C. R. Acad. Sci. Paris, Ser. I 353 (2015) 883–886

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Number theory/Harmonic analysis

On convergence almost everywhere of series of dilated functions

Sur la convergence presque partout des séries de fonctions dilatées

Michel J.G. Weber

IRMA, 10, rue du Général-Zimmer, 67084 Strasbourg cedex, France

ARTICLE INFO

Article history:
Received 28 July 2014
Accepted after revision 27 July 2015
Available online 29 August 2015

Presented by Jean-Pierre Kahane

ABSTRACT

Let $f(x) = \sum_{\ell\in\mathbb{Z}} a_\ell e^{2i\pi\ell x}$, where $\sum_{k\geq 1} a_k^2 d(k) < \infty$ and $d(k) = \sum_{d|k} 1$ and let $f_n(x) = f(nx)$. We show by using a new decomposition of squared sums that, for any $K \subset \mathbb{N}$ finite, $\|\sum_{k\in K} c_k f_k\|_2^2 \leq (\sum_{m=1}^\infty a_m^2 d(m)) \sum_{k\in K} c_k^2 d(k^2)$. If $f^s(x) = \sum_{j=1}^\infty \frac{\sin 2\pi jx}{j^s}$, $s > 1/2$, by only using elementary Dirichlet convolution calculus, we show that for $0 < \varepsilon \leq 2s-1$, $\zeta(2s)^{-1}\|\sum_{k\in K} c_k f_k^s\|_2^2 \leq \frac{1+\varepsilon}{\varepsilon}(\sum_{k\in K} |c_k|^2 \sigma_{1+\varepsilon-2s}(k))$, where $\sigma_h(n) = \sum_{d|n} d^h$. From this, we deduce that if $f \in BV(\mathbb{T})$, $\langle f, 1\rangle = 0$ and $\sum_{k=1}^\infty c_k^2 \frac{(\log\log k)^4}{(\log\log\log k)^2} < \infty$, then the series $\sum_k c_k f_k$ converges almost everywhere. This slightly improves a recent result, depending on a fine analysis on the polydisc ([1], th. 3) ($n_k = k$), where it was assumed that $\sum_{k=1}^\infty c_k^2 (\log\log k)^\gamma$ converges for some $\gamma > 4$.

RÉSUMÉ

Soit $f(x) = \sum_{\ell\in\mathbb{Z}} a_\ell e^{2i\pi\ell x}$ telle que la série $\sum_{k\geq 1} a_k^2 d(k)$ où $d(k) = \sum_{d|k} 1$ converge, et soit $f_n(x) = f(nx)$. Nous montrons à l'aide d'une nouvelle décomposition des sommes carrées que $\|\sum_{k\in K} c_k f_k\|_2^2 \leq (\sum_{m=1}^\infty a_m^2 d(m)) \sum_{k\in K} c_k^2 d(k^2)$, pour tout ensemble fini d'entiers K. Si $f^s(x) = \sum_{j=1}^\infty \frac{\sin 2\pi jx}{j^s}$, $s > 1/2$, nous montrons aussi, par un calcul simple sur les convolutions de Dirichlet, que $\zeta(2s)^{-1}\|\sum_{k\in K} c_k f_k^s\|_2^2 \leq \frac{1+\varepsilon}{\varepsilon}(\sum_{k\in K} |c_k|^2 \sigma_{1+\varepsilon-2s}(k))$, où $0 < \varepsilon \leq 2s-1$ et $\sigma_h(n) = \sum_{d|n} d^h$. Nous en déduisons que, pour tout $f \in BV(\mathbb{T})$ telle que $\langle f, 1\rangle = 0$, si la série $\sum_{k=1}^\infty c_k^2 \frac{(\log\log k)^4}{(\log\log\log k)^2}$ converge, alors la série $\sum_k c_k f_k$ converge presque partout. Cela améliore un résultat récent, dépendant d'une analyse fine sur le polydisque ([1], th. 3) ($n_k = k$), où l'on suppose que la série $\sum_{k=1}^\infty c_k^2 (\log\log k)^\gamma$ converge pour un réel $\gamma > 4$.

