 2.6.** A monomial \( z \) in \( P_k \) is called a spike if \( v_j(z) = 2^j i - 1 \) for \( d_j \) a non-negative integer and \( j = 1, 2, \ldots, k \). If \( z \) is a spike with \( d_1 > d_2 > \ldots > d_{k-1} \geq d_k > 0 \) and \( d_j = 0 \) for \( j > r \), then it is called the minimal spike.

In [11], Singer showed that if \( \alpha(n + k) \leq k \), then there exists uniquely a minimal spike of degree \( n \) in \( P_k \).

**Lemma 2.7.**

i) All the spikes in \( P_k \) are admissible and their weight vectors are weakly decreasing.

ii) If a weight vector \( \omega \) is weakly decreasing and \( \omega_1 \leq k \), then there is a spike \( z \) in \( P_k \) such that \( \omega(z) = \omega \).

**Theorem 2.8.** (See Singer [11]) Suppose \( x \in P_k \) is a monomial of degree \( n \), where \( \alpha(n + k) \leq k \). Let \( z \) be the minimal spike of degree \( n \). If \( \omega(x) < \omega(z) \), then \( x \) is hit.

The following theorem will be used in the next section.

**Theorem 2.9.** (See [13, 14]) Let \( n = \sum_{i=1}^{k-1} (2^i \cdot d_i - 1) \) with \( d_i \) positive integers such that \( d_1 > d_2 > \ldots > d_{k-2} \geq d_{k-1} \), and let \( m = \sum_{i=1}^{k-2} (2^i \cdot d_i - d_{i+1} - 1) \). If \( d_{k-1} \geq k - 1 \geq 3 \), then

\[
\dim(QP_k)_n = (2^k - 1) \dim(QP_{k-1})_m.
\]

Note that we correct Theorem 3 in [13] by replacing the condition \( d_{k-1} \geq k - 1 \geq 1 \) with \( d_{k-1} \geq k - 1 \geq 3 \).

3. Proof of the Main Theorem

Denote \( \mathcal{N}_k = \{(i; I); I = (i_1, i_2, \ldots, i_r), 1 \leq i < i_1 < \ldots < i_r \leq k, \ 0 \leq r < k\} \).

**Definition 3.1.** Let \( (i; I) \in \mathcal{N}_k \), let \( r = c(I) \) be the length of \( I \), and let \( u \) be an integer with \( 1 \leq u \leq r \). A monomial \( x \in P_{k-1} \) is said to be \( u \)-compatible with \( (i; I) \) if all of the following hold:

i) \( v_{i_1-1}(x) = v_{i_2-1}(x) = \ldots = v_{i_{r-1}-1}(x) = 2^r - 1 \),

ii) \( v_{i_{r-1}-1}(x) > 2^r - 1 \),

iii) \( \alpha_{r-t}(v_{i_{r-1}-1}(x)) = 1, \ \forall t, \ 1 \leq t \leq u \),

iv) \( \alpha_{r-t}(v_{i_{r-1}-1}(x)) = 1, \ \forall t, \ u < t \leq r \).
Clearly, a monomial $x$ can be $u$-compatible with a given $(i; I) \in \mathcal{N}_k$ for at most one value of $u$. By convention, $x$ is 1-compatible with $(i; \emptyset)$.

For $1 \leq i \leq k$, define the homomorphism $f_i : P_{k-1} \to P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} 
  x_j, & \text{if } 1 \leq j < i, \\
  x_{j+1}, & \text{if } i \leq j < k.
\end{cases}$$

**Definition 3.2.** Let $(i; I) \in \mathcal{N}_k$, $x_{(i,u)} = x_\mathcal{N}_k^{2^{i-1}+\ldots+2^{i-u}} \prod_{\mu \leq t \leq u} x_\mathcal{N}_k^{2^{\mu-t}}$ for $\ell = 1 > 0$, $x_{(i,\emptyset)} = 1$. For a monomial $x$ in $P_{k-1}$, we define the monomial $\phi_{(i; I)}(x)$ in $P_k$ by setting

$$\phi_{(i; I)}(x) = \begin{cases} 
  (x_\mathcal{N}_k^{2^{i-1}} f_i(x))/x_{(i,u)}, & \text{if there exists } u \text{ such that } x \text{ is } u \text{-compatible with } (i, I), \\
  0, & \text{otherwise}.
\end{cases}$$

Then we have an $\mathbb{F}_2$-linear map $\phi_{(i; I)} : P_{k-1} \to P_k$. In particular, $\phi_{(i; \emptyset)} = f_i$.

