C. R. Acad. Sci. Paris, Ser. I 353 (2015) 1035–1040

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Topology

On the generators of the polynomial algebra as a module over the Steenrod algebra

Sur les générateurs de l'algèbre polynomiale comme module sur l'algèbre de Steenrod

Đặng Võ Phúc, Nguyễn Sum

Department of Mathematics, Quy Nhơn University, 170 An Dương Vương, Quy Nhơn, Bình Định, Viet Nam

ARTICLE INFO

Article history:
Received 15 March 2015
Accepted after revision 3 September 2015
Available online 21 October 2015

Presented by the Editorial Board

Keywords:
Steenrod squares
Peterson hit problem
Polynomial algebra

ABSTRACT

Let $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ be the polynomial algebra over the prime field of two elements, $\mathbb{F}_2$, in k variables $x_1, x_2, \ldots, x_k$, each of degree 1. We are interested in the *Peterson hit problem* of finding a minimal set of generators for P_k as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. In this paper, we study the hit problem in degree $(k-1)(2^d-1)$, with d a positive integer. Our result implies the one of Mothebe [4,5].

RÉSUMÉ

Soient $\mathcal{A}$ l'algèbre de Steenrod mod-2 et $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ l'algèbre polynomiale graduée à k générateurs sur le corps à deux éléments $\mathbb{F}_2$, chaque générateur étant de degré 1. Nous étudions le problème suivant soulevé par F. Peterson : déterminer un système minimal de générateurs comme module sur l'algèbre de Steenrod pour P_k, problème appelé *hit problem* en anglais. Dans ce but, nous étudions le *hit problem* en degré $(k-1)(2^d-1)$, avec $d > 0$. Cette solution implique un résultat de Mothebe [4,5].

1. Introduction

Let P_k be the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_k]$, with the degree of each x_i being 1. This algebra arises as the cohomology with coefficients in $\mathbb{F}_2$ of an elementary Abelian 2-group of rank k. Then, P_k is a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The action of $\mathcal{A}$ on P_k is determined by the elementary properties of the Steenrod squares Sq^i and subject to the Cartan formula (see Steenrod and Epstein [12]).

An element g in P_k is called *hit* if it belongs to $\mathcal{A}^+P_k$, where $\mathcal{A}^+$ is the augmentation ideal of $\mathcal{A}$. This means that g can be written as a finite sum $g = \sum_{u \geqslant 0} Sq^{2^u}(g_u)$ for suitable polynomials $g_u \in P_k$.

E-mail addresses: dangphuc150497@gmail.com (Đ. Võ Phúc), nguyensum@qnu.edu.vn (N. Sum).

http://dx.doi.org/10.1016/j.crma.2015.09.002
1631-073X/

We are interested in the *hit problem*, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra P_k as a module over the Steenrod algebra. In other words, we want to find a basis of the $\mathbb{F}_2$-vector space $QP_k := P_k/\mathcal{A}^+P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$.

The hit problem was first studied by Peterson [7], Wood [16], Singer [10], and Priddy [8], who showed its relation to several classical problems respectively in cobordism theory, modular representation theory, the Adams spectral sequence for the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups.

The vector space QP_k was explicitly calculated by Peterson [7] for $k = 1, 2$, by Kameko [3] for $k = 3$, and recently by the second author [13,14] for $k = 4$. From the results of Wood [16] and Kameko [3], the hit problem is reduced to the case of degree n of the form

$$n = s(2^d - 1) + 2^d m, \tag{1.1}$$

where s, d, m are non-negative integers and $1 \leqslant s < k$ (see [14]). For $s = k - 1$ and $m > 0$, the problem was studied by Crabb and Hubbuck [2], Nam [6], Repka and Selick [9], and the second author [13,14].

In the present paper, we study the hit problem in degree n of the form (1.1) with $s = k - 1$, $m = 0$ and d an arbitrary positive integer.

Denote by $(QP_k)_n$ the subspace of QP_k consisting of the classes represented by the homogeneous polynomials of degree n in P_k. From the result of Carlisle and Wood [1] on the boundedness conjecture, one can see that for d big enough, the dimension of $(QP_k)_n$ does not depend on d; it depends only on k. In this paper, we prove the following.

