Probability theory

An improvement of the mixing rates in a counter-example to the weak invariance principle

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A B S T R A C T

In [1], the authors gave an example of absolutely regular strictly stationary process that satisfies the central limit theorem, but not the weak invariance principle. For each \( q < 1/2 \), the process can be constructed with mixing rates of order \( N^{-q} \). The goal of this note is to show that actually the same construction can give mixing rates of order \( N^{-q} \) for a given \( q < 1 \).

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R E S U M É

Dans [1], les auteurs ont fourni un exemple de processus strictement stationnaire \( \beta \)-mélangeant vérifiant le théorème limite central, mais pas le principe d’invariance faible. Pour tout \( q < 1/2 \), le processus peut être construit avec des taux de mélange de l’ordre de \( N^{-q} \). L’objectif de cette note est de montrer que la même construction peut fournir des taux de mélange de l’ordre de \( N^{-q} \) pour un \( q < 1 \) donné.

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1. Notations and main result

We recall some notations in order to make this note more self-contained. Let \((\Omega, \mathcal{F}, \mu)\) be a probability space. If \( T : \Omega \to \Omega \) is one-to-one, bi-measurable and measure preserving (in sense that \( \mu(T^{-1}(A)) = \mu(A) \) for all \( A \in \mathcal{F} \)), then the sequence \( (f \circ T^n)_{n \in \mathbb{Z}} \) is strictly stationary for any measurable \( f : \Omega \to \mathbb{R} \). Conversely, each strictly stationary sequence can be represented in this way.

For a zero mean square integrable \( f : \Omega \to \mathbb{R} \), we define \( S_n(f) := \sum_{j=0}^{n-1} f \circ T^j \), \( \sigma_n^2(f) := E(S_n(f)^2) \) and \( S_n^*(f, t) := S_{\lfloor nt \rfloor}(f) + (nt - \lfloor nt \rfloor) f \circ T^{\lfloor nt \rfloor} \), where \( \lfloor x \rfloor \) is the greatest integer, which is less than or equal to \( x \).

Define the \( \beta \)-mixing coefficients by

\[
\beta(A, B) := \frac{1}{2} \sup \sum_{i=1}^{j} \sum_{j=1}^{l} \left| \mu(A_i \cap B_j) - \mu(A_i) \mu(B_j) \right|,
\]

\[ (1) \]

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where the supremum is taken over the finite partitions \( \{ A_i, 1 \leq i \leq I \} \) and \( \{ B_j, 1 \leq j \leq J \} \) of \( \Omega \) of elements of \( \mathcal{A} \) (respectively of \( \mathcal{B} \)). They were introduced by Volkonskii and Rozanov [4].

For a strictly stationary sequence \( (X_k)_{k \geq 0} \) and \( n \geq 0 \), we define \( \beta_X(n) = \beta(n) = \beta(F^0_{-\infty}, F^\infty_n) \), where \( F^\infty_n \) is the \( \sigma \)-algebra generated by \( X_k \) with \( u \leq k \leq v \) (if \( u = -\infty \) or \( v = \infty \), the corresponding inequality is strict).

**Theorem 1.** Let \( \delta > 0 \). There exists a strictly stationary real valued process \( Y = (Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0} \) satisfying the following conditions:

a) the central limit theorem with normalization \( \sqrt{n} \) takes place;

b) the weak invariance principle with normalization \( \sqrt{n} \) does not hold;

c) \( \sigma_n(f)^2 \asymp n \);

d) for some positive \( C \) and each integer \( N \), \( \beta_Y(N) \leq C \cdot N^{-1+\delta} \);

e) \( Y_0 \in L^p \) for any \( p > 0 \).

We refer the reader to Remark 2 of [1] for a comparison with existing results about the weak invariance principle for strictly stationary mixing sequences.

**2. Proof**

We recall the construction given in [1]. Let us consider an increasing sequence of positive integers \( (n_k)_{k \geq 1} \) such that

\[
    n_1 \geq 2 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{n_k} < \infty, \tag{2}
\]

and for each integer \( k \geq 1 \), let \( A^+_k \), \( A^-_k \) be disjoint measurable sets such that \( \mu(A^+_k) = 1/(2n_k^2) = \mu(A^-_k) \).