E-mail address: m.j.g.weber@gmail.com.

http://dx.doi.org/10.1016/j.crma.2015.07.010
1631-073X/

1. Introduction – main result

One of the oldest and most central problems in the theory of systems of dilated sums is the study of the convergence in norm or almost everywhere of the series $\sum_{k=1}^{\infty} c_k f(n_k x)$, where f is a periodic function f and $\mathcal{N} = \{n_k, k \geq 1\}$ a sequence of positive integers (see [3]). Our main concern is the search of individual conditions ensuring convergence, a barely investigated part of the theory. We use an arithmetical approach based on elementary Dirichlet convolution calculus and on a new decomposition of squared sums, continuing the work in [7,4]. We show that this approach is strong enough to recover and even slightly improve a recent a.e. convergence result [1] (Theorem 3) in the case $\mathcal{N} = \mathbb{N}$ without using analysis on the polydisc. Results in [1] were recently developed in [2]. Our approach is also in the spirit of the work of Hilberdink [5] on some arithmetical mappings and extrema linked to arithmetical functions, with applications to Ω-results of the Riemann Zeta function. Denote $e(x) = e^{2i\pi x}$, $e_n(x) = e(nx)$, $n \geq 1$. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1[$. Let $f(x) \sim \sum_{j=1}^{\infty} a_j e_j(x)$. Let $f_n(x) = f(nx)$, $n \in \mathbb{N}$. We assume throughout that

$$f \in L^2(\mathbb{T}), \qquad \langle f, 1\rangle = 0. \tag{1}$$

A key preliminary step naturally consists in searching bounds of $\| \sum_{k\in K} c_k f_k\|_2$ integrating in their formulation the arithmetical structure of K. That question has received a satisfactory answer only for specific cases. We state our mean results. Let $d(n)$ be the divisor function, namely the number of divisors of n. Throughout, K denotes a finite set of natural numbers.

Theorem 1.1. *Assume that $\sum_{m=1}^{\infty} a_m^2 d(m) < \infty$. Then,*

$$\Big\| \sum_{k\in K} c_k f_k \Big\|_2^2 \leq \Big(\sum_{m=1}^{\infty} a_m^2 d(m)\Big) \sum_{k\in K} c_k^2 d(k^2).$$

In [7], using Hooley's Delta function, we recently showed a similar estimate however restricted to sets K such that $K \subset]e^r, e^{r+1}]$ for some integer r. Theorem 1.1 is deduced from a more general result. Introduce the necessary notation. Let $A_k = \sum_{\nu=1}^{\infty} a_{\nu k}^2$. Let ζ_h be defined by $\zeta_h(n) = n^h$ for all positive n. Let $\theta(n)$ denotes the number of squarefree divisors of n. Given $K \subset \mathbb{N}$, we note $F(K) = \{d \geq 1; \exists k \in K : d|k\}$. If K is factor closed ($d|k \Rightarrow d \in K$ for all $k \in K$), then $F(K) = K$.

Theorem 1.2. *Let ψ be any arithmetical function taking only positive values. Then,*

$$\Big\| \sum_{k\in K} c_k f_k \Big\|_2^2 \leq B \sum_{k\in K} c_k^2 \psi * \zeta_0(k), \quad \textit{where} \quad B = \sup_{d\in F(K)} \Big(\sum_{\substack{k\in K \\ d|k}} \frac{A_{\frac{k}{d}}}{\psi(\frac{k}{d})} \theta\big(\frac{k}{d}\big)\Big) < \infty.$$

Here $*$ denotes the Dirichlet convolution. By choosing $\psi = \theta$ and since $\psi * \zeta_0(k) = d(k^2)$, we check that $B \leq \sum_{m=1}^{\infty} a_m^2 d(m)$, whence Theorem 1.1. Consider now the class of functions introduced in [6], $f^s(x) = \sum_{j=1}^{\infty} \frac{\sin 2\pi jx}{j^s}$, $s > 1/2$ and recall that $\langle f_k^s, f_\ell^s\rangle = \zeta(2s) \frac{(k,\ell)^{2s}}{k^s \ell^s}$ where $(k, \ell) = \gcd(k, \ell)$.

