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On the defect of compactness in Banach spaces

Sur le défaut de compacité dans les espaces de Banach

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A B S T R A C T

Concentration compactness methods improve convergence for bounded sequences in Banach spaces beyond the weak-star convergence provided by the Banach–Alaoglu theorem. A further improvement of convergence, known as profile decomposition, is possible up to *defect of compactness*, a series of “elementary concentrations” defined relative to the action of some group of linear isometric operators. This note presents a general profile decomposition for uniformly convex and uniformly smooth Banach spaces, generalizing the result of one of the authors (S.S.) for Sobolev spaces and of the other (C.T. jointly with I. Schindler) for general Hilbert spaces. Unlike in the Hilbert space case, profile decomposition is based not on weak convergence, but on a different mode of convergence, called *polar convergence*, which coincides with weak convergence if and only if the norm satisfies the known Opial condition, used in the context of fixed point theory for nonexpansive maps.

R É S U M É

Les méthodes de concentration-compacité permettent l’amélioration de la convergence des (sous-)suites bornées dans les espaces de Banach au-delà de la convergence faible étoile fournie par le théorème de Banach–Alaoglu. Une amélioration de la convergence est possible en tenant compte de l’action d’un groupe d’isométries. P.-L. Lions a étudié ce type d’amélioration dans des espaces de fonctions spécifiques avec les groupes des translations et des dilatations. Une analyse complète en décomposition en profils (des suites bornées) pour les espaces de Sobolev $\dot{W}^{1,p}(\mathbb{R}^N)$ avec les groupes des translations et dilatations a été établie par l’un des auteurs (S.S.). L’autre auteur (avec I. Schindler) a démontré l’existence d’une décomposition en profils semblable dans un espace d’Hilbert général sur lequel agit un groupe d’isométrie satisfaisant certaines hypothèses. Ce cadre général a permis d’établir des décompositions en profils avec d’autres groupes d’isométries telles que des isométries sur des variétés sous-riemanniennes, des dilatations logarithmiques pour des espaces d’Orlicz et le groupe galiléen pour étudier l’équation de Schrödinger non linéaire (T. Tao). Ce travail généralise la décomposition en profils pour un espace de Banach uniformément convexe et uniformément régulier. Nous introduisons une autre définition de la convergence faible, dite convergence polaire. La convergence polaire coïncide avec la convergence faible si et seulement si la norme satisfait à la condition d’Opial, utilisée pour la théorie du point fixe dans le cas des applications non expansives. La convergence

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polaire $x_k \rightharpoondown x$ est caractérisée par la dualité $(x_k - x)^* \rightharpoonup 0$. Dans les espaces de Besov et de Triebel–Lizorkin (qui contiennent les espaces de Lebesgue et Sobolev), la condition d'Opial est satisfaite pour des normes équivalentes via la décomposition de Littlewood–Paley, ce qui donne une décomposition en profils avec un reste qui tend vers 0 en norme pour des suites bornées des espaces correspondants.

