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Probability theory

An improvement of the mixing rates in a counter-example to the weak invariance principle

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ABSTRACT

In [1], the authors gave an example of absolutely regular strictly stationary process that satisfies the central limit theorem, but not the weak invariance principle. For each $q < /1/2$, the process can be constructed with mixing rates of order N^{-q}. The goal of this note is to show that actually the same construction can give mixing rates of order N^{-q} for a given $q < 1$.

RÉSUMÉ

Dans [1], les auteurs ont fourni un exemple de processus strictement stationnaire β-mélangeant vérifiant le théorème limite central, mais pas le principe d'invariance faible. Pour tout $q < 1/2$, le processus peut être construit avec des taux de mélange de l'ordre de N^{-q}. L'objectif de cette note est de montrer que la même construction peut fournir des taux de mélange de l'ordre de N^{-q} pour un $q < 1$ donné.

1. Notations and main result

We recall some notations in order to make this note more self-contained. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. If $T\colon \Omega \to \Omega$ is one-to-one, bi-measurable and measure preserving (in sense that $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$), then the sequence $\left(f \circ T^k\right)_{k\in\mathbb{Z}}$ is strictly stationary for any measurable $f\colon \Omega \to \mathbb{R}$. Conversely, each strictly stationary sequence can be represented in this way.

For a zero mean square integrable $f\colon \Omega \to \mathbb{R}$, we define $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$, $\sigma_n^2(f) := \mathbb{E}(S_n(f)^2)$ and $S_n^*(f,t) := S_{\lfloor nt\rfloor}(f) + (nt - \lfloor nt\rfloor) f \circ T^{\lfloor nt\rfloor}$, where $\lfloor x\rfloor$ is the greatest integer, which is less than or equal to x.

Define the β-mixing coefficients by

$$\beta(\mathcal{A}, \mathcal{B}) := \frac{1}{2} \sup \sum_{i=1}^{I} \sum_{j=1}^{J} \left|\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)\right|, \tag{1}$$

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where the supremum is taken over the finite partitions $\{A_i, 1 \leqslant i \leqslant I\}$ and $\{B_j, 1 \leqslant j \leqslant J\}$ of Ω of elements of $\mathcal{A}$ (respectively of $\mathcal{B}$). They were introduced by Volkonskii and Rozanov [4].

For a strictly stationary sequence $(X_k)_{k\in\mathbb{Z}}$ and $n \geqslant 0$, we define $\beta_X(n) = \beta(n) = \beta(\mathcal{F}^0_{-\infty}, \mathcal{F}^{\infty}_n)$, where $\mathcal{F}^v_u$ is the σ-algebra generated by X_k with $u \leqslant k \leqslant v$ (if $u = -\infty$ or $v = \infty$, the corresponding inequality is strict).

Theorem 1. *Let $\delta > 0$. There exists a strictly stationary real valued process $Y = (Y_k)_{k\geqslant 0} = (f \circ T^k)_{k\geqslant 0}$ satisfying the following conditions:*

a) *the central limit theorem with normalization $\sqrt{n}$ takes place;*
b) *the weak invariance principle with normalization $\sqrt{n}$ does not hold;*
c) *$\sigma_N(f)^2 \asymp N$;*
d) *for some positive C and each integer N, $\beta_Y(N) \leqslant C \cdot N^{-1+\delta}$;*
e) *$Y_0 \in \mathbb{L}^p$ for any $p > 0$.*

We refer the reader to Remark 2 of [1] for a comparison with existing results about the weak invariance principle for strictly stationary mixing sequences.

2. Proof

We recall the construction given in [1]. Let us consider an increasing sequence of positive integers $(n_k)_{k\geqslant 1}$ such that

$$n_1 \geqslant 2 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{n_k} < \infty, \tag{2}$$

and for each integer $k \geqslant 1$, let A^-_k, A^+_k be disjoint measurable sets such that $\mu(A^-_k) = 1/(2n_k^2) = \mu(A^+_k)$.

Let the random variables e_k be defined by

$$e_k(\omega) := \begin{cases} 1 & \text{if } \omega \in A^+_k, \\ -1 & \text{if } \omega \in A^-_k, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We can choose the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ and the sets A^+_k, A^-_k in such a way that the family $(e_k \circ T^i)_{k\geqslant 1, i\in\mathbb{Z}}$ is independent. We define $A_k := A^+_k \cup A^-_k$ and

$$h_k := \sum_{i=0}^{n_k-1} U^{-i} e_k - U^{-n_k} \sum_{i=0}^{n_k-1} U^{-i} e_k, \quad h := \sum_{k=1}^{+\infty} h_k. \tag{4}$$

Let $i(N)$ denote the unique integer such that $n_{i(N)} \leqslant N < n_{i(N)+1}$.