Theorem 1.3. *Let $s > 0$, $0 \leq \tau \leq 2s$. Let also $\psi_1(u) > 0$ be non-decreasing and $\sigma_u(k) = \sum_{d|k} d^u$. Then,*

$$\sum_{k,\ell\in K} c_k c_\ell \frac{(k,\ell)^{2s}}{k^s \ell^s} \leq \Big(\sum_{u\in F(K)} \frac{1}{\psi_1(u)\sigma_\tau(u)}\Big)\Big(\sum_{\nu\in K} c_\nu^2 \psi_1(\nu)\sigma_{\tau-2s}(\nu)\Big).$$

In particular,

$$\sum_{k,\ell\in K} c_k c_\ell \frac{(k,\ell)^{2s}}{k^s \ell^s} \leq M(K)\Big(\sum_{k\in K} |c_k|^2 \sigma_{\tau-2s}(k)\Big) \qquad \textit{with} \quad M(K) = \sum_{k\in F(K)} \frac{1}{\sigma_\tau(k)}.$$

Remark 1. Let $s > 1/2$, $0 < \varepsilon \leq 2s - 1$ and take $\tau = 1 + \varepsilon$. Then

$$\zeta(2s)^{-1} \Big\| \sum_{k\in K} c_k f_k^s \Big\|_2^2 \leq \frac{1+\varepsilon}{\varepsilon} \Big(\sum_{k\in K} |c_k|^2 \sigma_{1+\varepsilon-2s}(k)\Big).$$

We use Theorem 1.3 to prove (with no analysis on the polydisc as in [1]) the following almost everywhere convergence results for functions with bounded variation.

Theorem 1.4. *Let $f \in \mathrm{BV}(\mathbb{T})$, $\langle f, 1\rangle = 0$. Assume that*

$$\sum_{k\ge 3} c_k^2 \frac{(\log\log k)^4}{(\log\log\log k)^2} < \infty. \tag{2}$$

Then the series $\sum_k c_k f_k$ converges almost everywhere.

Remark 2. This slightly improves Theorem 3 in [1] ($n_k = k$), where it was assumed that the series $\sum_{k=1}^{\infty} c_k^2 (\log\log k)^{\gamma}$ converges for some $\gamma > 4$.

We will also prove the following rather delicate result where multipliers have arithmetical factors.

Theorem 1.5. *Let $f \in \mathrm{BV}(\mathbb{T})$, $\langle f, 1\rangle = 0$. Assume that for some real $b > 0$,*

$$\sum_{k\ge 3} c_k^2 (\log\log k)^{2+b} \sigma_{-1+\frac{1}{(\log\log k)^{b/3}}}(k) < \infty. \tag{3}$$

Then the series $\sum_k c_k f_k$ converges almost everywhere.

These results and some others, notably on the Riemann Zeta function, are proved in [8].
In particular, the following Ω-result is established.

Theorem 1.6. *Let $\sigma > 1/2$. There exist a positive constant c_σ depending on σ only and a positive absolute constant c, such that for any integer $\nu \ge 2$ such that $\max_{[k,\ell]|\nu} \frac{(k\vee\ell)}{(k,\ell)} \ge c_\sigma$, and $0 \le \varepsilon < \sigma$, we have*

$$\max_{1\le t\le T} |\zeta(\sigma + it)| \ge c\zeta(2\sigma)\Big(\frac{1}{\sigma_{-2\varepsilon}(\nu)} \sum_{n|\nu} \frac{\sigma_{-s+\varepsilon}(n)^2}{n^{2\varepsilon}}\Big)^{1/2},$$

whenever ν and T are such that

$$\frac{\sigma_{-\varepsilon}(\nu)\sigma_{1-\sigma-\varepsilon}(\nu)\log(\nu T)}{\sum_{n|\nu} \frac{\sigma_{-s+\varepsilon}(n)^2}{n^{2\varepsilon}}} \le \frac{\zeta(2\sigma)^{1/2}}{4} T^{(2\sigma-1)}.$$

By taking ν a product of primes, it is easy to recover Theorem 3.3 in [5].
We only sketch the proof of Theorem 1.4, which uses Theorem 1.3.