Let the random variables \( e_k \) be defined by

\[
e_k(\omega) := \begin{cases} 
1 & \text{if } \omega \in A^+_k, \\
-1 & \text{if } \omega \in A^-_k, \\
0 & \text{otherwise.} \end{cases} \tag{3}
\]

We can choose the dynamical system \( (\Omega, \mathcal{F}, \mu, T) \) and the sets \( A^+_k, A^-_k \) in such a way that the family \( (e_k \circ T^i)_{k \geq 1, i \in \mathbb{Z}} \) is independent. We define \( A_k := A^+_k \cup A^-_k \) and

\[
h_k := \sum_{i=0}^{n_k-1} U^{-i}e_k - U^{-n_k} \sum_{i=0}^{n_k-1} U^{-i}e_k, \quad h := \sum_{k=1}^{\infty} h_k. \tag{4}
\]

Let \( i(N) \) denote the unique integer such that \( n_{i(N)} \leq N < n_{i(N)+1} \).

We shall show the following intermediate result.

**Proposition 1.** Assume that the sequence \( (n_k)_{k \geq 1} \) satisfies (2) and the following condition:

\[
\text{there exists } \eta > 0 \text{ such that for each } k, \quad n_{k+1} \geq n_k^{1+\eta}. \tag{5}
\]

Then:

a’) \( n^{-1/2} S_n(h) \to 0 \) in probability;

b’) the process \( (n^{-1/2} S_N(h, \cdot))_{N \geq 1} \) is not tight in \( C[0,1] \);

c’) \( \sigma_n(h)^2 \lesssim N \);

d’) for some positive \( C \), \( N \cdot \beta_Y(N) \leq C n_{i(N)+1}/n_{i(N)} \);

e’) \( h \in L^p \) for any \( p > 0 \).

Let us consider a bounded mean-zero function \( m \) of unit variance such that the sequence \( (m \circ T^i)_{i \geq 0} \) is independent and independent of the sequence \( (h \circ T^i)_{i \geq 0} \). We define \( f := m + h \).

**Corollary 2.** Assume the sequence \( (n_k)_{k \geq 1} \) satisfies (5). Then the sequence \( (f \circ T^i)_{i \geq 0} \) satisfies a), b), c), d’) and e).
For $k \geq 1$ and $N \geq n_k$, the $N$ partial sum of $h_k$ admits the expression

$$S_N(h_k) = \sum_{j=1}^{n_k} j U^{j+N-2n_k} e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{j+N-n_k} e_k$$

$$- \sum_{j=1}^{n_k} j U^{j-2n_k} e_k - \sum_{j=1}^{n_k-1} (n_k - j) U^{j-n_k} e_k.$$  \hspace{1cm} (6)

Let us prove Proposition 1. Item a') follows from the fact that $h$ is a coboundary (see the explanation before Section 2.2 of [1]).

For b'), we recall the following lemma (Lemma 10, [1]).

Lemma 3. There exists $N_0$ such that

$$\mu \left\{ \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq n_k \right\} > 1/4$$  \hspace{1cm} (7)

whenever $n_k \geq N_0$.

The following proposition improves Lemma 11 of [1] since the condition on the sequence $(n_k)_{k \geq 1}$ (namely, (5)) is weaker than both conditions (11) and (12) of [1].

Proposition 4. Assume that the sequence $(n_k)_{k \geq 1}$ satisfies (5). Then we have for $k$ large enough

$$\mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq 1/2 \right\} \geq 1/8.$$  \hspace{1cm} (8)

Proof. Let us fix an integer $k$. Let us define the events

$$A := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\},$$

$$B := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \geq k} h_j \right) \right| \geq 1 \right\} \text{ and}$$

$$C := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j=1}^{k-1} h_j \right) \right| \leq \frac{1}{2} \right\}.$$ \hspace{1cm} (9) (10) (11)

Since the family $\{e_k \circ T^i, k \geq 1, i \in \mathbb{Z}\}$ is independent, the events $B$ and $C$ are independent. Notice that $B \cap C \subset A$, hence

$$\mu(A) = \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\} \geq \mu(B) \mu(C).$$