Main Theorem. *Let $n = (k-1)(2^d - 1)$ with d a positive integer and let $p = \min\{k, d\}$, $q = \min\{k, d-1\}$. If $k \geqslant 3$, then*

$$\dim(QP_k)_n \geqslant c(k,d) := \sum_{t=1}^{p} \binom{k}{t} + (k-3)\binom{k}{2}\sum_{u=1}^{q}\binom{k}{u},$$

with equality if and only if either $k = 3$ or $k = 4$, $d \geqslant 5$ or $k = 5$, $d \geqslant 6$.

Note that $c(k,1) = \binom{k}{1} = k$. If $d > k$, then $c(k,d) = \big((k-3)\binom{k}{2} + 1\big)(2^k - 1)$. At the end of Section 3, we show that our result implies Mothebe's result in [4,5].

In Section 2, we recall the definition of an admissible monomial in P_k and Singer's criterion on the hit monomials. Our results will be presented in Section 3.

2. Preliminaries

In this section, we recall some needed information from Kameko [3] and Singer [11], which will be used in the next section.

Notation 2.1. We denote $\mathbb{N}_k = \{1, 2, \ldots, k\}$ and $X_{\mathbb{J}} = X_{\{j_1, j_2, \ldots, j_s\}} = \prod_{j \in \mathbb{N}_k \setminus \mathbb{J}} x_j$, $\mathbb{J} = \{j_1, j_2, \ldots, j_s\} \subset \mathbb{N}_k$. In particular, $X_{\mathbb{N}_k} = 1$, $X_{\emptyset} = x_1 x_2 \ldots x_k$, $X_j = x_1 \ldots \hat{x}_j \ldots x_k$, $1 \leqslant j \leqslant k$, and $X := X_k \in P_{k-1}$.

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a non-negative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots$, for $\alpha_i(a) = 0$ or 1 with $i \geqslant 0$. Set $\alpha(a) = \sum_{i \geqslant 0} \alpha_i(a)$.

Let $x = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} \in P_k$. Denote $\nu_j(x) = a_j$, $1 \leqslant j \leqslant k$. Set $\mathbb{J}_t(x) = \{j \in \mathbb{N}_k : \alpha_t(\nu_j(x)) = 0\}$, for $t \geqslant 0$. Then, we have $x = \prod_{t \geqslant 0} X_{\mathbb{J}_t(x)}^{2^t}$.

Definition 2.2. For a monomial x in P_k, define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots), \quad \sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_k(x)),$$

where $\omega_i(x) = \sum_{1 \leqslant j \leqslant k} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)}$, $i \geqslant 1$. The sequence $\omega(x)$ is called the weight vector of x.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of non-negative integers. The sequence ω is called the weight vector if $\omega_i = 0$ for $i \gg 0$.

The sets of the weight vectors and the sigma vectors are given the left lexicographical order.

For a weight vector ω, we define $\deg \omega = \sum_{i > 0} 2^{i-1}\omega_i$. If there are $i_0 = 0, i_1, i_2, \ldots, i_r > 0$ such that $i_1 + i_2 + \ldots + i_r = m$, $\omega_{i_1 + \ldots + i_{s-1} + t} = b_s$, $1 \leqslant t \leqslant i_s$, $1 \leqslant s \leqslant r$, and $\omega_i = 0$ for all $i > m$, then we write $\omega = (b_1^{(i_1)}, b_2^{(i_2)}, \ldots, b_r^{(i_r)})$. Denote $b_u^{(1)} = b_u$. For example, $\omega = (3, 3, 2, 1, 1, 1, 0, \ldots) = (3^{(2)}, 2, 1^{(3)})$.

Denote by $P_k(\omega)$ the subspace of P_k spanned by monomials y such that $\deg y = \deg \omega$, $\omega(y) \leqslant \omega$, and by $P_k^-(\omega)$ the subspace of P_k spanned by monomials $y \in P_k(\omega)$ such that $\omega(y) < \omega$.

Definition 2.3. Let ω be a weight vector and f, g two polynomials of the same degree in P_k.

i) $f \equiv g$ if and only if $f - g \in \mathcal{A}^+P_k$. If $f \equiv 0$ then f is called hit.
ii) $f \equiv_\omega g$ if and only if $f - g \in \mathcal{A}^+P_k + P_k^-(\omega)$.

Obviously, the relations $\equiv$ and $\equiv_\omega$ are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation $\equiv_\omega$. Then, we have $QP_k(\omega) = P_k(\omega)/((\mathcal{A}^+P_k \cap P_k(\omega)) + P_k^-(\omega))$ and $(QP_k)_n \cong \bigoplus_{\deg\omega=n} QP_k(\omega)$ (see Walker and Wood [15]).