2. Proof of Theorem 1.4

Choose $N_j = \mathrm{e}^{\mathrm{e}^{j^B}}$, with $B = 2\beta/\delta$ and δ is a (small) positive real. Let $\beta > 1$. Write

$$\sum_{N_j\le k<N_{j+1}} c_k f_k = \sum_{N_j\le k<N_{j+1}} c_k R_k^J + \sum_{N_j\le k<N_{j+1}} c_k r_k^J,$$

where $R^J(x) = \sum_{\ell=1}^{J} \frac{\sin 2\pi\ell x}{\ell}$, $r^J(x) = f(x) - R^J(x)$ (J being defined later as a function of j). As $f \in \mathrm{BV}(\mathbb{T})$, $a_j = \mathcal{O}(j^{-1})$, and so by Carleson–Hunt's maximal inequality,

$$\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \Big| \sum_{u\le k\le v} c_k R_k \Big| \Big\|_2 \le C(\log J)\Big(\sum_{N_j\le u\le N_{j+1}} c_k^2\Big)^{1/2}.$$

We now combine our Theorem 1.3 with the $(\varepsilon, 1-\varepsilon)$ argument introduced in [1]. Let $0 < \varepsilon < 1/2$. From the bound

$$\delta_{k,\ell}^J := \sum_{\substack{i,j>J\\ jk=i\ell}} \frac{1}{ij} \le C\min\Big(\frac{(k,\ell)}{(k\vee\ell)J}, \frac{(k,\ell)^2}{k\ell}\Big) \le C\Big(\frac{(k,\ell)}{(k\vee\ell)J}\Big)^{\varepsilon}\Big(\frac{(k,\ell)^2}{k\ell}\Big)^{1-\varepsilon} \le \frac{C}{J^\varepsilon}\langle f_k^{1-\varepsilon/2}, f_\ell^{1-\varepsilon/2}\rangle,$$

and since $\|\sum_{u\le k\le v} c_k r_k^J\|_2^2 = \sum_{u\le k,\ell\le v} c_k c_\ell \delta_{k,\ell}^J$, we get, choosing $\tau = 1+\varepsilon$, next using Gronwall's estimate,

$$\Big\|\sum_{u\le k\le v} c_k r_k^J\Big\|_2^2 \le \frac{C}{J^\varepsilon}\Big\|\sum_{u\le k\le v} |c_k| f_k^{1-\varepsilon/2}\Big\|_2^2 \le \frac{C}{\varepsilon J^\varepsilon}\Big(\sum_{u\le k\le v} c_k^2 \sigma_{-1+2\varepsilon}(k)\Big) \le \frac{C}{\varepsilon J^\varepsilon}\exp\Big\{\frac{\varrho}{2\varepsilon}\frac{(\log N_{j+1})^{2\varepsilon}}{\log\log N_{j+1}}\Big\},$$

where ϱ is some positive number. By a well-known variant of Rademacher–Menshov's maximal inequality,

$$\Big\| \sup_{N_j \le u \le v \le N_{j+1}} \Big| \sum_{u \le k \le v} c_k r_k^J \Big| \Big\|_2^2 \le \frac{C}{\varepsilon J^\varepsilon} (\log N_{j+1})^2 \exp\Big\{ \frac{\varrho}{2\varepsilon} \frac{(\log N_{j+1})^{2\varepsilon}}{\log\log N_{j+1}} \Big\} \Big(\sum_{N_j \le k \le N_{j+1}} c_k^2 \Big).$$

Choose $\varepsilon J^\varepsilon = (\log N_{j+1})^2 \exp\{ \frac{\varrho}{\varepsilon} \frac{(\log N_{j+1})^{2\varepsilon}}{\log\log N_{j+1}} \}$ with $\varepsilon = \frac{\log\log\log N_{j+1}}{2\log\log N_{j+1}}$. Then $\log J \le C \frac{(\log\log N_{j+1})^2}{(\log\log\log N_{j+1})}$, and by combining

$$\Big\| \sup_{N_j \le u \le v \le N_{j+1}} \Big| \sum_{u \le k \le v} c_k f_k \Big| \Big\|_2^2 \le C \sum_{N_j \le u \le N_{j+1}} c_k^2 \frac{(\log\log k)^4}{(\log\log\log k)^2}. \tag{4}$$

The assumption made implies that the oscillation of the sequence $\{\sum_{k=1}^N c_k f_k, N \ge 1\}$ around the subsequence $\{\sum_{k=1}^{N_j} c_k f_k, j \ge 1\}$ tends to zero almost everywhere. Now, by Tchebycheff's inequality,

$$\lambda\Big\{ \sup_{N_j \le u \le v \le N_{j+1}} \Big| \sum_{u \le k \le v} c_k r_k^J \Big| > j^{-\beta} \Big\} \le C j^{2\beta} \sum_{N_j \le k \le N_{j+1}} c_k^2 \le C \sum_{N_j \le u \le N_{j+1}} c_k^2 (\log\log k).$$