For a positive integer $b$, denote $\omega_{(k,b)} = ((k - 1)^{(b)})$ and $\tilde{\omega}_{(k,b)} = ((k - 1)^{(b-1)}, k - 3, 1)$.

**Lemma 3.3.** (See [14] ) Let $b$ be a positive integer and let $j_0, j_1, \ldots, j_b-1 \in \mathcal{N}_k$. We set $i = \min\{j_0, \ldots, j_b-1\}$, $I = (i_1, \ldots, i_k)$ with $\{i_1, \ldots, i_k\} = \{j_0, \ldots, j_b-1\} \setminus \{i\}$. Then, we have $\prod_{0 \leq t \leq \ell} x_\mathcal{N}_k^{2^t} = \omega_{(k,b)}(X_\mathcal{N}_k^{2^{\ell-1}})$.

**Definition 3.4.** For any $(i; I) \in \mathcal{N}_k$, we define the homomorphism $p_{(i; I)} : P_k \to P_{k-1}$ of algebras by substituting

$$p_{(i; I)}(x_j) = \begin{cases} 
  x_j, & \text{if } 1 \leq j < i, \\
  \sum_{s \leq I} x_{s-1}, & \text{if } j = i, \\
  x_{j-1}, & \text{if } i \leq j \leq k.
\end{cases}$$

Then, $p_{(i; I)}$ is a homomorphism of $A$-modules. In particular, for $I = \emptyset$, $p_{(i; \emptyset)}(x_i) = 0$ and $p_{(i; I)}(f_i(y)) = y$ for any $y \in P_{k-1}$.

**Lemma 3.5.** If $x$ is a monomial in $P_k$, then $p_{(i; I)}(x) \in P_{k-1}(\omega(x))$.

**Proof.** Set $y = p_{(i; I)}(x/x_\mathcal{N}_k^{v_i(x)})$. Then, $y$ is a monomial in $P_{k-1}$. If $v_i(x) = 0$, then $y = p_{(i; I)}(x)$ and $\omega(y) = \omega(x)$. Suppose $v_i(x) > 0$ and $v_i(x) = 2^1 + \ldots + 2^c$, where $0 \leq t_1 < \ldots < t_c$, $c \geq 1$.

If $I = \emptyset$, then $p_{(i; I)}(x) = 0$. If $I \neq \emptyset$, then $p_{(i; I)}(x)$ is a sum of monomials of the form $\tilde{y} := \left( \prod_{u=1}^{c} x_\mathcal{N}_k^{2^{t_u}} \right) y$, where $s_u \in I$, $1 \leq u \leq c$. If $\omega_{(k,b)}(V_{u_0-1}(y)) = 0$ for all $u$, then $\omega(\tilde{y}) = \omega(x)$. Suppose there is an index $u$ such that $\omega_{(k,b)}(V_{u_0-1}(y)) = 1$. Let $u_0$ be the smallest index such that $\omega_{(k,b)}(V_{u_0-1}(y)) = 1$. Then, we have

$$\omega(\tilde{y}) = \begin{cases} 
  \omega_{(k,b)}(x), & \text{if } i \leq t_{u_0}, \\
  \omega_{(k,b)}(x) - 2, & \text{if } i = t_{u_0} + 1.
\end{cases}$$

Hence, $\omega(\tilde{y}) < \omega(x)$ and $\tilde{y} \in P_{k-1}(\omega(x))$. The lemma is proved. \hfill \Box

**Lemma 3.5** implies that if $\omega$ is a weight vector and $x \in P_k(\omega)$, then $p_{(i; I)}(x) \in P_{k-1}(\omega)$. Moreover, $p_{(i; I)}$ passes to a homomorphism from $Q\mathcal{P}_k(\omega)$ to $Q\mathcal{P}_{k-1}(\omega)$. In particular, we have

**Lemma 3.6.** (See [14] ) Let $b$ be a positive integer and let $(j; J), (i; I) \in \mathcal{N}_k$ with $\ell(I) < b$.

i) If $(i; I) \subset (j; J)$, then $p_{(j; J)}(\phi_{(i; I)}(X_\mathcal{N}_k^{2^{\ell-1}})) = X_\mathcal{N}_k^{2^{\ell-1}} \mod(P_{k-1}^-(\omega_{(k,b)}))$.

ii) If $(i; I) \not\subset (j; J)$, then $p_{(j; J)}(\phi_{(i; I)}(X_\mathcal{N}_k^{2^{\ell-1}})) \in P_{k-1}^-(\omega_{(k,b)})$.