We note that the weight vector of a monomial is invariant under the permutation of the generators x_i, hence $QP_k(\omega)$ has an action of the symmetric group Σ_k.

For a polynomial $f \in P_k(\omega)$, we denote by $[f]_\omega$ the class in $QP_k(\omega)$ represented by f. Denote by $|S|$ the cardinal of a set S.

Definition 2.4. Let x, y be monomials of the same degree in P_k. We say that $x < y$ if and only if one of the following holds:

i) $\omega(x) < \omega(y)$;
ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.5. A monomial x is said to be inadmissible if there exist monomials $y_1, y_2, \dots, y_m$ such that $y_t < x$ for $t = 1, 2, \dots, m$ and $x - \sum_{t=1}^m y_t \in \mathcal{A}^+P_k$.

A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of the admissible monomials of degree n in P_k is a minimal set of $\mathcal{A}$-generators for P_k in degree n. Now, we recall a result of Singer [11] on the hit monomials in P_k.

Definition 2.6. A monomial z in P_k is called a spike if $\nu_j(z) = 2^{d_j} - 1$ for d_j a non-negative integer and $j = 1, 2, \dots, k$. If z is a spike with $d_1 > d_2 > \dots > d_{r-1} \geqslant d_r > 0$ and $d_j = 0$ for $j > r$, then it is called the minimal spike.

In [11], Singer showed that if $\alpha(n+k) \leqslant k$, then there exists uniquely a minimal spike of degree n in P_k.

Lemma 2.7.

i) *All the spikes in P_k are admissible and their weight vectors are weakly decreasing.*
ii) *If a weight vector ω is weakly decreasing and $\omega_1 \leqslant k$, then there is a spike z in P_k such that $\omega(z) = \omega$.*

The proof of this lemma is elementary. The following is a criterion for the hit monomials in P_k.

Theorem 2.8. *(See Singer [11].) Suppose $x \in P_k$ is a monomial of degree n, where $\alpha(n+k) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$, then x is hit.*

The following theorem will be used in the next section.

Theorem 2.9. *(See [13,14].) Let $n = \sum_{i=1}^{k-1}(2^{d_i} - 1)$ with d_i positive integers such that $d_1 > d_2 > \dots > d_{k-2} \geqslant d_{k-1}$, and let $m = \sum_{i=1}^{k-2}(2^{d_i - d_{k-1}} - 1)$. If $d_{k-1} \geqslant k-1 \geqslant 3$, then*

$$\dim(QP_k)_n = (2^k - 1)\dim(QP_{k-1})_m.$$

Note that we correct Theorem 3 in [13] by replacing the condition $d_{k-1} \geqslant k-1 \geqslant 1$ with $d_{k-1} \geqslant k-1 \geqslant 3$.

3. Proof of the Main Theorem

Denote $\mathcal{N}_k = \big\{(i; I); I = (i_1, i_2, \dots, i_r), 1 \leqslant i < i_1 < \dots < i_r \leqslant k,\ 0 \leqslant r < k\big\}$.

Definition 3.1. Let $(i; I) \in \mathcal{N}_k$, let $r = \ell(I)$ be the length of I, and let u be an integer with $1 \leqslant u \leqslant r$. A monomial $x \in P_{k-1}$ is said to be u-compatible with $(i; I)$ if all of the following hold:

i) $\nu_{i_1-1}(x) = \nu_{i_2-1}(x) = \dots = \nu_{i_{(u-1)}-1}(x) = 2^r - 1$,
ii) $\nu_{i_u-1}(x) > 2^r - 1$,
iii) $\alpha_{r-t}(\nu_{i_u-1}(x)) = 1,\ \forall t,\ 1 \leqslant t \leqslant u$,
iv) $\alpha_{r-t}(\nu_{i_t-1}(x)) = 1,\ \forall t,\ u < t \leqslant r$.

Clearly, a monomial x can be u-compatible with a given $(i;I) \in \mathcal{N}_k$ for at most one value of u. By convention, x is 1-compatible with $(i;\emptyset)$.