In order to give a lower bound for $\mu(B)$, we define $E_k := \bigcup_{N=2n_k}^{n_k^2} \bigcup_{j \geq k+1} \{S_N(h_j) \neq 0\}$; then

$$\mu(B) \geq \mu(B \cap E_k^c)$$

$$= \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \cap E_k^c$$

$$\geq \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} - \mu(E_k).$$ \hspace{1cm} (12) (13) (14)

Let us give an estimate of the probability of $E_k$. As noted in [1] (proof of Lemma 11 therein), the inclusion
\[ \bigcup_{N=2n_k}^{n_k^2} \{ S_N(h_j) \neq 0 \} \subset \bigcup_{i=0}^{n_k^2} T^{-i}A_j \]

(15)

takes place for \( j > k \), hence

\[ \mu \left( \bigcup_{N=2n_k}^{n_k^2} \{ S_N(h_j) \neq 0 \} \right) \leq \frac{n_k^2 + 2n_j}{n_j^2} . \]

(16)

and it follows that

\[ \mu(E_k) \leq \sum_{j=k+1}^{+\infty} \frac{2n_k}{n_j} . \]

(17)

By (5), we have \( n_k \leq n_j^{1/(1+\eta)} \) for \( j > k \), hence by (17),

\[ \mu(E_k) \leq 2 \sum_{j=k+1}^{+\infty} n_j^{-\frac{n}{1+\eta}} . \]

(18)

As condition (5) implies that \( n_k \geq 2^k \) for \( k \) large enough, we conclude that the following inequality holds for \( k \) large enough:

\[ \mu(E_k) \leq 2 \sum_{j=k+1}^{+\infty} 2^{-j\frac{n}{1+\eta}} . \]

(19)

Thus, by Lemma 3 and (19), we have for \( k \) large enough

\[ \mu \left( \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N(h) \right| \geq \frac{1}{2} \right) \]

\[ \geq \left( \frac{1}{4} - 2 \sum_{j=k+1}^{+\infty} 2^{-j\frac{n}{1+\eta}} \right) \left( 1 - \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j=k-1}^h \right) \right| \geq \frac{1}{2} \right\} \right) . \]

(20)

Defining \( c_k := \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j=k-1}^h \right) \right| > \frac{1}{2} \right\} \), it is enough to prove that

\[ \lim_{k \to +\infty} c_k = 0 . \]

(21)

Using (6) (accounting \( N \geq 2n_k \geq n_j \) for \( j < k \)), we get the inequalities

\[ c_k \leq \sum_{j=1}^{k-1} \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N(h_j) \right| > \frac{1}{2(k-1)} \right\} \]

\[ \leq \sum_{j=1}^{k-1} \mu \left\{ \sum_{i=1}^{n_j} iU^i e_j > \frac{n_k}{8k} \right\} + \sum_{j=1}^{k-1} \mu \left\{ \sum_{i=1}^{n_j-1} iU^i e_j > \frac{n_k}{8k} \right\} \]

\[ + \sum_{j=1}^{k-1} \mu \left\{ \max_{2n_k \leq N \leq n_k^2} \sum_{i=1}^{n_j} iU^i e_j \right\} ) + \]

\[ + \sum_{j=1}^{k-1} \mu \left\{ \max_{2n_k \leq N \leq n_k^2} \sum_{i=1}^{n_j-1} iU^i e_j \right\} \]

\[ \leq \frac{n_k}{8k} \left( \sum_{j=1}^{k-1} \mu \left\{ \sum_{i=1}^{n_j} iU^i e_j > \frac{n_k}{8k} \right\} + \sum_{j=1}^{k-1} \mu \left\{ \sum_{i=1}^{n_j-1} iU^i e_j > \frac{n_k}{8k} \right\} \right) . \]

(24)

Notice that for each \( j \leq k-1 \),

\[ \mu \left\{ \sum_{i=1}^{n_j-1} iU^i e_j > \frac{n_k}{8k} \right\} \leq \mu \left\{ \sum_{i=1}^{n_j} iU^i e_j > \frac{n_k}{16k} \right\} + \mu \left\{ \sum_{i=1}^{n_j-1} iU^i e_j > \frac{n_k}{8k} \right\} . \]

