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Dynamical systems

On the hyperbolicity of C¹-generic homoclinic classes



Sur l'hyperbolicité des classes homoclines C¹-génériques

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ARTICLE INFO

Article history: Received 8 December 2014 Accepted 10 July 2015 Available online 21 October 2015

Presented by Étienne Ghys

ABSTRACT

The works of Liao, Mañé, Franks, Aoki, and Hayashi characterized a lack of hyperbolicity for diffeomorphisms by the existence of weak periodic orbits. In this note, we announce a result that can be seen as a local version of these works: for C^1 -generic diffeomorphisms, a homoclinic class either is hyperbolic or contains a sequence of periodic orbits that have a Lyapunov exponent arbitrarily close to 0.

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RÉSUMÉ

Des travaux de Liao, Mañé, Franks, Aoki et Hayashi ont caractérisé le manque d'hyperbolicité des difféomorphismes par l'existence d'orbites périodiques faibles. Dans cette note, nous annonçons un résultat qui peut être considéré comme une version locale de ces travaux : pour les difféomorphismes C¹-génériques, une classe homocline, ou bien est hyperbolique, ou bien contient une suite d'orbites périodiques qui ont un exposant de Lyapunov arbitrairement proche de 0.

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1. Introduction

It is known for a long time that hyperbolic invariant compact sets have many nice properties, like shadowing properties, stability properties, the existence of uniform stable and unstable manifolds, etc. So it is important to understand dynamics beyond hyperbolicity and to characterize hyperbolicity. The first works in this direction were devoted to the stability conjecture, which tells that hyperbolic diffeomorphisms are the only ones that are Ω -stable. For surface diffeomorphisms, this conjecture has been solved independently by Liao and Mañé in [12] and [13]. In their proofs, Liao's selecting lemma and Mañé's ergodic closing lemma played an important role. For higher dimensions, Mañé solved it in [14]. From [7], it is easy to see that a C¹ Ω -stable diffeomorphism f satisfies the *star condition*: there is a C¹-neighborhood \mathcal{U} of f, such that any $g \in \mathcal{U}$ has no non-hyperbolic periodic point. Mañé conjectured that if a diffeomorphism satisfied the star condition, then it is hyperbolic, that is to say, its chain recurrent set is hyperbolic. This conjecture was proved by Aoki and Hayashi, see [1] and [10]. Then one would ask the following question naturally, which is a local version of this conjecture. Recall that a

http://dx.doi.org/10.1016/j.crma.2015.07.017

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homoclinic class H(p) of a hyperbolic periodic point p is the closure of the union of hyperbolic periodic orbits that are homoclinically related to orb(p).

Question 1 (Problem 1.8 in [6]). For C^1 -generic $f \in \text{Diff}^1(M)$, if a homoclinic class H(p) is not hyperbolic, does it contain periodic orbits that have a Lyapunov exponent arbitrarily close to 0?

The works of Liao, Mañé imply the existence of weak periodic orbits close to a non-hyperbolic homoclinic class for C^1 -generic diffeomorphisms. The difficulty of Question 1 is to link the weak periodic orbits to the homoclinic class. More precisely, if H(p) is not hyperbolic, we can get weak periodic orbits arbitrarily close to it by perturbation with the classical arguments, but we do not know whether they are contained in the homoclinic class. In this paper, we can prove that, generically, they are in fact contained in the homoclinic class.

2. Precise statements

Let *M* be a compact connected smooth Riemannian manifold without boundary, of dimension *d*. Denote by Diff¹(*M*) the space of C¹-diffeomorphisms from *M* to *M*. For a diffeomorphism $f \in \text{Diff}^1(M)$, for any number $\varepsilon > 0$, we call a sequence of points $\{x_i\}_{i=a}^b$ an ε -pseudo orbit of *f*, if $d(f(x_i), x_{i+1}) < \varepsilon$ for any $i = a, a + 1, \dots, b - 1$, where $-\infty \le a < b \le \infty$. We say that *y* is a *chain-attainable* form *x*, denoted by $x \dashv y$, if for any $\varepsilon > 0$, there is an ε -pseudo orbit $\{x = x_0, x_1, \dots, x_n = y\}$ of *f*. The *chain-recurrent set* of a diffeomorphism *f*, denoted by R(f), is the union of points that are chain attainable from itself. We say that two points *x*, *y* are *chain related*, denoted by $x \sim y$, if $x \dashv y$ and $y \dashv x$. The relation \sim is an equivalent relation to R(f), and every equivalent class of \sim is called a *chain-recurrence class*. For a point $x \in R(f)$, denote by C(x, f) the chain recurrence class that contains *x*.