For $1 \leqslant i \leqslant k$, define the homomorphism $f_i : P_{k-1} \to P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i, \\ x_{j+1}, & \text{if } i \leqslant j < k. \end{cases}$$

Definition 3.2. Let $(i;I) \in \mathcal{N}_k$, $x_{(I,u)} = x_{i_u}^{2^{r-1}+\ldots+2^{r-u}} \prod_{u<t\leqslant r} x_{i_t}^{2^{r-t}}$ for $r = \ell(I) > 0$, $x_{(\emptyset,1)} = 1$. For a monomial x in P_{k-1}, we define the monomial $\phi_{(i;I)}(x)$ in P_k by setting

$$\phi_{(i;I)}(x) = \begin{cases} (x_i^{2^r-1} f_i(x))/x_{(I,u)}, & \text{if there exists } u \text{ such that } x \text{ is } u\text{-compatible with } (i,I), \\ 0, & \text{otherwise.} \end{cases}$$

Then we have an $\mathbb{F}_2$-linear map $\phi_{(i;I)} : P_{k-1} \to P_k$. In particular, $\phi_{(i;\emptyset)} = f_i$.

For a positive integer b, denote $\omega_{(k,b)} = ((k-1)^{(b)})$ and $\bar{\omega}_{(k,b)} = ((k-1)^{(b-1)}, k-3, 1)$.

Lemma 3.3. *(See [14].) Let b be a positive integer and let $j_0, j_1, \ldots, j_{b-1} \in \mathbb{N}_k$. We set $i = \min\{j_0, \ldots, j_{b-1}\}$, $I = (i_1, \ldots, i_r)$ with $\{i_1, \ldots, i_r\} = \{j_0, \ldots, j_{b-1}\} \setminus \{i\}$. Then, we have $\prod_{0\leqslant t<b} X_{j_t}^{2^t} \equiv_{\omega_{(k,b)}} \phi_{(i;I)}(X^{2^b-1})$.*

Definition 3.4. For any $(i;I) \in \mathcal{N}_k$, we define the homomorphism $p_{(i;I)} : P_k \to P_{k-1}$ of algebras by substituting

$$p_{(i;I)}(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i, \\ \sum_{s\in I} x_{s-1}, & \text{if } j = i, \\ x_{j-1}, & \text{if } i < j \leqslant k. \end{cases}$$

Then, $p_{(i;I)}$ is a homomorphism of $\mathcal{A}$-modules. In particular, for $I = \emptyset$, $p_{(i;\emptyset)}(x_i) = 0$ and $p_{(i;I)}(f_i(y)) = y$ for any $y \in P_{k-1}$.

Lemma 3.5. *If x is a monomial in P_k, then $p_{(i;I)}(x) \in P_{k-1}(\omega(x))$.*

Proof. Set $y = p_{(i;I)}\left(x/x_i^{\nu_i(x)}\right)$. Then, y is a monomial in P_{k-1}. If $\nu_i(x) = 0$, then $y = p_{(i;I)}(x)$ and $\omega(y) = \omega(x)$. Suppose $\nu_i(x) > 0$ and $\nu_i(x) = 2^{t_1} + \ldots + 2^{t_c}$, where $0 \leqslant t_1 < \ldots < t_c$, $c \geqslant 1$.

If $I = \emptyset$, then $p_{(i;I)}(x) = 0$. If $I \neq \emptyset$, then $p_{(i;I)}(x)$ is a sum of monomials of the form $\bar{y} := \left(\prod_{u=1}^{c} x_{s_u-1}^{2^{t_u}}\right) y$, where $s_u \in I$, $1 \leqslant u \leqslant c$. If $\alpha_{t_u}(\nu_{s_u-1}(y)) = 0$ for all u, then $\omega(\bar{y}) = \omega(x)$. Suppose there is an index u such that $\alpha_{t_u}(\nu_{s_u-1}(y)) = 1$. Let u_0 be the smallest index such that $\alpha_{t_{u_0}}(\nu_{s_{u_0}-1}(y)) = 1$. Then, we have

$$\omega_i(\bar{y}) = \begin{cases} \omega_i(x), & \text{if } i \leqslant t_{u_0}, \\ \omega_i(x) - 2, & \text{if } i = t_{u_0} + 1. \end{cases}$$

Hence, $\omega(\bar{y}) < \omega(x)$ and $\bar{y} \in P_{k-1}(\omega(x))$. The lemma is proved. $\square$

Lemma 3.5 implies that if ω is a weight vector and $x \in P_k(\omega)$, then $p_{(i;I)}(x) \in P_{k-1}(\omega)$. Moreover, $p_{(i;I)}$ passes to a homomorphism from $QP_k(\omega)$ to $QP_{k-1}(\omega)$. In particular, we have