For $0 \leq h \leq k$, set $\mathcal{N}_{k,h} = \{(i; I) \in \mathcal{N}_k : \ell(I) < h\}$. Then, $|\mathcal{N}_{k,h}| = \sum_{t=1}^{h} \binom{k}{t}$.

**Proposition 3.7.** Let $d$ be a positive integer and let $p = \min(k, d)$. Then, the set $B(d) := \{\phi_{(i; I)}(X_\mathcal{N}_k^{2^{\ell-1}})_{(\omega_{(k,d)})} : (i; I) \in \mathcal{N}_{k,p}\}$ is a basis of the $\mathbb{F}_2$-vector space $Q\mathcal{P}_k(\omega_{(k,d)})$. Consequently $\dim Q\mathcal{P}_k(\omega_{(k,d)}) = \sum_{t=1}^{p} \binom{k}{t}$. 

Proof. Let $x$ be a monomial in $P_k(\omega_{k,d})$ and $[x]_{\omega_{k,d}} \neq 0$. Then, we have $\omega(x) = \omega_{k,d}$. So, there exist $j_0, j_1, \ldots, j_{d-1} \in \mathbb{N}_k$ such that $x = \prod_{0 \leq t < d} X_t^{\ell_t}$. According to Lemma 3.3, there is $(i; I) \in \mathcal{N}_k$ such that $x = \prod_{0 \leq t < d} X_t^{\ell_t} \equiv \omega_{k,d} \phi(i, t)(X^{2^{d-1}})$, where $r = \ell(t) < p = \min(k, d)$. Hence, $QP_k(\omega_{k,d})$ is spanned by the set $B(d)$.

Now, we prove that the set $B(d)$ is linearly independent in $QP_k(\omega_{k,d})$. Suppose that there is a linear relation

$$\sum_{(i; I) \in \mathcal{N}_k} \gamma_{(i; I)} \phi(i, t)(X^{2^{d-1}}) \equiv \omega_{k,d} \phi(i, t)(X^{2^{d-1}}),$$

where $\gamma_{(i; I)} \in \mathbb{F}_2$. By induction on $t(I)$, using Lemma 3.5 and Lemma 3.6 with $b = d$, we can easily show that $\gamma_{(i; I)} = 0$ for all $(i; I) \in \mathcal{N}_k$. The proposition is proved.

Set $C_k = \{x_{j_1}x_{j_2} \ldots x_{j_{k-3}} x_j^2 : 1 \leq j_1 < j_2 < \ldots < j_{k-3} < k, \ j_1 \leq j < k \} \subset P_{k-1}$. It is easy to see that $|C_k| = (k - 3)(\frac{d}{2})$.

Lemma 3.8. $C_k$ is the set of the admissible monomials in $P_{k-1}$ such that their weight vectors are $\omega_{(k,1)} = (k - 3, 1)$. Consequently, $\dim QP_k(\omega_{(k,1)}) = (k - 3)(\frac{d}{2})$.

Proof. Let $z$ be a monomial in $P_{k-1}$ such that $\omega(z) = (k - 3, 1)$. Then, $z = x_{j_1}x_{j_2} \ldots x_{j_{k-3}} x_j^2$ with $1 \leq j_1 < j_2 < \ldots < j_{k-3} < k$ and $1 \leq j < k$. If $z \notin C_k$, then $j < j_1$. Then, we have $z = \sum_{i=1}^{k-3} x_{j_1}x_{j_2} \ldots \hat{x}_j \ldots x_{j_{k-3}} x_j + Sq_1(x_{j_1}x_{j_2} \ldots x_{j_{k-3}} x_j)$. Since $x_{j_1}x_{j_2} \ldots \hat{x}_j \ldots x_{j_{k-3}} x_j < z$ for $1 \leq i < k - 3, z$ is inadmissible.