1. Analysis of concentration: profile decompositions

Banach–Alaoglu theorem is useful for applications requiring convergence in a larger topological space, when the imbedding into this space is compact. Concentration compactness approach addresses the case when the imbedding of interest is not compact by isolating the asymptotically singular behavior of a sequence in a structured "defect of compactness". A general formalization of defect of compactness can be made in terms of a profile decomposition, which was constructed by Michael Struwe in 1984 for a particular class of sequences in Sobolev spaces. Initially called *global compactness* by Struwe or *splitting lemma* by Benci and Cerami, *profile decomposition* (a term probably introduced first by Gallagher in the title of [6]) of a sequence (x_k) in a Banach space X, relative to a group D of isometries of X, is a representation of x_k as a sum of elementary concentrations, i.e. of terms of the form $g_k w$, $g_k \in D$, $w \in X$, and a remainder r_k satisfying $g_k r_k \rightharpoonup 0$ for any sequence $(g_k) \subset D$. In the latter case, one says that r_k converges to zero D-weakly. We refer the reader for motivation of profile decomposition as an extension of the Banach–Alaoglu theorem, and a proof of both via non-standard analysis, to Tao [14]. For *general* bounded sequences in Sobolev spaces $\dot{H}^{1,p}(\mathbb{R}^N)$, profile decompositions were proved in [12], and the D-weak convergence of the remainder was identified as convergence in the Lorentz spaces $L^{p^*,q}$, $q > p$, where $p^* = \frac{pN}{N-p}$, and $p < N$ (which includes L^{p^*} but excludes $L^{p^*,p}$). The result of [12] (although with a weaker form of asymptotics) was subsequently given an independent and insightful proof by Gérard [7], and then extended by Jaffard [8] to the case of fractional Sobolev spaces, by a powerful reduction of the problem to sequence spaces by means of the wavelet transformation. For general Hilbert spaces, equipped with a non-compact group of isometries of particular type, the existence of profile decomposition was proved in [11]. This, in turn, stimulated the search for new concentration mechanisms, which include inhomogeneous dilations $j^{-1/2}u(z^j)$, $j \in \mathbb{N}$, with z^j denoting an integer power of a complex number, for problems in the Sobolev space $H_0^{1,2}(B)$ of the unit disk, related to the Trudinger–Moser functional; for the action of the Galilean invariance, together with shifts and rescalings, involved in the loss of compactness in Strichartz's imbeddings for the nonlinear Schrödinger equations. It is impossible to list here all important applications of profile decompositions based on usual translations and dilations and, for a more comprehensive summary of known profile decompositions, we refer the reader to a recent survey [16].

This note introduces a general theory of concentration analysis in Banach space. The difference between the Hilbert space and the Banach space cases is of essence and is based on the difficulty, in a general Banach space, to find a simple "energy" inequality that controls the total bulk of profiles and which in turn results in the convergence of the sum of elementary concentrations. Such energy estimates can be obtained if the profiles are defined not as weak limits, but as limits relative to a new mode of convergence, polar convergence (which is a slight modification of Δ-convergence introduced by Lim [9]), defined below. The underlying inequality says that if u is a polar limit of a sequence $(u_k) \subset X$, $\|u_k\| \le 1$, then $\|u_k\| \ge \|u\| + \delta(\|u_k - u\|) + o(1)$, where $\delta(\|\cdot\|)$ denotes the modulus of convexity of X.

2. Polar convergence

Δ-convergence of sequences in metric spaces is related to the notion of asymptotic center involved in the fixed point theory [5,10]. It coincides with weak convergence in Hilbert spaces, and more generally, in case of uniformly convex and uniformly smooth spaces, coincidence of weak and Δ-convergence happens to be equivalent to the well-known Opial condition, as elaborated below.

Definition 2.1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). One says that $x \in X$ is a Δ-*limit* of $(x_n)_{n\in\mathbb{N}}$ if for every $y \in X$

$$d(x_n, x) \le d(x_n, y) + o_{n\to\infty}(1). \tag{2.1}$$

Furthermore, one says that x is a polar limit if for every $y \neq x$ there exists $N_y \in \mathbb{N}$ such that

$$d(x_n, x) < d(x_n, y) \text{ for all } n \ge N_y. \tag{2.2}$$

For a polar limit, we will use notation $x_n \rightharpoondown x$.

Obviously, every polar limit is a Δ-limit. It is immediate that the polar limit is unique and that the usual limit of the sequence is also its polar limit. It is also immediate that, given $p \in [1, \infty)$, if the assertion of the important Brezis–Lieb Lemma holds for a given sequence $(u_k) \subset L^p$ in the sense that $\int |u_k + v|^p = \int |u_k - u|^p + \int |u + v|^p + o(1)$ for every $v \in L^p$, then u is the polar limit of (u_k). As we see below, in a Hilbert space, polar convergence coincides with weak convergence, but this is not the case for L^p of Euclidean domains with $p \neq 2$, see the suitably interpreted example in [10, Section 5].

The statement below (see [13]) is an elementary consequence of uniform convexity.