Lemma 3.6. *(See [14].) Let b be a positive integer and let $(j;J), (i;I) \in \mathcal{N}_k$ with $\ell(I) < b$.*

i) *If $(i;I) \subset (j;J)$, then $p_{(j;J)}\phi_{(i;I)}(X^{2^b-1}) = X^{2^b-1} \mod(P_{k-1}^-(\omega_{(k,b)}))$.*
ii) *If $(i;I) \not\subset (j;J)$, then $p_{(j;J)}\phi_{(i;I)}(X^{2^b-1}) \in P_{k-1}^-(\omega_{(k,b)})$.*

For $0 < h \leqslant k$, set $\mathcal{N}_{k,h} = \{(i;I) \in \mathcal{N}_k : \ell(I) < h\}$. Then, $|\mathcal{N}_{k,h}| = \sum_{t=1}^{h} \binom{k}{t}$.

Proposition 3.7. *Let d be a positive integer and let $p = \min\{k, d\}$. Then, the set $B(d) := \{[\phi_{(i;I)}(X^{2^d-1})]_{\omega_{(k,d)}} : (i;I) \in \mathcal{N}_{k,p}\}$ is a basis of the $\mathbb{F}_2$-vector space $QP_k(\omega_{(k,d)})$. Consequently $\dim QP_k(\omega_{(k,d)}) = \sum_{t=1}^{p} \binom{k}{t}$.*

Proof. Let x be a monomial in $P_k(\omega_{(k,d)})$ and $[x]_{\omega_{(k,d)}} \neq 0$. Then, we have $\omega(x) = \omega_{(k,d)}$. So, there exist $j_0, j_1, \dots, j_{d-1} \in \mathbb{N}_k$ such that $x = \prod_{0\leqslant t<d} X_{j_t}^{2^t}$. According to Lemma 3.3, there is $(i; I) \in \mathcal{N}_k$ such that $x = \prod_{0\leqslant t<d} X_{j_t}^{2^t} \equiv_{\omega_{(k,d)}} \phi_{(i;I)}(X^{2^d-1})$, where $r = \ell(I) < p = \min\{k, d\}$. Hence, $QP_k(\omega_{(k,d)})$ is spanned by the set $B(d)$.

Now, we prove that the set $B(d)$ is linearly independent in $QP_k(\omega_{(k,d)})$. Suppose that there is a linear relation $\sum_{(i;I)\in\mathcal{N}_{k,p}} \gamma_{(i;I)}\phi_{(i;I)}(X^{2^d-1}) \equiv_{\omega_{(k,d)}} 0$, where $\gamma_{(i;I)} \in \mathbb{F}_2$. By induction on $\ell(I)$, using Lemma 3.5 and Lemma 3.6 with $b = d$, we can easily show that $\gamma_{(i;I)} = 0$ for all $(i; I) \in \mathcal{N}_{k,p}$. The proposition is proved. $\square$

Set $C_k = \{x_{j_1}x_{j_2}\dots x_{j_{k-3}}x_j^2 : 1 \leqslant j_1 < j_2 < \dots < j_{k-3} < k,\ \ j_1 \leqslant j < k\} \subset P_{k-1}$. It is easy to see that $|C_k| = (k-3)\binom{k}{2}$.

Lemma 3.8. *C_k is the set of the admissible monomials in P_{k-1} such that their weight vectors are $\bar\omega_{(k,1)} = (k-3, 1)$. Consequently, $\dim QP_{k-1}(\bar\omega_{(k,1)}) = (k-3)\binom{k}{2}$.*

Proof. Let z be a monomial in P_{k-1} such that $\omega(z) = (k-3, 1)$. Then, $z = x_{j_1}x_{j_2}\dots x_{j_{k-3}}x_j^2$ with $1 \leqslant j_1 < j_2 < \dots < j_{k-3} < k$ and $1 \leqslant j < k$. If $z \notin C_k$, then $j < j_1$. Then, we have $z = \sum_{s=1}^{k-3} x_{j_s}^2 x_{j_1}x_{j_2}\dots\hat{x}_{j_s}\dots x_{j_{k-3}}x_j + Sq^1(x_{j_1}x_{j_2}\dots x_{j_{k-3}}x_j)$. Since $x_{j_s}^2 x_{j_1}x_{j_2}\dots\hat{x}_{j_s}\dots x_{j_{k-3}}x_j < z$ for $1 \leqslant s \leqslant k-3$, z is inadmissible.