Let Λ be an invariant compact set. We say Λ is *hyperbolic* if there is a continuous splitting $T_{\Lambda}M = E^s \oplus E^u$, such that E^s is *contracted* and E^u is *expanded*, that is to say, there are two constants C > 0 and $\lambda \in (0, 1)$, such that, for any $x \in \Lambda$ and any integer $n \in \mathbb{N}$, we have $\|Df^n|_{E(x)}\| < C\lambda^n$ and $\|Df^{-n}|_{F(x)}\| < C\lambda^n$. If the orbit of a periodic point p is hyperbolic, then we call p a *hyperbolic periodic point*, and the dimension of E^s is called the *index* of p, denoted by ind(p).

The set Λ is said to have a *dominated splitting* if there are a continuous splitting $T_{\Lambda}M = E \oplus F$, an integer $m \in \mathbb{N}$ and a constant $\lambda \in (0, 1)$, such that $\|Df^m|_{E(X)}\| \cdot \|Df^{-m}|_{F(f^m_X)}\| < \lambda$ for all $x \in \Lambda$. Sometimes, we call a dominated splitting associated with the two numbers m and λ an (m, λ) -dominated splitting.

For an ergodic measure μ of f, there are d numbers $\chi_1 \le \chi_2 \le \cdots \times \chi_d$, such that, for μ -almost every point $x \in M$, and for any non-zero vector $v \in T_x M$, one has $\lim_{n\to+\infty} \frac{1}{n} \log \|Df^n(v)\| = \chi_i$ for some $i = 1, 2, \cdots, d$. These numbers are called the *Lyapunov exponents* of the measure μ . Particularly, we call the Lyapunov exponents of the Dirac measure of a periodic orbit the Lyapunov exponents of the periodic orbit. Hence, a periodic point p is hyperbolic if and only if all the Lyapunov exponents of orb(p) are non-zero.

For any point $x \in M$, any number $\delta > 0$, we define the *local stable manifold* and *local unstable manifold* of x of size δ , respectively, as follows:

$$W^{s}_{\delta}(x) = \{y : d(f^{n}(x), f^{n}(y)) \le \delta, \forall n \ge 0; \text{ and } \lim_{n \to +\infty} d(f^{n}(x), f^{n}(y)) = 0\}; \\ W^{u}_{\delta}(x) = \{y : d(f^{-n}(x), f^{-n}(y)) \le \delta, \forall n \ge 0; \text{ and } \lim_{n \to +\infty} d(f^{-n}(x), f^{-n}(y)) = 0\},$$

and the stable manifold and unstable manifold of x respectively as follows:

$$W^{s}(x) = \{y : \lim_{n \to +\infty} d(f^{n}(x), f^{n}(y)) = 0\};$$

$$W^{u}(x) = \{y : \lim_{n \to +\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}.$$

By [11], for a hyperbolic invariant compact set Λ of f, there is a number $\delta > 0$, such that for any $x \in \Lambda$, the local stable manifold $W^s_{\delta}(x)$ of x is an embedding disk with dimension dim (E^s) and is tangent to E^s at x, where $T_{\Lambda}M = E^s \oplus E^u$ is the hyperbolic splitting. Moreover, the stable manifold $W^s(x)$ of x is an immersing submanifold of M. Symmetrically, we have similar statements for $W^u_{\delta}(x)$ and $W^u(x)$.