Proposition 2.2. *If X is a uniformly convex Banach space, then every bounded sequence that satisfies (2.1) also satisfies (2.2). Consequently, any Δ-limit of a sequence is its polar limit.*

It is well known that if X is a uniformly smooth Banach space, the duality map $X \setminus \{0\} \to X^*$, $x \mapsto x^*$, such that $\|x^*\| = 1$ and $\langle x^*, x\rangle = \|x\|$, is uniquely defined. We have [13] the following characterization of polar convergence. The proof is essentially a minimization argument, based on the fact that for uniformly smooth Banach spaces, the norm is Frechet differentiable away from zero with the derivative given by the duality map.

Theorem 2.3. *Let E be a uniformly convex and uniformly smooth Banach space. Let $x \in X$ and let $(x_k) \subset X$ be a bounded sequence such that $\liminf \|x_k - x\| > 0$. Then $x_k \rightharpoondown x$ if and only if $(x_k - x)^* \rightharpoonup 0$.*

Theorem 2.4. *If X is a uniformly smooth Banach space, then every sequence $(x_k) \subset X$, which has a polar limit, is bounded.*

Sketch of the proof. It suffices to prove the theorem for the case $x_k \rightharpoondown 0$. Assume that $\|x_k\| \to \infty$. Let $\omega(x, y) = \|x + y\| - \|x\| - \langle x^*, y\rangle$. An elementary identity gives then

$$0 \le \|x_k + z\|^2 - \|x_k\|^2 = \alpha_k^2 + 2\|x_k\|\alpha_k$$

for all k sufficiently large, where $\alpha_k = \|x_k\|\omega(\frac{x_k}{\|x_k\|}, \frac{z}{\|x_k\|}) + \langle x_k^*, z\rangle$ is a bounded sequence. Consequently, $\alpha_k \ge 0$. Since X is uniformly smooth, $\omega(x, y) = o(\|y\|)$ as $\|y\| \to 0$, locally uniformly in x. Then $\alpha_k \ge 0$ implies $|\langle \psi(\|x_k\|)x_k^*, z\rangle| \le 1$ with some function ψ such that $\psi(t) \to \infty$ when $t \to \infty$. By the Uniform Boundedness Principle, the sequence $\psi(\|x_k\|)$ is bounded, but this contradicts the assumption $\|x_k\| \to \infty$. □

We also have the following analog of the Banach–Alaoglu Theorem.

Theorem 2.5. *Let X be a uniformly convex Banach space and let $(x_k) \subset X$ be a bounded sequence. Then (x_k) has a polar-convergent subsequence.*

Proof. By Theorem 1 in [5], every bounded sequence has an asymptotic center, i.e., in the terminology of [9], uniformly convex Banach spaces are Δ-complete. Then by Theorem 3 in [9], (x_k) has a Δ-convergent subsequence. By Proposition 2.2, this subsequence is polar-convergent. □

In [13] we verify an equivalent form of the classical Opial condition (condition (2) in [10]), which is satisfied, in particular, by Hilbert spaces and by ℓ^p, $p \in (1, \infty)$.

Theorem 2.6. *If E is a uniformly convex and uniformly smooth Banach space, then the Opial's condition holds if and only if*

$$x_n \rightharpoonup x \text{ in } E \iff x_n \rightharpoondown x. \tag{2.3}$$

Remark 2.7. It was shown by van Dulst [4] that a separable uniformly convex Banach space can be provided with an equivalent norm satisfying the Opial condition. Applications of van Dulst's result are somewhat limited: in the fixed point theory, mappings may cease to be nonexpansive with respect to the new norm, and in concentration analysis the new norm remains invariant with respect to the given group only if van Dulst's construction is using a wavelet basis associated with this group. In Besov and Triebel–Lizorkin spaces considered with Euclidean translations and dilations, such basis indeed exists. On the other hand, as shown by Cwikel [3], Besov and Triebel–Lizorkin spaces equipped with a well-known equivalent norm, invariant with respect to translations and dilations and based on Littlewood–Paley decomposition, satisfy the Opial's condition.