Suppose that $z \in C_k$. If there is an index s such that $j = j_s$, then z is a spike. Hence, by Lemma 2.7, it is admissible. Assume that $j \neq j_s$ for all s. If z is inadmissible, then there exist monomials $y_1, \dots, y_m$ in P_{k-1} such that $y_t < z$ for all t and $z = \sum_{t=1}^m y_t + \sum_{u\geqslant 0} Sq^{2^u}(g_u)$, where g_u are suitable polynomials in P_{k-1}. Since $y_t < z$ for all t, z is a term of $\sum_{u\geqslant 0} Sq^{2^u}(g_u)$ (recall that a monomial x in P_k is called *a term* of a polynomial f if it appears in the expression of f in terms of the monomial basis of P_k). Based on the Cartan formula, we see that z is not a term of $Sq^{2^u}(g_u)$ for all $u > 0$. If z is a term of $Sq^1(y)$ with y a monomial in P_{k-1}, then $y = x_{j_1}x_{j_2}\dots x_{j_{k-3}}x_j := \tilde y$. So, $\tilde y$ is a term of g_0. Then, we have

$$\bar y := x_{j_1}^2 x_{j_2}\dots x_{j_{k-3}}x_j = \sum_{s=2}^{k-3} x_{j_s}^2 x_{j_1}x_{j_2}\dots\hat{x}_{j_s}\dots x_{j_{k-3}}x_j + \sum_{t=1}^m y_t + Sq^1(g_0 + \tilde y) + \sum_{u\geqslant 1} Sq^{2^u}(g_u).$$

Since $j_1 < j$, we have $y_t < z < \bar y$ for all t. Hence, $\bar y$ is a term of $Sq^1(g_0 + \tilde y) + \sum_{u\geqslant 1} Sq^{2^u}(g_u)$. By an argument analogous to the previous one, we see that $\tilde y$ is a term of $g_0 + \tilde y$. This contradicts the fact that $\tilde y$ is a term of g_0. The lemma is proved. $\square$

Proposition 3.9. *Let d be a positive integer and let $q = \min\{k, d-1\}$. Then, the set $\bar B(d) := \bigcup_{z\in C_k}\{[\phi_{(i;I)}(X^{2^{d-1}-1}z^{2^{d-1}})]_{\bar\omega_{(k,d)}} : (i; I) \in \mathcal{N}_{k,q}\}$ is linearly independent in $QP_k(\bar\omega_{(k,d)})$. If $d > k$, then $\bar B(d)$ is a basis of $QP_k(\bar\omega_{(k,d)})$. Consequently $\dim QP_k(\bar\omega_{(k,d)}) \geqslant (k-3)\binom{k}{2}\sum_{u=1}^q \binom{k}{u}$ with equality if $d > k$.*

Proof. We prove the first part of the proposition. Suppose there is a linear relation $\mathcal{S} := \sum_{((i;I),z)\in\mathcal{N}_{k,q}\times C_k} \gamma_{(i;I),z} \times \phi_{(i;I)}(X^{2^{d-1}-1}z^{2^{d-1}}) \equiv_{\bar\omega_{(k,d)}} 0$, where $\gamma_{(i;I),z} \in \mathbb{F}_2$. We prove $\gamma_{(j;J),z} = 0$ for all $(j; J) \in \mathcal{N}_{k,q}$ and $z \in C_k$. The proof proceeds by induction on $m = \ell(J)$. Let $(i; I) \in \mathcal{N}_{k,q}$. Since $r = \ell(I) < q = \min\{k, d-1\}$, $X^{2^{d-1}-1}z^{2^{d-1}}$ is 1-compatible with $(i; I)$ and $x_i^{2^r-1}f_i(X^{2^{d-1}-1})$ is divisible by $x_{(I,1)}$. Hence, using Definition 3.2, we easily obtain $\phi_{(i;I)}(X^{2^{d-1}-1}z^{2^{d-1}}) = \phi_{(i;I)}(X^{2^{d-1}-1})f_i(z^{2^{d-1}})$. A simple computation shows that if $g \in P_{k-1}^-(\omega_{(k,d-1)})$, then $gz^{2^{d-1}} \in P_{k-1}^-(\bar\omega_{(k,d)})$; if $(i; I) \subset (j; \emptyset)$, then $(i; I) = (j; \emptyset)$; by Lemma 3.5, $p_{(j;\emptyset)}(\mathcal{S}) \equiv_{\bar\omega_{(k,d)}} 0$. Hence, applying Lemma 3.6 with $b = d-1$, we get $p_{(j,\emptyset)}(\mathcal{S}) \equiv_{\bar\omega_{(k,d)}} \sum_{z\in C_k}\gamma_{(j;\emptyset),z}X^{2^{d-1}-1}z^{2^{d-1}} \equiv_{\bar\omega_{(k,d)}} 0$. Since z is admissible in P_{k-1}, $X^{2^{d-1}-1}z^{2^{d-1}}$ is also admissible in P_{k-1}. Hence, the last relation implies $\gamma_{(j;\emptyset),z} = 0$ for all $z \in C_k$. Suppose $0 < m < q$ and $\gamma_{(i;I),z} = 0$ for all $z \in C_k$ and $(i; I) \in \mathcal{N}_{k,q}$ with $\ell(I) < m$. Let $(j; J) \in \mathcal{N}_{k,q}$ with $\ell(J) = m$. Note that by Lemma 3.5, $p_{(j;J)}(\mathcal{S}) \equiv_{\bar\omega_{(k,d)}} 0$; if $(i; I) \in \mathcal{N}_{k,q}$, $\ell(I) \geqslant m$ and $(i; I) \subset (j; J)$, then $(i; I) = (j; J)$. So, using Lemma 3.6 with $b = d-1$ and the inductive hypothesis, we obtain $p_{(j,J)}(\mathcal{S}) \equiv_{\bar\omega_{(k,d)}} \sum_{z\in C_k}\gamma_{(j;J),z}X^{2^{d-1}-1}z^{2^{d-1}} \equiv_{\bar\omega_{(k,d)}} 0$.