Two hyperbolic periodic points p and q of f are called *homoclinic related*, if their stable and unstable manifolds respectively intersect transversely, that is to say, $W^u(\operatorname{orb}(p)) \pitchfork W^s(\operatorname{orb}(q)) \neq \emptyset$ and $W^s(\operatorname{orb}(p)) \pitchfork W^u(\operatorname{orb}(q)) \neq \emptyset$. For a hyperbolic periodic point p, the closure of the set of periodic points that are homoclinically related to p is called the *homoclinic class* of p, denoted by H(p). Also, it is well known that H(p) is the closure of all transverse intersections of its stable and unstable manifolds, that is to say, $H(p) = \overline{W^u(\operatorname{orb}(p))} \pitchfork W^s(\operatorname{orb}(p))$.

For an invariant compact set Λ of f, a Df-invariant sub-bundle $E \subset T_{\Lambda}M$, an integer $m \in \mathbb{N}$, and any number $\lambda \in (0, 1)$, we call $x \in \Lambda$ an (m, λ) -E-Pliss point, if $\prod_{i=0}^{n-1} \|Df^{im}|_{E(f^{im}(X))}\| \leq \lambda^n$, for any integer n > 0. If Λ does not contain any (m, λ) -E-Pliss point, we call Λ an (m, λ) -E-weak set. We call two (m, λ) -E-Pliss points $(f^n(x), f^l(x))$ on a single orbit consecutive (m, λ) -E-Pliss points, if n < l and for all n < k < l, $f^k(x)$ is not an (m, λ) -E-Pliss point. And if there is a dominated splitting $T_{\Lambda}M = E \oplus F$ on Λ , we call $x \in \Lambda$ an (m, λ) -b-Pliss point, if it is an (m, λ) -E-Pliss point for f and an (m, λ) -F-Pliss point or λ -E-Pliss point for f and an (m, λ) -F-Pliss point for f^{-1} . If m = 1, we will just write λ -E-Pliss point or λ -E-weak set.

A subset *R* of a topological space *X* is called a *residual* set, if *R* contains a dense G_{δ} set of *X*. We say that a property is a *generic* property of *X*, if there is a residual set $R \subset X$, such that each element contained in *R* satisfies the property. We now announce an answer to Question 1.

Main Theorem. For C^1 -generic $f \in \text{Diff}^1(M)$, a homoclinic class H(p) either is hyperbolic, or contains periodic orbits with arbitrarily long periods that are homoclinically related to $\operatorname{orb}(p)$ and have a Lyapunov exponent arbitrarily close to 0.

We point out that the Main Theorem is not true without the genericity assumption. Refs. [16,4] construct a nonhyperbolic homoclinic class which is uniformly hyperbolic in the measure sense: all Lyapunov exponents of all invariant measures supported on the homoclinic class are uniformly bounded away from 0.

From [9] and Lemma II.3 of [13], we have the fact that for generic $f \in \text{Diff}^1(M)$, if all Lyapunov exponents of periodic orbits that are homoclinically related to $\operatorname{orb}(p)$ are uniformly away from 0, then H(p) has a dominated splitting $T_{H(p)}M = E \oplus F$, with dim $E = \operatorname{ind}(p)$. Our main theorem is thus a consequence of the following theorem.

Theorem A. For C^1 -generic $f \in \text{Diff}^1(M)$, assume that p is a hyperbolic periodic point of f. If the homoclinic class H(p) has a dominated splitting $T_{H(p)}M = E \oplus F$, with dim $E \leq \text{ind}(p)$, such that the bundle E is not contracted, then there are periodic orbits in H(p) with index dim(E) and with arbitrarily long periods whose maximal Lyapunov exponent along E is arbitrarily close to 0.

Ref. [3] obtains the conclusion of Theorem A under the hypothesis that the bundle F is expanded, but without the genericity assumption. By a standard argument, we can control the norm of the product by controlling the product of the norm with the perturbations. Thus to prove the main theorem, we only have to prove the following.