3. Profile decompositions

Let D be a subset of a group D_0 of linear isometries on a Banach space E. We say that a sequence (r_k) vanishes in the D-polar sense ($r_k \overset{D}{\rightharpoondown} 0$) if for any sequence $(g_k) \subset D$, $g_k^{-1} r_k \rightharpoondown 0$. We have in [13] the following result.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, and let the group D_0 satisfy*

$$\{g_k\} \subset D_0,\ g_k \not\rightharpoonup 0 \Longrightarrow \exists \{k_j\} \subset \mathbb{N}: \{g_{k_j}^{-1}\} \text{ and } \{g_{k_j}\} \text{ converge operator-strongly} \tag{3.1}$$

(i.e. pointwise) and

$$u_k \rightharpoondown 0,\ w \in X,\ (g_k) \subset D_0,\ g_k \rightharpoonup 0 \Longrightarrow u_k + g_k w \rightharpoondown 0. \tag{3.2}$$

Then every bounded sequence $(u_k) \subset E$ *admits a polar profile decomposition relative to a set* $D \subset D_0$*, namely, there exist sequences* $(g_k^{(n)})_{k\in\mathbb{N}} \subset D$ *with* $g_k^{(1)} = I$*, elements* $w^{(n)} \in X$*,* $n \in \mathbb{N}$*, and a sequence* $r_k \overset{D}{\rightharpoondown} 0$*, such that*

$$(g_k^{(n)})^{-1} g_k^{(m)} \rightharpoonup 0 \text{ whenever } m \neq n \text{ (asymptotic decoupling of gauges)}, \tag{3.3}$$

and a renamed subsequence of u_k *can be represented in the form*

$$u_k = \sum_{j=1}^{\infty} g_k^{(j)} w^{(j)} + r_k, \tag{3.4}$$

where the series $\sum_{j=1}^{\infty} g_k^{(j)} w^{(j)}$ *converges in the norm of* X*, uniformly in* k*. In this case, we also have*

$$(g_k^{(n)})^{-1} u_k \rightharpoondown w^{(n)},\ n \in \mathbb{N}. \tag{3.5}$$

Moreover, if $\|u_k\| \leq 1$, and δ is the modulus of convexity of X, then $\|w^{(n)}\| \leq 2$ for all $n \in \mathbb{N}$ and

$$\limsup \|r_k\| + \sum_n \delta(\|w^{(n)}\|) \leq 1. \tag{3.6}$$

Idea of the proof. The construction of the profile decomposition is similar to the one for the Hilbert space in [11], but with polar convergence taking the place of weak convergence, and with allusion to Theorem 2.5 instead of the Banach–Alaoglu theorem. Polar convergence is essential for the stability estimate $\sum_n \delta(\|w^{(n)}\|) \leq 1$ (which is not known if the profiles $w^{(n)}$ were defined as weak, rather than polar, limits in (3.5)). A subtle diagonalization argument based on identifying consequent clusters $\sum_{n=p_j}^{q_j} g_k^{(n)} w^{(n)}$ of elementary concentrations with values of the modulus of convexity subordinate to the sequence (2^{-j}) assures then the convergence of the sequence $\sum_{j=1}^{\infty} g_k^{(j)} w^{(j)}$. □

In applications, one is often able to find a continuous embedding of X into another Banach space Y such that D-polar vanishing of the remainder is equivalent to convergence in the norm of Y. This property of embedding is called D-cocompactness. In Besov and Triebel–Lizorkin spaces, in face of Remark 2.7, polar convergence in the equivalent norm given in terms of Littlewood–Paley decomposition coincides with weak convergence. Cocompactness of Sobolev embeddings relative to Euclidean shifts and dilations was proved by Lieb (1983) in the subcritical case, and Lions (1985) in the critical case, extended, with a new proof, to embeddings into Lorentz spaces by Solimini [12], and vastly extended to embeddings of Besov and Triebel–Lizorkin spaces by Bahouri, Cohen and Koch [2], who do not use the term cocompactness, but verify it as their Condition 1.1. Cocompactness of embeddings of Sobolev spaces of Moser–Trudinger type into Orlicz spaces relative to logarithmic dilations was proved in [1], and cocompactness of Strichartz embeddings for Schrödinger equations relative to Galilean transformations was proved in [15]. The question of the characterization of D-cocompact embeddings is akin to the question of characterization of compact imbeddings (when $D = \{\mathrm{id}\}$ and X is a Hilbert space, polar D-cocompactness is the usual compactness), but more general results can be established, for example, on the lines of [2] when the group D generates a wavelet basis of X as a subset of an orbit.

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