From this equality, one gets $\gamma_{(j;J),z} = 0$ for all $z \in C_k$. The first part of the proposition follows.

The proof of the second part is similar to the one of Proposition 3.3 in [14]. However, the relation $\equiv_{\bar\omega_{(k,d)}}$ is used in the proof instead of $\equiv$. $\square$

For $k = 5$, we have the following result.

Theorem 3.10. *Let $n = 4(2^d - 1)$ with d a positive integer. The dimension of the $\mathbb{F}_2$-vector space $(QP_5)_n$ is determined by the following table:*

$n = 4(2^d-1)$	$d=1$	$d=2$	$d=3$	$d=4$	$d\geqslant 5$
$\dim(QP_5)_n$	45	190	480	650	651

Since $n = 4(2^d - 1) = 2^{d+1} + 2^d + 2^{d-1} + 2^{d-1} - 4$, for $d \geqslant 5$, the theorem follows from Theorem 2.9 and a result in [14]. For $1 \leqslant d \leqslant 4$, the proof of this theorem is based on Theorem 2.8 and some results of Kameko [3]. It is long and very technical. The detailed proof of it will be published elsewhere.

Proof of Main Theorem. For $k = 3$, the theorem follows from the results of Kameko [3]. For $k = 4$, it follows from the results in [13,14]. Theorem 3.10 implies immediately this theorem for $k = 5$.

Suppose $k \geqslant 6$. Lemma 3.8 implies that $QP_k(\bar{\omega}_{(k,1)}) \neq 0$. Hence,

$$\dim(QP_k)_{k-1} \geqslant \dim QP_k(\omega_{(k,1)}) + \dim QP_k(\bar{\omega}_{(k,1)}) > \dim QP_k(\omega_{(k,1)}) = k = c(k,1).$$

So, the theorem holds for $d = 1$.

Now, let $d > 1$ and $\widetilde{\omega}_{(k,d)} = ((k-1)^{(d-2)}, k-3, k-4, 2)$. Since $\widetilde{\omega}_{(k,d)}$ is weakly decreasing, by Lemma 2.7, $QP_k(\widetilde{\omega}_{(k,d)}) \neq 0$. We have $\deg(\omega_{(k,d)}) = \deg(\bar{\omega}_{(k,d)}) = \deg(\widetilde{\omega}_{(k,d)}) = (k-1)(2^d - 1) = n$ and $(QP_k)_n \cong \bigoplus_{\deg \omega = n} QP_k(\omega)$. Hence, using Propositions 3.7 and 3.9, we get

$$\begin{aligned}\dim(QP_k)_n &= \sum_{\deg \omega = n} \dim QP_k(\omega) \geqslant \dim QP_k(\omega_{(k,d)}) + \dim QP_k(\bar{\omega}_{(k,d)}) + \dim QP_k(\widetilde{\omega}_{(k,d)}) \\ &> \dim QP_k(\omega_{(k,d)}) + \dim QP_k(\bar{\omega}_{(k,d)}) \geqslant c(k,d).\end{aligned}$$

The theorem is proved. □

Denote by $N(k,n)$ the number of spikes of degree n in P_k. Note that if $(i;I) \in \mathcal{N}_k$ and $I \neq \emptyset$, then $\phi_{(i;I)}(x)$ is not a spike for any monomial x. Hence, using Propositions 3.7 and 3.9, we easily obtain the following.

