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Dynamical systems

Exponential decay of correlations for a real-valued dynamical system embedded in $\mathbb{R}^2$

Décroissance des corrélations pour une récurrence à deux termes

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ABSTRACT

We study the real valued process $\{X_t, t \in \mathbb{N}\}$ defined by $X_{t+2} = \varphi(X_t, X_{t+1})$, where the X_t are bounded. We aim at proving the decay of correlations for this model, under regularity assumptions on the transformation φ.

RÉSUMÉ

On étudie le processus réel $\{X_t, t \in \mathbb{N}\}$ défini par $X_{t+2} = \varphi(X_t, X_{t+1})$, les X_t étant bornés. Sous des hypothèses de régularité sur la transformation φ, on établit la décroissance des corrélations pour ce modèle.

1. Introduction

Since the 1980s, the study by statisticians of nonlinear time series has allowed one to model a great number of phenomena in Physics, Economics, and Finance [5], [6]. Then, in the 1990s, the theory of Chaos became an essential axis of research for the study of these processes [5]. For an exhaustive review on this subject, one can consult Collet–Eckmann [2] about chaos theory and Chan–Tong [8,9] about nonlinear time series. Within this framework, a general model could be written as

$$X_{t+1} = \varphi(X_t, \dots, X_{t-d+1}) + \varepsilon_t,$$

where φ is nonlinear and ε_t is a noise. We propose a first study of the "skeleton" of this model, as Tong calls it, beginning with $d = 2$ and, more precisely, of the dynamical system induced by this model. Indeed, we consider the model with bounded variables, $X_{t+2} = \varphi(X_t, X_{t+1})$, with $\varphi : \mathcal{U}^2 \to \mathcal{U}$ for $\mathcal{U} = [-L, L]$ and $L \in \mathbb{R}_+^*$, φ being defined piecewisely on $\mathcal{U}^2$. This model gives rise to a dynamical system (Ω, τ, μ, T) where μ is a measure on the σ-algebra τ, invariant under the transformation $T : \Omega \to \Omega$ and Ω is a compact subset of $\mathbb{R}^2$. Under hypotheses on φ, which imply that T satisfies the hypotheses of Saussol [7], and if we suppose that T is mixing, we obtain the exponential decay of correlations. More precisely, for well-chosen applications f and h, there exist constants $C = C(f, h) > 0$, $0 < \rho < 1$ such that:

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$$\left|\int_\Omega f\circ T^k h\,\mathrm{d}\mu - \int_\Omega f\,\mathrm{d}\mu \int_\Omega h\,\mathrm{d}\mu\right| \leqslant C\,\rho^k.$$

This result can be seen as a covariance inequality of the following kind:

$$|\,\mathrm{Cov}(\,f(X_k), h(X_0)\,)\,| \leqslant C\,\rho^k.$$

Other ways could certainly be used to get the same result, under different hypotheses on the induced system, for example the method of Young's towers [10]. To have a general view on these different technics, one can read the article of Alves–Freitas–Luzzato–Vaienti [1] and [3], [4], [6].

2. Hypotheses and results

Let $L \in \mathbb{R}_+^*$. Let $\varphi : [-L, L]^2 \to [-L, L]$ be piecewisely defined on[1] $[-L, L]^2$. To study the process $\{X_t, t \in \mathbb{N}\}$ defined by $X_{t+2} = \varphi(X_t, X_{t+1})$, there exist different ways of choosing the induced dynamical system $Z_{t+1} = T(Z_t)$ with $Z_t \in \mathbb{R}^2$. We tried two different approaches, on the one hand the canonical method, setting $T(x, y) = (y, \varphi(x, y))$, and on the other hand a double iteration, which comes down to setting $T(x, y) = (\varphi(x, y), \varphi(y, \varphi(x, y)))$. The first approach, up to a conjugation, is the most fruitful, the second one requiring stronger hypotheses and yielding weaker results. We therefore set $T(x, y) = (\frac{y}{\gamma}, \gamma\varphi(x, \frac{y}{\gamma}))$ with $Z_t = (X_t, \gamma X_{t+1})$, for a suitable positive γ. It then became possible to work in spaces similar to Saussol's V_α and to use his results.

More precisely, we suppose that the following hypotheses are fulfilled.

(H1) There exists $d \in \mathbb{N}^*$ such that $[-L, L]^2 = \bigcup_{k=1}^{d} O_k \cup \mathcal{N}$, where the O_k are nonempty open sets, $\mathcal{N}$ is negligible for the Lebesgue measure and the union is disjoint. The edges of the O_k can be split into a finite number of smooth components, each one included in a C^1, compact and one-dimensional submanifold of $\mathbb{R}^2$.