Theorem B. For C^1 -generic $f \in \text{Diff}^1(M)$, assume that p is a hyperbolic periodic point of f and that the homoclinic class H(p) has a dominated splitting $T_{H(p)}M = E \oplus F$, with dim $E \leq \text{ind}(p)$, such that the bundle E is not contracted. Then there are a constant $\lambda_0 \in (0, 1)$, and an integer $m_0 \in \mathbb{N}$, satisfying: for any $m \in \mathbb{N}$ with $m \geq m_0$, any constants $\lambda_1, \lambda_2 \in (\lambda_0, 1)$ with $\lambda_1 < \lambda_2$, there is a sequence of periodic orbits $\mathcal{O}_k = \text{orb}(q_k)$ with period $\tau(q_k)$ contained in H(P), such that

$$\lambda_1^{\tau(q_k)} < \prod_{0 \le i < \tau(q_k)/m} \|Df^m|_{E(f^{im}(q_k))}\| < \lambda_2^{\tau(q_k)}$$

In the next sections, we give a sketch of the proof of the above theorem.

3. Existence of a bi-Pliss point accumulating backward to an E-weak set

We assume that f is a C¹-generic diffeomorphism in Diff¹(M) and H(p) is a homoclinic class of f that satisfies the hypothesis of Theorem B. We can choose two numbers $\lambda_0 \in (0, 1)$ and $m_0 \in \mathbb{N}$, such that, for any $m \ge m_0$, the splitting $E \oplus F$ is (m, λ_0) -dominated, and for the hyperbolic periodic orbit orb(p),

$$\prod_{0 \leq i < \tau(p)/m_0} \|Df^{m_0}|_{E(f^{im_0}(p))}\| < \lambda_0^{\tau(p)/m_0},$$

where $\tau(p)$ is the period of orb(p). In the following, we fix $m \ge m_0$. In order to simplify the notations, we will assume that m = 1 and that p is a fixed point of f, but the general case is identical.

Since the bundle *E* is not contracted, there is a point $b \in H(p)$ such that, for any $n \ge 1$, we have $\prod_{i=0}^{n-1} \|Df|_{E(f^i(b))}\| \ge 1$. For any number $\lambda \in (\lambda_0, 1)$, by Liao's selecting lemma (see [12,18]), there is a λ -*E*-weak set contained in H(p) (otherwise, we will get periodic orbits that satisfy the conclusions of Theorem B). Now we fix three numbers $\lambda_1 < \lambda_2 < \lambda_3 \in (\lambda_0, 1)$, take the closure of the union of all the λ_2 -*E*-weak sets and denote it by \hat{K} . Then there are two cases: either \hat{K} is a λ_2 -*E*-weak set or not. With the arguments related to the Pliss lemma [15] and the selecting lemma [12,18], we can get the following lemma under the hypothesis of Theorem B.

Lemma 1. There are a λ_2 -*E*-weak set $K \subset H(p)$, a λ_3 -bi-Pliss point $x \in H(p) \setminus K$ satisfying: $\alpha(x) = K$.

4. The perturbation to make $W^{u}(p)$ accumulate to the weak set K

Since the λ_2 -*E*-weak set *K* is contained in H(p), and $W^u(p)$ is dense in H(p), with the technics in the proof of Proposition 10 in [5], we can prove that there is a point on $W^u(p)$ that accumulates the weak set *K* for a diffeomorphism g_1 , that is that C¹ close to *f* and coincides with *f* on *K*, orb(*p*) and the backward orbit of *x*. Moreover, the key point here is that, by the generic assumption of *f*, we can assure that there is a λ_3 -*E*-Pliss point close to *x* that is contained in the chain-recurrence class of *p* for g_1 ; hence *K* is still chain related with *p* for g_1 . More precisely, we have the following lemma.

Lemma 2. There is a residual set $\mathcal{R} \subset \text{Diff}^1(M)$, such that, if $f \in \mathcal{R}$ and satisfy the hypothesis of *Theorem B*, then for any neighborhood \mathcal{U} of f in $\text{Diff}^1(M)$, there are a diffeomorphism $g_1 \in \mathcal{U}$ and a point $y \in M$, such that,

1. $y \in W^u(p, f)$ and g_1 coincides with f on the set $K \cup \operatorname{orb}(p) \cup \operatorname{orb}^-(y)$, hence $y \in W^u(p, g_1)$, 2. $\omega(y, g_1) \subset K \subset \mathcal{C}(p, g_1)$.