Corollary 3.11. *Under the hypotheses of the Main Theorem,*

$$\dim(QP_k)_n \geqslant N(k,n) + \sum_{t=2}^{p} \binom{k}{t} + (k-3)\binom{k}{2} \sum_{u=2}^{q} \binom{k}{u}.$$

This corollary implies Mothebe's result.

Corollary 3.12. *(See Mothebe [4,5].) Under the above hypotheses,*

$$\dim(QP_k)_n \geqslant N(k,n) + \sum_{t=2}^{p} \binom{k}{t}.$$

Acknowledgements

We would like to express our warmest thanks to the referee for many helpful suggestions and detailed comments, which have led to the improvement of the paper's exposition. The second author was supported in part by the Research Project Grant No. B2013.28.129 and by the Viet Nam Institute for Advanced Study in Mathematics.

References

[1] D.P. Carlisle, R.M.W. Wood, The boundedness conjecture for the action of the Steenrod algebra on polynomials, in: N. Ray, G. Walker (Eds.), Adams Memorial Symposium on Algebraic Topology 2, Manchester, 1990, in: Lond. Math. Soc. Lect. Note Ser., vol. 176, Cambridge University Press, Cambridge, 1992, pp. 203–216, MR1232207.
[2] M.C. Crabb, J.R. Hubbuck, Representations of the homology of BV and the Steenrod algebra II, in: Algebraic Topology: New Trend in Localization and Periodicity, Sant Feliu de Guíxols, 1994, in: Prog. Math., vol. 136, Birkhäuser Verlag, Basel, Switzerland, 1996, pp. 143–154, MR1397726.
[3] M. Kameko, Products of projective spaces as Steenrod modules, PhD thesis, The Johns Hopkins University, ProQuest LLC, Ann Arbor, MI, 1990, 29 p., MR2638633.
[4] M.F. Mothebe, Generators of the polynomial algebra $\mathbb{F}_2[x_1, \ldots, x_n]$ as a module over the Steenrod algebra, PhD thesis, The University of Manchester, UK, 1997.
[5] M.F. Mothebe, Dimension result for the polynomial algebra $\mathbb{F}_2[x_1, \ldots, x_n]$ as a module over the Steenrod algebra, Int. J. Math. Math. Sci. (2013) 150704, MR3144989.
[6] T.N. Nam, $\mathcal{A}$-générateurs génériques pour l'algèbre polynomiale, Adv. Math. 186 (2004) 334–362, MR2073910.
[7] F.P. Peterson, Generators of $H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$ as a module over the Steenrod algebra, Abstr. Amer. Math. Soc. 833 (April 1987) 55–89.
[8] S. Priddy, On characterizing summands in the classifying space of a group, I, Amer. J. Math. 112 (1990) 737–748, MR1073007.
[9] J. Repka, P. Selick, On the subalgebra of $H_*((\mathbb{R}P^\infty)^n; \mathbb{F}_2)$ annihilated by Steenrod operations, J. Pure Appl. Algebra 127 (1998) 273–288, MR1617199.
[10] W.M. Singer, The transfer in homological algebra, Math. Z. 202 (1989) 493–523, MR1022818.
[11] W.M. Singer, On the action of the Steenrod squares on polynomial algebras, Proc. Amer. Math. Soc. 111 (1991) 577–583, MR1045150.
[12] N.E. Steenrod, D.B.A. Epstein, Cohomology Operations, Ann. Math. Stud., vol. 50, Princeton University Press, Princeton, NJ, 1962, MR0145525.
[13] N. Sum, On the hit problem for the polynomial algebra, C. R. Acad. Sci. Paris, Ser. I 351 (2013) 565–568, MR3095107.
[14] N. Sum, On the Peterson hit problem, Adv. Math. 274 (2015) 432–489, MR3318156.
[15] G. Walker, R.M.W. Wood, Weyl modules and the mod 2 Steenrod algebra, J. Algebra 311 (2007) 840–858, MR2314738.
[16] R.M.W. Wood, Steenrod squares of polynomials and the Peterson conjecture, Math. Proc. Camb. Philos. Soc. 105 (1989) 307–309, MR0974986.