(H2) There exists $\varepsilon_1 > 0$ such that, for all $k \in \{1, \dots d\}$, there exists an application φ_k defined on $B_{\varepsilon_1}(\overline{O_k}) = \{(x, y) \in \mathbb{R}^2,\ d((x, y), \overline{O_k}) \le \varepsilon_1\}$, with values in $\mathbb{R}$, such that $\varphi_k|_{O_k} = \varphi|_{O_k}$.

(H3) The application φ_k is bounded, belongs to the Hölder class $C^{1,\alpha}$ on $B_{\varepsilon_1}(\overline{O_k})$ for a real $\alpha \in]0, 1]$.[2]

We moreover suppose that there exist $A > 1$ and $M \in]0, A - 1[$ such that:

$$\forall (u, v) \in B_{\varepsilon_1}(\overline{O_k}), \qquad \left|\frac{\partial \varphi_k}{\partial u}(u, v)\right| \ge A, \quad \left|\frac{\partial \varphi_k}{\partial v}(u, v)\right| \le M,$$

to ensure the expanding properties.

(H4) The open sets O_k satisfy the following geometrical condition[3]: for all (u, v) and (u', v) in $B_{\varepsilon_1}(\overline{O_k})$, there exists a C^1 path $\Gamma = (\Gamma_1, \Gamma_2) : [0, 1] \to B_{\varepsilon_1}(\overline{O_k})$ C^1 joining (u, v) and (u', v), whose gradient does not vanish, and which satisfies

$$\forall t \in]0, 1[,\ \left|\Gamma_1'(t)\right| > \frac{M}{A}\left|\Gamma_2'(t)\right|.$$

(H5) Let $Y \in \mathbb{N}^*$ be the maximal number of C^1 components of $\mathcal{N}$ meeting at one point and set

$$s = \left(\frac{2A + M^2 - M\sqrt{M^2 + 4A}}{2}\right)^{-1/2} < 1.$$

One supposes that

$$\eta := s^\alpha + \frac{8s}{\pi(1-s)} Y < 1.$$

[1] To get similar results on $[a, b]$ instead of $[-L, L]$, it suffices to conjugate by an affine application.

[2] If φ_k is C^2 on $B_{\varepsilon_1}(\overline{O_k})$, it is $C^{1,\alpha}$ on $B_{\varepsilon_1}(\overline{O_k})$ with $\alpha = 1$.

[3] In suitable cases, this hypothesis can be replaced by a weaker but simpler one: for all points (u, v) and (u', v) in $B_{\varepsilon_1}(\overline{O_k})$, the segment $[(u, v), (u', v)]$ is included in $B_{\varepsilon_1}(\overline{O_k})$.

We set $\gamma = \frac{1}{\sqrt{A}} < 1$ and, for all $k \in \{1, \ldots, d\}$, we denote by U_k (resp. W_k, $\mathcal{N}'$) the image of O_k (resp. $B_{\varepsilon_1}(\overline{O_k})$, $\mathcal{N}$) under the compression that associates $(u, \gamma v)$ with each $(u, v) \in \mathbb{R}^2$. The set $\Omega = [-L, L] \times [-\gamma L, \gamma L]$, on which we shall be working, is the image of $[-L, L]^2$ under the same compression.

For every non-negligible Borel set S of $\mathbb{R}^2$, for every $f \in L^1_m(\mathbb{R}^2, \mathbb{R})$, set

$$\mathrm{Osc}(f, S) = Esup_S f - Einf_S f,$$

where $Esup_S$ and $Einf_S$ are the essential supremum and infimum with respect to the Lebesgue measure m. One then defines:

$$|f|_\alpha = \sup_{0<\varepsilon<\varepsilon_1} \varepsilon^{-\alpha} \int_{\mathbb{R}^2} \mathrm{Osc}(f, B_\varepsilon(x, y))\, \mathrm{d}x\, \mathrm{d}y\ ,\ \|f\|_\alpha = \|f\|_{L^1_m} + |f|_\alpha$$

and the set $V_\alpha = \{f \in L^1_m(\mathbb{R}^2, \mathbb{R}),\ \|f\|_\alpha < +\infty\}$.

Let us introduce similar notions on Ω: for every $0 < \varepsilon_0 < \gamma\varepsilon_1$, for every $g \in L^\infty_m(\Omega, \mathbb{R})$, one defines

$$N(g, \alpha, L) = \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\alpha} \int_\Omega \mathrm{Osc}(g, B_\varepsilon(x, y) \cap \Omega)\, \mathrm{d}x\, \mathrm{d}y.$$

One then sets:

$$||g||_{\alpha,L} = N(g, \alpha, L) + 16(1 + \gamma)\varepsilon_0^{1-\alpha} L||g||_\infty + ||g||_{L^1_m}.$$

The function g is said to belong to $V_\alpha(\Omega)$ if the above expression is finite. The set $V_\alpha(\Omega)$ does not depend on the choice of ε_0, whereas N and $\|.\|_{\alpha,L}$ do.