Suppose that $z \in C_k$. If there is an index $s$ such that $j = j_s$, then $z$ is a spike. Hence, by Lemma 2.7, it is admissible. Assume that $j \neq j_s$ for all $s$. If $z$ is inadmissible, then there exist monomials $y_1, \ldots, y_m$ in $P_{k-1}$ such that $y_1 < z$ for all $t = 2$ and $z = \sum_{i=1}^{m} y_1 + \sum_{u \geq 1} Sq_1(g_u)$, where $g_u$ are suitable polynomials in $P_{k-1}$. Since $y_1 < z$ for all $t = 2$, $z$ is a term of $\sum_{u \geq 1} Sq_1(g_u)$ (recall that a monomial $x$ in $P_k$ is called a term of a polynomial $f$ if it appears in the expression of $f$ in terms of the monomial basis of $P_k$). Based on the Cartan formula, we see that $z$ is not a term of $\sum_{u \geq 1} Sq_1(g_u)$ for all $u > 0$. If $z$ is a term of $\sum_{u \geq 1} Sq_1(g_u)$ with $y$ a monomial in $P_{k-1}$, then $y = x_{j_1}x_{j_2} \ldots x_{j_{k-3}} x_j := \tilde{y}$. So, $\tilde{y}$ is a term of $g_0$. Then, we have

$$\tilde{y} := x_{j_1}^2 x_{j_2} x_{j_3} \ldots x_{j_{k-3}} x_j = \sum_{i=2}^{k-3} x_{j_1} x_{j_2} \ldots \hat{x}_j \ldots x_{j_{k-3}} x_j + \sum_{t=1}^{m} y_t + \sum_{u \geq 1} Sq_1(g_0 + \tilde{y}) + \sum_{u \geq 1} Sq_1(g_u).$$

Since $j_1 < j$, we have $y_t < z < \tilde{y}$ for all $t$. Hence, $\tilde{y}$ is a term of $\sum_{u \geq 1} Sq_1(g_0 + \tilde{y}) + \sum_{u \geq 1} Sq_1(g_u)$. By an argument analogous to the previous one, we see that $\tilde{y}$ is a term of $g_0 + \tilde{y}$. This contradicts the fact that $\tilde{y}$ is a term of $g_0$. The lemma is proved.

Proposition 3.9. Let $d$ be a positive integer and let $q = \min(k, d - 1)$. Then, the set $B(d) := \bigcup_{i \in C_k} \{\phi(i, t)(X^{2^{d-1}}) \mid \omega_{k,d,p} \}$ is linearly independent in $QP_k(\omega_{k,d})$. If $d > k$, then $B(d)$ is a basis of $QP_k(\omega_{k,d})$. Consequently $\dim QP_k(\omega_{k,d}) = (k - 3)(\frac{d}{2})$ with equality if $d > k$.

Proof. We prove the first part of the proposition. Suppose there is a linear relation $S := \sum_{(i; I) \in \mathcal{N}_k} \gamma_{(i; I)} \phi(i, t)(X^{2^{d-1}}) \equiv \omega_{k,d,p}$, where $\gamma_{(i; I)} \in \mathbb{F}_2$. We prove $\gamma_{(i; I)} = 0$ for all $(i; J) \in \mathcal{N}_k,q$ and $z \in C_k$. The proof proceeds by induction on $m = \ell(J)$. Let $(i; I) \in \mathcal{N}_k,q$. Since $r = \ell(I) < q = \min(k, d - 1)$, $X^{2^{d-1}} \equiv X^{2^{d-1}}$ is 1-compatible with $(i; I)$ and $X^{2^{d-1}}$ is divisible by $x_{j_1}$. Hence, using Definition 3.2, we easily obtain $\phi(i, t)(X^{2^{d-1}}) = \omega_{k,d,p}(X^{2^{d-1}}) f_1(z^{2^{d-1}})$. A simple computation shows that if $g \in P_{k-1}(\omega_{k,d-1})$, then $g z^{2^{d-1}} \in P_{k-1}(\omega_{k,d})$. So, $(i; I) = (j; \emptyset)$; by Lemma 3.5, $p_{(j;m)}(S) = \omega_{k,d,p}$. Hence, applying Lemma 3.6 with $b = d - 1$, we get $p_{(j;m)}(S) = \omega_{k,d,p}$. Since $z$ is admissible in $P_{k-1}, X^{2^{d-1}} \equiv X^{2^{d-1}}$ is also admissible in $P_{k-1}$. Hence, the last relation implies $\gamma_{(j; \emptyset)} = 0$ for all $z \in C_k$. Suppose $0 < m < q$ and $\gamma_{(i; I)} = 0$ for all $z \in C_k$ and $(i; I) \in \mathcal{N}_k,q$ with $\ell(I) < m$. Let $(i; J) \in \mathcal{N}_k,q$ with $\ell(J) = m$. Note that by Lemma 3.5, $p_{(j;m)}(S) = \omega_{k,d,p}$; if $(i; I) \subset (j; \emptyset)$, then $(i; I) = (j; \emptyset)$; so, by Lemma 3.6 with $b = d - 1$ and the inductive hypothesis, we obtain $p_{(j;m)}(S) = \omega_{k,d,p}$.