We shall show the following intermediate result.

Proposition 1. *Assume that the sequence $(n_k)_{k\geqslant 1}$ satisfies (2) and the following condition:*

$$\textit{there exists } \eta > 0 \textit{ such that for each } k, \quad n_{k+1} \geqslant n_k^{1+\eta}. \tag{5}$$

Then:

a') *$n^{-1/2} S_n(h) \to 0$ in probability;*
b') *the process $(N^{-1/2} S^*_N(h, \cdot))_{N\geqslant 1}$ is not tight in $C[0, 1]$;*
c') *$\sigma_N(h)^2 \lesssim N$;*
d') *for some positive C, $N \cdot \beta_Y(N) \leqslant C n_{i(N)+1}/n_{i(N)}$;*
e') *$h \in \mathbb{L}^p$ for any $p > 0$.*

Let us consider a bounded mean-zero function m of unit variance such that the sequence $(m \circ T^i)_{i\geqslant 0}$ is independent and independent of the sequence $(h \circ T^i)_{i\geqslant 0}$. We define $f := m + h$.

Corollary 2. *Assume the sequence $(n_k)_{k\geqslant 1}$ satisfies (5). Then the sequence $(f \circ T^i)_{i\geqslant 0}$ satisfies a), b), c), d') and e).*

For $k \geqslant 1$ and $N \geqslant n_k$, the N partial sum of h_k admits the expression

$$S_N(h_k) = \sum_{j=1}^{n_k} jU^{j+N-2n_k}e_k + \sum_{j=1}^{n_k-1}(n_k-j)U^{j+N-n_k}e_k - \sum_{j=1}^{n_k} jU^{j-2n_k}e_k - \sum_{j=1}^{n_k-1}(n_k-j)U^{j-n_k}e_k. \tag{6}$$

Let us prove Proposition 1. Item a') follows from the fact that h is a coboundary (see the explanation before Section 2.2 of [1]).

For b'), we recall the following lemma (Lemma 10, [1]).

Lemma 3. *There exists N_0 such that*

$$\mu\left\{\max_{2n_k\leqslant N\leqslant n_k^2}|S_N(h_k)|\geqslant n_k\right\} > 1/4 \tag{7}$$

whenever $n_k \geqslant N_0$.

The following proposition improves Lemma 11 of [1] since the condition on the sequence $(n_k)_{k\geqslant 1}$ (namely, (5)) is weaker than both conditions (11) and (12) of [1].

Proposition 4. *Assume that the sequence $(n_k)_{k\geqslant 1}$ satisfies (5). Then we have for k large enough*

$$\mu\left\{\frac{1}{n_k}\max_{2n_k\leqslant N\leqslant n_k^2}|S_N(h)|\geqslant 1/2\right\}\geqslant 1/8. \tag{8}$$

Proof. Let us fix an integer k. Let us define the events

$$A := \left\{\frac{1}{n_k}\max_{2n_k\leqslant N\leqslant n_k^2}|S_N(h)|\geqslant \frac{1}{2}\right\}, \tag{9}$$

$$B := \left\{\frac{1}{n_k}\max_{2n_k\leqslant N\leqslant n_k^2}\left|S_N\left(\sum_{j\geqslant k}h_j\right)\right|\geqslant 1\right\} \text{ and} \tag{10}$$

$$C := \left\{\frac{1}{n_k}\max_{2n_k\leqslant N\leqslant n_k^2}\left|S_N\left(\sum_{j\leqslant k-1}h_j\right)\right|\leqslant \frac{1}{2}\right\}. \tag{11}$$

Since the family $\left\{e_k\circ T^i, k\geqslant 1, i\in\mathbb{Z}\right\}$ is independent, the events B and C are independent. Notice that $B\cap C\subset A$, hence

$$\mu(A) = \mu\left\{\frac{1}{n_k}\max_{2n_k\leqslant N\leqslant n_k^2}|S_N(h)|\geqslant\frac{1}{2}\right\}\geqslant \mu(B)\mu(C).$$

In order to give a lower bound for $\mu(B)$, we define $E_k := \bigcup_{N=2n_k}^{n_k^2}\bigcup_{j\geqslant k+1}\left\{S_N(h_j)\neq 0\right\}$; then

$$\mu(B) \geqslant \mu(B\cap E_k^c) \tag{12}$$

$$= \mu\left(\left\{\frac{1}{n_k}\max_{2n_k\leqslant N\leqslant n_k^2}|S_N(h_k)|\geqslant 1\right\}\cap E_k^c\right) \tag{13}$$

$$\geqslant \mu\left(\left\{\frac{1}{n_k}\max_{2n_k\leqslant N\leqslant n_k^2}|S_N(h_k)|\geqslant 1\right\}\right) - \mu(E_k). \tag{14}$$