5. The perturbations to get a heteroclinic connection between *p* and *K*

Denote $K_0 = \omega(y, g_1) \subset K$, then for the diffeomorphism g_1 , K_0 is a λ_2 -*E*-weak set, and the orbit of *y* connects the hyperbolic fixed point *p* to K_0 . By two additional perturbations, we can obtain furthermore an orbit connecting K_0 to *p*. First, since $K_0 \subset C(p, g_1)$, with the technics for the connecting of pseudo-orbits in [2,5], we can connect K_0 by a true orbit to any neighborhood of *p* by a C¹ small perturbation. Then, by the hyperbolicity of *p*, we use the uniform connecting lemma (see [17,19]) to "push" this orbit inside the stable manifold of *p*. In these two steps, the orbit orb(*y*) that connects *p* to K_0 is not changed. We point out that the proof here is delicate (in fact the most delicate part of the whole proof) and one has to go back in the arguments of [2,5].

Lemma 3. For the diffeomorphism $f \in \mathcal{R}$, for any neighborhood \mathcal{U} of f in Diff¹(M), there are a diffeomorphism $g_2 \in \mathcal{U}$ and two points $y, y' \in M$, such that,

- 1. $y \in W^u(p, g_2)$ and $\omega(y, g_2) \subset K$,
- 2. $y' \in W^{s}(p, g_2)$ and $\alpha(y', g_2) \subset \omega(y, g_2)$,
- 3. g_2 coincides with f on the set $\omega(y, g_2) \cup \operatorname{orb}(p)$.

6. Last perturbation to get a weak periodic orbit

Now we have obtained heteroclinic connections between the hyperbolic fixed point p and a subset $K_0 = \omega(y, g_2)$ of the weak set K. Then using the connecting lemma, we can get a periodic orbit that spends a given proportion of time close to orb(p) and K_0 by a C^1 small perturbation. More precisely, the periodic orbit that we get spends a long time close to the weak set K_0 , and spends another long time (which can be controlled) close to p, hence the average of the product of the norm along the bundle E of this periodic orbit is larger than λ_1 (controlled by the norm of points close to K_0) and smaller than λ_2 (modified by the norm of points close to p). The key point in the connecting process is that, for the hyperbolic fixed point p, and the two points y and y', by the λ -Lemma, there are a number l and two small neighborhoods U_y and $U_{y'}$ of y and y', respectively, such that, for any $n \ge l$, there is an orbit segment with length n that connects $U_{y'}$ to U_y , and moreover, only the two endpoints of the segment are contained in $U_{y'} \cup U_y$, and the other part of the segment is close to the point p.

Lemma 4. For the diffeomorphism $f \in \mathcal{R}$, for any neighborhood \mathcal{U} of f in Diff¹(M), for any integer L > 0, any neighborhood U_p of p, there is $g \in \mathcal{U}$, such that, $g = f|_{orb(p)}$ and g has a periodic orbit $\mathcal{O} = orb(q)$ with period $\tau > L$ such that $q \in U_p$ and

$$\lambda_1^{\tau} \leq \prod_{0 \leq i \leq \tau-1} \|Dg|_{E(g^i(q))}\| \leq \lambda_2^{\tau}.$$

Finally, by a standard Baire argument (see for example [8]), for the C^1 -generic diffeomorphism f, there is a sequence of periodic orbits that are homoclinically related with each other and accumulates to a subset of H(p), and the product of the norms along the bundle E of these periodic orbits satisfies the inequality in Lemma 4. Therefore, these periodic orbits are contained in H(p). This finishes the proof of Theorem B.

Acknowledgements

The author is grateful to Sylvain Crovisier for many useful suggestions both in solving the problem and in the writing and to Lan Wen for carefully listening to the proof and for the useful comments. The author would like to thank Shaobo Gan, Dawei Yang and Rafael Potrie for listening to the proof and for the helpful discussions. The author would also like to thank University Paris-Sud 11 for the hospitality and China Scholarship Council (CSC) for financial support (201306010008).

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