Borel–Cantelli's lemma implies that the series $\sum_j |\sum_{N_j < u \le N_{j+1}} c_k r_k^J|$ converges almost everywhere. The treatment of the other sum is more tricky. Let h and H be such that $J^h < N_j \le J^{h+1} \le \ldots \le J^{h+H-1} \le N_{j+1} < J^{h+H}$. One first observe that

$$\begin{aligned} \Big\| \sum_{N_j < k \le N_{j+1}} c_k R_k^J \Big\|_2^2 &\le \zeta(2) \sum_{\substack{N_j < k, \ell \le N_{j+1} \\ (k \vee \ell) \le J(k \wedge \ell)}} |c_k||c_\ell| \frac{(k,\ell)^2}{k\ell} \le \big(4\zeta(2)\log J\big) \sum_{\mu=h}^{H} \sum_{J^{\mu-1} \le k \le J^{\mu+2}} c_k^2 \sigma_{-1}(k) \\ &\le C \sum_{J^{-1} N_j < k \le N_{j+1} J^2} c_k^2 \frac{(\log\log k)^2}{\log\log\log k} \sigma_{-1}(k). \end{aligned} \tag{5}$$

By Tchebycheff's inequality,

$$\begin{aligned} \lambda\Big\{ \Big| \sum_{N_j < k \le N_{j+1}} c_k R_k^J \Big| > j^{-\beta} \Big\} &\le C j^{2\beta} \sum_{J^{-1} N_j < k \le N_{j+1} J^2} c_k^2 \frac{(\log\log k)^2}{\log\log\log k} \sigma_{-1}(k) \\ &\le C \sum_{J^{-1} N_j < k \le N_{j+1} J^2} c_k^2 \frac{(\log\log k)^{2+\delta}}{\log\log\log k} \sigma_{-1}(k). \end{aligned}$$

Treating separately sums with odd indices and sums with even indices allows us to show, by Borel–Cantelli's lemma, that the series

$$\sum_j \Big| \sum_{N_j < k \le N_{j+1}} c_k R_k^J \Big|$$

converges almost everywhere. This allows us to conclude.

Final note. In a very recent work, Lewko and Radziwill (arXiv:1408.2334v1) proposed a new approach to Gál's theorem. They could also reduce the condition $\gamma > 4$ in Remark 2 to $\gamma > 2$. This naturally includes our Theorem 1.4, but not our Theorem 1.5 with arithmetical multipliers. Further, the new argument we introduced in the proof of Theorem 1.4 suggests a possibility to improve Lewko and Radziwill's convergence condition by requiring only that $\sum_k c_k^2 (\log\log k)^2/(\log\log\log k)^2 < \infty$.

References

[1] C. Aistleitner, I. Berkes, K. Seip, GCD sums from Poisson integrals and systems of dilated functions, J. Eur. Math. Soc. 17 (6) (2015) 1517–1546.
[2] C. Aistleitner, I. Berkes, K. Seip, M. Weber, Convergence of series of dilated functions and spectral norms of GCD matrices, Acta Arith. 168 (3) (2015) 221–246.
[3] I. Berkes, M. Weber, On the convergence of $\sum c_k f(n_k x)$, Mem. Amer. Math. Soc. 201 (943) (2009), vi+72.
[4] I. Berkes, M. Weber, On series of dilated functions, Q. J. Math. 65 (1) (2014) 2552.
[5] T. Hilberdink, An arithmetical mapping and applications to Ω-results for the Riemann Zeta function, Acta Arith. 139 (2009) 341–367.
[6] P. Lindqvist, K. Seip, Note on some greatest common divisor matrices, Acta Arith. LXXXIV (2) (1998) 149–154.
[7] M. Weber, On systems of dilated functions, C. R. Acad. Sci. Paris, Ser. I 349 (2011) 1261–1263.
[8] M. Weber, An arithmetical approach to the convergence problem of series of dilated functions and its connection with the Riemann Zeta function, http://arxiv.org/abs/1407.2722v2, 2014.