Let us give an estimate of the probability of E_k. As noted in [1] (proof of Lemma 11 therein), the inclusion

$$\bigcup_{N=2n_k}^{n_k^2} \left\{S_N(h_j) \neq 0\right\} \subset \bigcup_{i=-2n_j+1}^{n_k^2} T^{-i} A_j \tag{15}$$

takes place for $j > k$, hence

$$\mu\left(\bigcup_{N=2n_k}^{n_k^2} \left\{S_N(h_j) \neq 0\right\}\right) \leqslant \frac{n_k^2 + 2n_j}{n_j^2}, \tag{16}$$

and it follows that

$$\mu(E_k) \leqslant \sum_{j=k+1}^{+\infty} \frac{2n_k}{n_j}. \tag{17}$$

By (5), we have $n_k \leqslant n_j^{1/(1+\eta)}$ for $j > k$, hence by (17),

$$\mu(E_k) \leqslant 2 \sum_{j=k+1}^{+\infty} n_j^{-\frac{\eta}{1+\eta}}. \tag{18}$$

As condition (5) implies that $n_k \geqslant 2^k$ for k large enough, we conclude that the following inequality holds for k large enough:

$$\mu(E_k) \leqslant 2 \sum_{j=k+1}^{+\infty} 2^{-j\frac{\eta}{1+\eta}} \tag{19}$$

Thus, by Lemma 3 and (19), we have for k large enough

$$\mu\left\{\frac{1}{n_k} \max_{2n_k \leqslant N \leqslant n_k^2} \left|S_N(h)\right| \geqslant \frac{1}{2}\right\}$$
$$\geqslant \left(\frac{1}{4} - 2 \sum_{j=k+1}^{+\infty} 2^{-j\frac{\eta}{1+\eta}}\right)\left(1 - \mu\left\{\frac{1}{n_k} \max_{2n_k \leqslant N \leqslant n_k^2} \left|S_N\left(\sum_{j \leqslant k-1} h_j\right)\right| > \frac{1}{2}\right\}\right). \tag{20}$$

Defining $c_k := \mu\left\{\frac{1}{n_k} \max_{2n_k \leqslant N \leqslant n_k^2} \left|S_N\left(\sum_{j \leqslant k-1} h_j\right)\right| > \frac{1}{2}\right\}$, it is enough to prove that

$$\lim_{k \to \infty} c_k = 0. \tag{21}$$

Using (6) (accounting $N \geqslant 2n_k \geqslant n_j$ for $j < k$), we get the inequalities

$$c_k \leqslant \sum_{j=1}^{k-1} \mu\left\{\frac{1}{n_k} \max_{2n_k \leqslant N \leqslant n_k^2} \left|S_N(h_j)\right| > \frac{1}{2(k-1)}\right\} \tag{22}$$

$$\leqslant \sum_{j=1}^{k-1} \mu\left\{\left|\sum_{i=1}^{n_j} iU^i e_j\right| > \frac{n_k}{8k}\right\} + \sum_{j=1}^{k-1} \mu\left\{\left|\sum_{i=1}^{n_j-1} iU^i e_j\right| > \frac{n_k}{8k}\right\} \tag{23}$$

$$+ \sum_{j=1}^{k-1} \mu\left\{\max_{2n_k \leqslant N \leqslant n_k^2} U^N \left|\sum_{i=1}^{n_j} iU^i e_j\right| > \frac{n_k}{8k}\right\} +$$

$$+ \sum_{j=1}^{k-1} \mu\left\{\max_{2n_k \leqslant N \leqslant n_k^2} U^N \left|\sum_{i=1}^{n_j-1} iU^i e_j\right| > \frac{n_k}{8k}\right\}$$

$$\leqslant n_k^2 \left(\sum_{j=1}^{k-1} \mu\left\{\left|\sum_{i=1}^{n_j} iU^i e_j\right| > \frac{n_k}{8k}\right\} + \sum_{j=1}^{k-1} \mu\left\{\left|\sum_{i=1}^{n_j-1} iU^i e_j\right| > \frac{n_k}{8k}\right\}\right). \tag{24}$$

Notice that for each $j \leqslant k-1$,

$$\mu\left\{\left|\sum_{i=1}^{n_j-1} iU^i e_j\right| > \frac{n_k}{8k}\right\} \leqslant \mu\left\{\left|\sum_{i=1}^{n_j} iU^i e_j\right| > \frac{n_k}{16k}\right\} + \mu\left\{\left|n_j U^{n_j} e_j\right| > \frac{n_k}{16k}\right\}. \tag{25}$$