There exist relationships between these two sets. Indeed, thanks to Proposition 3.4 of [7], one can prove the following result.

Proposition 2.1.

(i) *If $g \in V_\alpha(\Omega)$ and if one extends g as a function denoted by f, setting $f(x, y) = 0$ if $(x, y) \notin \Omega$, then $f \in V_\alpha$ and*

$$\|f\|_\alpha \le \|g\|_{\alpha,L}.$$

(ii) *Let f be in V_α. Set $g = f\mathbf{1}_\Omega$. Then $g \in V_\alpha(\Omega)$ and one has*

$$\|g\|_{\alpha,L} \le \left(1 + 16(1 + \gamma)L \frac{\max(1, \varepsilon_0^\alpha)}{\pi\, \varepsilon_0^{1+\alpha}}\right) \|f\|_\alpha.$$

Under the above hypotheses (H1) to (H5), one obtains a first result.

Theorem 2.2. *Let T be the transformation defined on Ω by: $\forall (x, y) \in U_k$:*

$$T(x, y) = T_k(x, y) = \left(\frac{y}{\gamma},\ \gamma\varphi_k(x, \frac{y}{\gamma})\right).$$

Keeping the same formula, one extends the definition of T_k to W_k. Then

(i) *the Frobenius–Perron operator $P : L^1_m(\Omega) \to L^1_m(\Omega)$ associated with T has a finite number of eigenvalues $\lambda_1, \ldots, \lambda_r$ of modulus one;*

(ii) *for each $i \in \{1, \ldots, r\}$, the eigenspace $E_i = \{f \in L^1_m(\Omega)\ :\ Pf = \lambda_i f\}$ associated with the eigenvalue λ_i is finite dimensional and included in $V_\alpha(\Omega)$;*

(iii) *the operator P decomposes as*

$$P = \sum_{i=1}^{r} \lambda_i P_i + Q\,,$$

where the P_i are projections on the spaces E_i, $|||P_i|||_1 \le 1$ and Q is a linear operator defined on $L^1_m(\Omega)$, satisfying $Q(V_\alpha(\Omega)) \subset V_\alpha(\Omega)$, $\sup_{n\in\mathbb{N}^*} |||Q^n|||_1 < \infty$ and $|||Q^n|||_{\alpha,L} = O(q^n)$ when $n \to +\infty$ for an exponent $q \in]0, 1[$. Moreover, $P_i P_j = 0$ if $i \neq j$, $P_i Q = Q P_i = 0$ for all i;

(iv) *the number 1 is an eigenvalue of P. Set $\lambda_1 = 1$, let $h_* = P_1\mathbf{1}_\Omega$ and let $\mathrm{d}\mu = h_*\, \mathrm{d}m$. Then μ is the greatest absolutely continuous invariant measure (ACIM) of T, that is to say: if $\nu << m$ and if ν is T-invariant, then $\nu << \mu$;*

(v) *the support of* μ *can be decomposed into a finite number of disjoint measurable sets, on which a power of* T *is mixing. More precisely, for all* $j \in \{1, 2, \dots, \dim(E_1)\}$*, there exist an integer* $L_j \in \mathbb{N}^*$ *and* L_j *disjoint sets* $W_{j,l}$ $(0 \le l \le L_j - 1)$ *satisfying* $T(W_{j,l}) = W_{j,l+1 \bmod (L_j)}$ *and* T^{L_j} *is mixing on every* $W_{j,l}$*. We denote by* $\mu_{j,l}$ *the normalized restriction of* μ *to* $W_{j,l}$*, defined by*

$$\mu_{j,l}(B) = \frac{\mu(B \cap W_{j,l})}{\mu(W_{j,l})},\ \mathrm{d}\mu_{j,l} = \frac{h^* \mathbf{1}_{W_{j,l}}}{\mu(W_{j,l})} \mathrm{d}m.$$

The fact that T^{L_j} *is mixing on every* $W_{j,l}$ *means that, for all* $f \in L^1_{\mu_{j,l}}(W_{j,l})$ *and all* $h \in L^\infty_{\mu_{j,l}}(W_{j,l})$*,*

$$\lim_{t \to +\infty} < T^{tL_j} f, h >_{\mu_{j,l}} = < f, 1 >_{\mu_{j,l}} < 1, h >_{\mu_{j,l}}$$

with the notations (indifferently employed) $< f, g >_{\mu'} = \mu'(fg) = \int fg \, \mathrm{d}\mu'$*;*