(25)
Condition (5) implies the inequality $16k \cdot n_{k-1} < n_k$ for $k$ large enough, hence keeping in mind that $U^n e_j$ is bounded by 1, inequality (25) becomes for such $k$'s:

$$\mu \left\{ \sum_{i=1}^{n-1} iU^i e_j > \frac{n_k}{16k} \right\} \leq \mu \left\{ \sum_{i=1}^{n_j} iU^i e_j > \frac{n_k}{16k} \right\}. \quad (26)$$

Combining (24) with (26), we obtain

$$c_k \leq 2n^2 k \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{16k} \right\} \leq 2n^2 (16k)^p \sum_{j=1}^{k-1} \left| \sum_{i=1}^{n_j} iU^i e_j \right|^p. \quad (27)$$

where $p > 2 + 1/\eta$. By Rosenthal's inequality (see [3], Theorem 1), we have

$$\mathbb{E} \left| \sum_{i=1}^{n_j} iU^i e_j \right|^p \leq C_p \left( \sum_{i=1}^{n_j} \mathbb{E} |e_j| \right)^p = C_p n_j^{p+1-2} + n_j^{3p/2}/n_j^p \leq 2C_p n_j^{p-1} \quad (29)$$

as $p > 2$. Therefore, for some constant $K$ depending only on $p$,

$$c_k \leq K \cdot n^2_{k-1} p^{p-1} \sum_{j=1}^{n_j} n_j^{p-1} \leq K \cdot k^{p+1} n_{k-1}^{p-1}/n_k^{p-2}. \quad (30)$$

and by (5),

$$c_k \leq K \cdot k^{p+1} n_{k-1}^{p-1-(p-2)(1+\eta)}. \quad (31)$$

Since $p - 1 - (p - 2)(1 + \eta) = 1 - (p - 2)\eta < 0$ and $n_{k-1} \geq 2^{k-1}$ for each $k \geq 2$, we get:

$$c_k \leq K \cdot k^{p+1} 2^{1-(p-2)\eta}(k-1). \quad (32)$$

This concludes the proof of Proposition 4 hence that of b').

For c'), we follow the computation in the proof of Proposition 13 of [1], using the fact that $\sup_n \sum_{j=1}^{n_j} n_j/n_k$ is finite. We now provide a bound for the mixing rates. Corollary 6 of [1] states the following.

**Proposition 5.** For each integer $k$, we have

$$\beta(N) \leq \sum_{j:2n_j \geq N} \frac{4}{n_j}. \quad (33)$$

Then d') follows from the bounds

$$\beta(2N) \leq \frac{4}{n_i(N)} + \sum_{k \geq i+1} \frac{4}{n_{k+1}} = \frac{4}{n_i(N)} \left( 1 + \sum_{j \geq i+1} \frac{n_j}{n_{j+1}} \right). \quad (34)$$

In Proposition 14 of [1], it was proved that for each $q \geq 2$, there exists a constant $C_q$ such that for each $k \geq 1$, $\|h_k\|_p \leq C_q n_k^{-1/q}$. Condition (5) implies that $n_k \geq 2^{k}$ for $k$ large enough, hence e') is satisfied.

This concludes the proof of Proposition 1 and that of Corollary 2.

**Proof of Theorem 1.** Let $\eta$ be an arbitrary positive real and let

$$n_k := \lfloor 2^{(1+\eta)k+1} \rfloor. \quad (35)$$
The sequence \((n_k)_{k \geq 1}\) satisfies (5) (and (2)). Consequently, if \(h\) is defined by (4), then the sequence of partial sums \((S_n(h))_{n \geq 1}\) satisfies the conclusions of Proposition 1. It follows that the sequence \((f \circ T^k)_{k \geq 0}\) satisfies the conclusions of Corollary 2. Now, by Proposition 11 of [2], we have \(\beta(N) \leq CN^{-1/(1+\eta)}\) for some universal positive constant \(C\), which completes the proof of Theorem 1. \(\square\)

References