From this equality, one gets $\gamma_{(j; \emptyset)} = 0$ for all $z \in C_k$. The first part of the proposition follows.

The proof of the second part is similar to the one of Proposition 3.3 in [14]. However, the relation $\equiv \omega_{k,d,p}$ is used in the proof instead of $\equiv$.

For $k = 5$, we have the following result.

Theorem 3.10. Let $n = 4(2^d - 1)$ with $d$ a positive integer. The dimension of the $\mathbb{F}_2$-vector space $QP_n$ is determined by the following table:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$4(2^d - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>1 2 3 4 5</td>
</tr>
<tr>
<td>dim(QP_n)</td>
<td>45 190 480 650 651</td>
</tr>
</tbody>
</table>
Since \( n = 4(2^d - 1) = 2^d + 2^d + 2^{d-1} + 2^{d-1} - 4 \), for \( d \geq 5 \), the theorem follows from Theorem 2.9 and a result in [14]. For \( 1 \leq d \leq 4 \), the proof of this theorem is based on Theorem 2.8 and some results of Kameko [3]. It is long and very technical. The detailed proof of it will be published elsewhere.

**Proof of Main Theorem.** For \( k = 3 \), the theorem follows from the results of Kameko [3]. For \( k = 4 \), it follows from the results in [13,14]. Theorem 3.10 implies immediately this theorem for \( k = 5 \).

Suppose \( k \geq 6 \). Lemma 3.8 implies that \( Q_P(\omega(k,1)) \neq 0 \). Hence,

\[
\dim(Q_P)_k \geq \dim Q_P(\omega(k,1)) + \dim Q_P(\tilde{\omega}(k,1)) > \dim Q_P(\omega(k,1)) = k = c(k,1).
\]

So, the theorem holds for \( d = 1 \).

Now, let \( d > 1 \) and \( \tilde{\omega}(k,d) = ((k-1)(d-2), k-3, k-4, 2) \). Since \( \tilde{\omega}(k,d) \) is weakly decreasing, by Lemma 2.7, \( Q_P(\tilde{\omega}(k,d)) \neq 0 \). We have \( \deg(\omega(k,1)) = \deg(\omega(k,d)) = \deg(\tilde{\omega}(k,d)) = (k-1)(2^d - 1) = n \) and \( (Q_P)_n = \bigoplus_{\deg = n} Q_P(\omega) \). Hence, using Propositions 3.7 and 3.9, we get

\[
\dim(Q_P)_n = \sum_{\deg = n} \dim Q_P(\omega) \geq \dim Q_P(\omega(k,d)) + \dim Q_P(\tilde{\omega}(k,d)) \geq c(k,d).
\]

The theorem is proved.

Denote by \( N(k,n) \) the number of spikes of degree \( n \) in \( P_k \). Note that if \( (i: I) \in N_k \) and \( I \neq \emptyset \), then \( \phi_{i:1}(x) \) is not a spike for any monomial \( x \). Hence, using Propositions 3.7 and 3.9, we easily obtain the following.

**Corollary 3.11. Under the hypotheses of the Main Theorem,**

\[
\dim(Q_P)_n \geq N(k,n) + \sum_{t=2}^p \binom{k}{t} + (k-3) \binom{k}{2} \sum_{u=2}^q \binom{k}{u}.
\]

This corollary implies Mothebe’s result.

**Corollary 3.12. (See Mothebe [4,5]) Under the above hypotheses,**

\[
\dim(Q_P)_n \geq N(k,n) + \sum_{t=2}^p \binom{k}{t}.
\]

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**References**