Condition (5) implies the inequality $16k \cdot n_{k-1} < n_k$ for k large enough, hence keeping in mind that $U^{n_j} e_j$ is bounded by 1, inequality (25) becomes for such k's:

$$\mu\left\{\left|\sum_{i=1}^{n_j-1} iU^i e_j\right| > \frac{n_k}{8k}\right\} \leqslant \mu\left\{\left|\sum_{i=1}^{n_j} iU^i e_j\right| > \frac{n_k}{16k}\right\}. \tag{26}$$

Combining (24) with (26), we obtain

$$c_k \leqslant 2n_k^2 \sum_{j=1}^{k-1} \mu\left\{\left|\sum_{i=1}^{n_j} iU^i e_j\right| > \frac{n_k}{16k}\right\} \tag{27}$$

$$\leqslant 2n_k^2 \frac{(16k)^p}{n_k^p} \sum_{j=1}^{k-1} \mathbb{E}\left|\sum_{i=1}^{n_j} iU^i e_j\right|^p, \tag{28}$$

where $p > 2 + 1/\eta$. By Rosenthal's inequality (see [3], Theorem 1), we have

$$\mathbb{E}\left|\sum_{i=1}^{n_j} iU^i e_j\right|^p \leqslant C_p\left(\sum_{i=1}^{n_j} i^p \mathbb{E}\left|e_j\right| + \left(\sum_{i=1}^{n_j} \mathbb{E}[i^2 e_j^2]\right)^{p/2}\right) \tag{29}$$

$$\leqslant C_p(n_j^{p+1-2} + n_j^{3p/2}/n_j^p) \tag{30}$$

$$\leqslant 2C_p n_j^{p-1} \tag{31}$$

as $p > 2$. Therefore, for some constant K depending only on p,

$$c_k \leqslant K \cdot n_k^{2-p} k^p \sum_{j=1}^{k-1} n_j^{p-1} \leqslant K \cdot k^{p+1} \frac{n_{k-1}^{p-1}}{n_k^{p-2}}, \tag{32}$$

and by (5),

$$c_k \leqslant K \cdot k^{p+1} n_{k-1}^{p-1-(p-2)(1+\eta)}. \tag{33}$$

Since $p - 1 - (p-2)(1+\eta) = 1 - (p-2)\eta < 0$ and $n_{k-1} \geqslant 2^{k-1}$ for each $k \geqslant 2$, we get:

$$c_k \leqslant K \cdot k^{p+1} 2^{(1-(p-2)\eta)(k-1)}. \tag{34}$$

This concludes the proof of Proposition 4 hence that of b'). □

For c'), we follow the computation in the proof of Proposition 13 of [1], using the fact that $\sup_k \sum_{j=1}^{k-1} n_j/n_k$ is finite.
We now provide a bound for the mixing rates. Corollary 6 of [1] states the following.

Proposition 5. *For each integer k, we have*

$$\beta(N) \leqslant \sum_{j:2n_j \geqslant N} \frac{4}{n_j}. \tag{35}$$

Then d') follows from the bounds

$$\beta(2N) \leqslant \frac{4}{n_{i(N)}} + \sum_{k \geqslant i(N)} \frac{4}{n_{k+1}} = \frac{4}{n_{i(N)}}\left(1 + \sum_{j \geqslant 1} \frac{n_j}{n_{j+1}}\right). \tag{36}$$

In Proposition 14 of [1], it was proved that for each $q \geqslant 2$, there exists a constant C_q such that for each $k \geqslant 1$, $\|h_k\|_q \leqslant C_q n_k^{-1/q}$. Condition (5) implies that $n_k \geqslant 2^k$ for k large enough, hence e') is satisfied.
This concludes the proof of Proposition 1 and that of Corollary 2.

Proof of Theorem 1. Let η be an arbitrary positive real and let

$$n_k := \lfloor 2^{(1+\eta)^k + 1} \rfloor. \tag{37}$$

The sequence $(n_k)_{k\geqslant 1}$ satisfies (5) (and (2)). Consequently, if h is defined by (4), then the sequence of partial sums $(S_n(h))_{n\geqslant 1}$ satisfies the conclusions of Proposition 1. It follows that the sequence $(f\circ T^k)_{k\geqslant 0}$ satisfies the conclusions of Corollary 2. Now, by Proposition 11 of [2], we have $\beta(N)\leqslant CN^{-1/(1+\eta)}$ for some universal positive constant C, which completes the proof of Theorem 1. □

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