(vi) *moreover, there exist* $C > 0$ *and* $0 < \rho < 1$ *such that, for all h in* $V_\alpha(\Omega)$ *and* $f \in L^1_\mu(\Omega)$*, one has*

$$\left| \int_\Omega f \circ T^{k \times \mathrm{ppcm}(L_i)} h \, \mathrm{d}\mu - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j - 1} \mu(W_{j,l}) < f, 1 >_{\mu_{j,l}} < 1, h >_{\mu_{j,l}} \right| \le C ||h||_{\alpha,\Omega} ||f||_{L^1_\mu(\Omega)} \rho^k;$$

(vii) *if, moreover,* T *is mixing,*[4] *then the preceding result can be written as follows: there exist* $C > 0$ *and* $0 < \rho < 1$ *such that, for all h in* $V_\alpha(\Omega)$ *and* $f \in L^1_\mu(\Omega)$*, one has:*

$$\left| \int_\Omega f \circ T^k h \, \mathrm{d}\mu - \int_\Omega f \mathrm{d}\mu \int_\Omega h \, \mathrm{d}\mu \right| \le C ||h||_{\alpha,\Omega} \, ||f||_{L^1_\mu(\Omega)} \rho^k.$$

We now come back to the initial problem and deduce from this result the invariant law associated with X_t. If $(X_t)_t$ is defined by X_0, X_1 (valued in $[-L, L]$) and the recurrence relation $X_{t+2} = \varphi(X_t, X_{t+1})$, one sets $Z_t = (X_t, \gamma X_{t+1})$. Then $(Z_t)_t$ satisfies the recurrence relation $Z_{t+1} = T(Z_t)$, which implies the following result (by comparing the marginal distributions):

Theorem 2.3. *Suppose that the random variable* $Z_0 = (X_0, \gamma X_1)$ *has the density* h_**. Then, for all* $t \in \mathbb{N}$*,* Z_t *has the density* h_* *and* X_t *has the density*

$$f : x \mapsto \int_{[-\gamma L, \gamma L]} h_*(x, v) \, \mathrm{d}v = \gamma \int_{[-L, L]} h_*(u, \gamma x) \, \mathrm{d}u.$$

If F is defined on $[-L, L]$, let $\mathrm{Tr}\ F$ be the function defined on Ω by $\mathrm{Tr}\ F(x, y) = F(x)$.

One then obtains the following result, which is a direct consequence of the sixth point of Theorem 2.2, applied to Tr F and Tr H:

Theorem 2.4. *For every Borel set B and every interval I, if* (X_0, X_1) *has the invariant distribution, then*

$$\left| P\left(X_{k \times \mathrm{ppcm}(L_i)} \in B, X_0 \in I\right) - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j - 1} \mu(W_{j,l}) < \mathrm{Tr}\ \mathbf{1}_B, 1 >_{\mu_{j,l}} < 1, \mathrm{Tr}\ \mathbf{1}_I >_{\mu_{j,l}} \right|$$
$$\le 16 \, (1 + \gamma) \, C \, L^3 \, (10 \varepsilon_0^{1-\alpha} + L) \, \rho^k.$$

More generally, let F, defined and measurable on $[-L, L]$*, be such that* $\mathrm{Tr}\ F$ *belongs to* $L^1_\mu(\Omega)$*. Let* $H \in L^\infty_m([-L, L])$ *be such that*

$$\sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\alpha} \int_{[-L, L]} \mathrm{Osc}(H,]x - \varepsilon, x + \varepsilon[\cap [-L, L]) \, (\mathrm{d}x < +\infty.$$

Then $\mathrm{Tr}\ H \in V_\alpha(\Omega)$ *and*

$$\left| E(F(X_{k \times \mathrm{ppcm}(L_i)}) H(X_0)) - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j - 1} \mu(W_{j,l}) \mu_{j,l}(\mathrm{Tr}\ F) \mu_{j,l}(\mathrm{Tr}\ H) \right| \le C(F, H) \, \rho^k$$

with

[4] Which is equivalent to: if 1 is the only eigenvalue of P with modulus one and if it is simple.

$$C(F,H) = C\ L\ ||\mathrm{Tr}\ F||_{L^1_\mu}\left(2\gamma \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\alpha} \int_{[-L,L]} \mathrm{Osc}(H,]x-\varepsilon, x+\varepsilon[\cap[-L,L])\ (\mathrm{d}x\right.$$

$$\left. + 16(1+\gamma)\ \varepsilon_0^{1-\alpha} ||H||_{L^\infty_m([-L,L])} + 2\gamma\ ||H||_{L^1_m([-L,L])}\right).$$

If, moreover, T is mixing, then:

$$|\mathrm{Cov}(F(X_k), H(X_0))| \le C(F,H)\ \rho^k.$$

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